Weak prime $L$–fuzzy filters of semilattices

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Abstract

The concept of weak prime $L$–fuzzy filter of a semilattice $S$ is introduced and example are given. A characterization of weak prime $L$–fuzzy filters is established and prime filters of $S$ are identified with weak prime $L$–fuzzy filters. Also, minimal weak prime $L$–fuzzy filters are characterized.

Keywords: Bounded semilattice, $L$–fuzzy filter, prime $L$–fuzzy filter, weak prime $L$–fuzzy filter, frame.

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1. Introduction

Zadeh, in his pioneering work [11] introduced the notion of a fuzzy subset $A$ of a non-empty set $X$ as a function from $X$ into $[0, 1]$. Rosenfield [6] applied this notion to develop the theory of groups. Goguen [1] generalized and continued the work of Zadeh and realized that the unit interval $[0, 1]$ is not sufficient to take the truth values of general fuzzy statements. Therewith, several researchers took interest to the fuzzyfication of algebraic structures. In which, Kuroki [2], Liu [3], Malik and Mordersan [4], and Mukherjee and Sen [5] are engaged in fuzzifying various concepts and obtained significant results of algebras.

Further, Swamy and Swamy [10] have introduced the concept of a fuzzy prime ideal of a ring and developed the theory of fuzzy ideals by assuming truth values in a complete lattice $L$ satisfying the infinite meet distributive law, such lattices are called frames. The concept of prime ideal is vital in the study of structure theory of distributive lattices. In [8], the authors have introduced and studied the notion of $L$–fuzzy filters of a semilattice $S$ with truth values in a frame $L$. It is proved that $S$ is distributive iff the lattice $\mathcal{F}(S)$ of all filters of $S$ is distributive iff the lattice $\mathcal{F}_L(\mathcal{F}(S))$ of all $L$–fuzzy filters of $S$ is distributive. In [9], the authors have introduced the concept of prime $L$–fuzzy filters of a bounded semilattice $S$, which are meet-prime elements in the lattice $\mathcal{F}_L(\mathcal{F}(S))$. Further, in [7] the authors have introduced the notion of $L$–fuzzy ideals of a semilattice $S$ and obtained significant results on this.

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The aim of this paper is to study the $L$–fuzzy filters $A$ of a bounded semilattice $S$ for which each $\alpha$-cut $A_\alpha$, i.e., $A_\alpha = \{x \in S : A(x) \geq \alpha\}$ is either a prime filter of $S$ or whole semilattice $S$. This paper consists of four sections. In the second section, we recall some definitions and certain results. In third section, we introduce the concept of a weak prime $L$–fuzzy filter (WPLF) of a bounded semilattice $S$ and characterize these. Fourth section deals with minimal weak prime $L$–fuzzy filters (Minimal WPLFs).

Throughout this paper, $S$ stands for a bounded semilattice $(S, \wedge, 0, 1)$ unless otherwise stated. And, $L$ stands for a non-trivial frame $(L, \wedge, \vee, 0, 1)$; i.e., a complete lattice satisfying the infinite meet distributive law

$$\alpha \wedge (\bigvee_{\beta \in T} \beta) = \bigvee_{\beta \in T} (\alpha \wedge \beta),$$

for all $\alpha \in L$ and any $T \subseteq L$. Here the operations $\wedge$ and $\vee$ are supremum and infimum in the lattice $L$. An element $1 \neq c \in L$ is said to be meet-prime if, for any $a, b \in L$ and $a \wedge b \leq c$ imply $a \leq c$ or $b \leq c$.

2. Preliminaries

In this section, we collect some definitions and certain results from [8, 9], that we need in sequel.

A semilattice (meet-semilattice) is an algebra $S = (S, \wedge)$ satisfying the axioms

1. $x \wedge x = x$;
2. $x \wedge y = y \wedge x$; and
3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, for all $x, y, z \in S$.

If we define $x \leq y$ iff $x \wedge y = x$, then $\leq$ is a partial order on $S$ in which $x \wedge y$ is the inf$(x, y)$ in $S$. A non-empty subset $F$ of $S$ is said to be a final segment of $S$ if, for any $x \in F, y \in S$ and $x \leq y$ implies $y \in F$. A filter of a semilattice $S$ is a final segment $F$ of $S$ such that $x \wedge y \in F$ for all $x, y \in F$. The principal filter generated by an element $a$ of $S$, i.e., the set $\{x \in S : x \geq a\}$ will be denoted by $[a]$. A proper filter $P$ of a semilattice $S$ is said to be prime if whenever two filters $G$ and $H$ are such that $\phi \neq G \cap H \subseteq P$ imply either $G \subseteq P$ or $H \subseteq P$ (or equivalently, if, for any $a, b$ are such that $a \notin P$ and $b \notin P$ imply the existence of $x \in S$ such that $a \leq x, b \leq x$ and $x \notin P$).

Definition 2.1. Let $X$ be any non-empty set and $L$ a frame. Any function $A : X \to L$ is called an $L$–fuzzy subset of $X$. For any $L$–fuzzy subset $A$ of $X$ and $\alpha \in L, A_\alpha$ denotes $\alpha$-cut of $A$, i.e.,

$$A_\alpha = \{x \in X : \alpha \leq A(x)\}.$$

Definition 2.2. For any $L$–fuzzy subsets $A$ and $B$ of $X$, define

$$A \leq B \iff A(x) \leq B(x), \text{ for all } x \in X.$$ 

Then $\leq$ is a partial order on the set of $L$–fuzzy subsets of $X$ and is called the point wise ordering.

Result 1. Let $A$ and $B$ be $L$–fuzzy subsets of $X$. Then

$$A \leq B \iff A_\alpha \subseteq B_\alpha, \text{ for all } \alpha \in L.$$ 

Definition 2.3. A proper $L$–fuzzy subset $A$ of $X$ is a non-constant $L$–fuzzy subset of $X$, i.e., $A(x) \neq 1$ for some $x \in X$.

Definition 2.4. An $L$–fuzzy subset $A$ of $S$ is said to be an $L$–fuzzy filter of $S$ if,

$$A(x_0) = 1, \text{ for some } x_0 \in S,$$

and

$$A(x \wedge y) = A(x) \wedge A(y), \text{ for all } x, y \in S.$$
Result 2. The following are equivalent to each other, for any $L$–fuzzy subset $A$ of $S$,

1. $A$ is an $L$–fuzzy filter of $S$.
2. $A(x_0) = 1$ for some $x_0 \in S$, $A(x \land y) \geq A(x) \land A(y)$ and $x \leq y \Rightarrow A(y) \geq A(x)$.
3. $A_\alpha$ is a filter of $S$, for all $\alpha \in L$.

Result 3. Let $A$ be a fuzzy filter of $S$ and $X$ a non-empty subset of $S$, and $x, y \in S$. We have

1. $x \in [X] \Rightarrow A(x) \geq \bigwedge_{i=1}^{m} A(a_i)$ for some $a_1, a_2, \ldots, a_m \in X$, where

\[ [X] = \{ a \in S : \bigwedge_{i=1}^{n} x_i \leq a \text{ for some } x_i \in X \} \]

2. $x \in [y] \Rightarrow A(x) \geq A(y)$.
3. If $S$ is bounded then $A(0) < 1$ and $A(1) = 1$.

Result 4. Let $(S, \land)$ be a bounded semilattice and $\mathcal{F}_L(\mathcal{F}(S))$ denote the lattice all $L$–fuzzy filters of $S$. Then the following are equivalent to each other:

1. $\mathcal{F}_L(\mathcal{F}(S))$ is a distributive.
2. $\mathcal{F}(S)$ is a distributive, where $\mathcal{F}(S)$ denotes the lattice of filters of $S$.
3. $S$ is distributive.

Definition 2.5. A proper $L$–fuzzy filter $A$ of a bounded semilattice $S$ is said to be prime $L$–fuzzy filter of $S$ if, for any $L$–fuzzy filters $B$ and $C$ of $S$,

\[ B \land C \leq A \Rightarrow B \leq A \text{ or } C \leq A, \]

where $(B \land C)(x) = B(x) \land C(x)$.

Result 5. Let $A$ be an $L$–fuzzy filter of $S$. Then $A$ is prime $L$–fuzzy filter of $S$ if and only if, the following are satisfied.

1. $|\text{Im}(A)| = 2$, i.e., $A$ is two-valued.
2. For any $x \in S$, either $A(x) = 1$ or $A(x)$ is meet-prime element in $L$.
3. $A_1$ is a prime filter of $S$.

Result 6. Let $A$ be an $L$–fuzzy filter of $S$. Then $A$ is a prime $L$–fuzzy filter of $S$ if there exists a prime filter $P$ of $S$ and a meet-prime element $\alpha$ in $L$ such that $A = A_\alpha^P$, where

\[ A_\alpha^P(x) = \begin{cases} 1 & \text{if } x \in P, \\ \alpha & \text{if } x \notin P. \end{cases} \]

3. Weak prime $L$–Fuzzy filters (WPLF)

Let us recall that an $L$–fuzzy subset $A$ of $S$ is an $L$–fuzzy filter of $S$ iff $A_\alpha$ is a filter of $S$ for each $\alpha \in L$.

Definition 3.1. A proper $L$–fuzzy filter $A$ of $S$ is called a weak prime $L$–fuzzy filter (WPLF), if for each $\alpha \in L$, $A_\alpha$ is a prime filter of $S$ or $A_\alpha = S$. 
Example 3.2. Consider the semilattice $S$ whose Hasse-diagram is as depicted in Figure 1 and $L = [0, 1]$, the closed interval of real numbers which is a frame in which, for any $x, y \in L$,

$$x \lor y = \max\{x, y\}, \ x \land y = \min\{x, y\}.$$  

![Figure 1: Hasse-diagram of Semilattice $S$.]

Clearly $\{a\}, \{b\}$ and $\{a, b, c\}$ are all prime filters of $S$. Now, define $A : S \to L$ as follows:

$$A = \{(0, 0), (c, 0.5), (b, 0.5), (a, 1)\}.$$  

Then $A$ is a WPLF; since, the $\alpha$-cuts of $A$ are

- $A_0 = S$
- $A_1 = \{a\}$
- $A_{0.5} = \{a, b, c\}$
- $A_\alpha = \{a\}$, for any $\alpha \in (0.5, 1)$,

and

$A_\alpha = \{a, b, c\}$, for any $\alpha \in (0, 0.5)$.

Theorem 3.3. Let $A$ be a proper $L$-fuzzy filter of $S$. If $A$ is a WPLF of $S$, then $\text{Im}(A)$ is a chain.

Proof. Let $a$ and $b \in S$ and put $\alpha = A(a) \lor A(b)$. Then,

$$x \in [a] \cap [b] \Rightarrow a \leq x \text{ and } b \leq x$$

$$\Rightarrow A(a) \leq A(x) \text{ and } A(b) \leq A(x)$$

$$\Rightarrow \alpha = A(a) \lor A(b) \leq A(x)$$

$$\Rightarrow x \in A_\alpha.$$  

Therefore $[a] \cap [b] \subseteq A_\alpha$. Since $A_\alpha$ is prime, $[a] \subseteq A_\alpha$ or $[b] \subseteq A_\alpha$.

$$[a] \subseteq A_\alpha \Rightarrow a \in A_\alpha \Rightarrow \alpha = A(a) \lor A(b) \leq A(a)$$

$$\Rightarrow A(b) \leq A(a).$$

Similarly, $[b] \subseteq A_\alpha \Rightarrow A(a) \leq A(b)$. Thus $\text{Im}(A)$ is a chain in $L$. \qed

The converse of above theorem is not true. For, consider the following example.

Example 3.4. Consider two lattices $S$ and $L$ whose Hasse-diagrams are given in Figure 2 and Figure 3 respectively, where $S = \{0, c, a, b, 1\}$ and $L = \{0, s, 1\}$. 


Clearly for any L-fuzzy filter $A$ of $S$, $\text{Im}(A)$ is a chain. Define $A : S \rightarrow L$ as

$$A = \{(0,0), (c,s), (a,s), (b,s), (1,1)\}.$$ 

Then the $\alpha$-cuts of $A$ are $A_0 = S$, $A_s = \{c,a,b,1\}$ and $A_1 = \{1\}$, which are filters of $S$. Therefore $A$ is an L–fuzzy filter of $S$. However $A$ is not WPLF because $A_1$ is not prime since $[a] \cap [b] = \{1\}$.

The following gives a characterization of WPLFs.

**Theorem 3.5.** For any L–fuzzy filter $A$ of $S$, the following are equivalent:

1. $A$ is a WPLF of $S$.
2. For any $a$ and $b \in S$, 
   $$\bigwedge \{A(x) : x \in [a] \cap [b]\} = A(a) \sqcap A(b).$$
3. For any $a$ and $b \in S$, 
   $$\bigwedge \{A(x) : x \in [a] \cap [b]\} = A(a) \sqcup A(b),$$

and 

$$\text{Im}(A)$$ is a chain in $L$.

**Proof.** First note that for any $a$ and $b \in S$,

$$A(a) \text{ and } A(b) \leq \bigwedge \{A(x) : x \in [a] \cap [b]\}.$$ 

(1) $\Rightarrow$ (2): Let $a$ and $b \in S$ and put $\alpha = \bigwedge \{A(x) : x \in [a] \cap [b]\}$. Then $\alpha \leq A(x)$ for all $x \in [a] \cap [b]$, so that $[a] \cap [b] \subseteq \alpha$. By (1) $A_\alpha$ is a prime filter of $S$ and hence $[a] \subseteq A_\alpha$ or $[b] \subseteq A_\alpha$. So that $a \in A_\alpha$ or $b \in A_\alpha$, i.e., $\alpha \leq A(a)$ or $\alpha \leq A(b)$. This implies $\alpha = A(a)$ or $A(b)$.

(2) $\Rightarrow$ (3): Let $a$ and $b \in S$. Then, by (2),

$$\bigwedge \{A(x) : x \in [a] \cap [b]\} = A(a) \text{ or } A(b),$$

and hence $A(b) \leq A(a) \text{ or } A(a) \leq A(b)$. Therefore $\text{Im}(A)$ is a chain in $L$. Also, by (2) and since $A(a)$, $A(b)$ are lower bounds of $\{A(x) : x \in [a] \cap [b]\}$, it follows that 

$$\bigwedge \{A(x) : x \in [a] \cap [b]\} = \max \{A(a), A(b)\} = A(a) \sqcup A(b).$$
(3) \(\Rightarrow\) (1): Let \(\alpha \in L\). Such that \(A_\alpha \neq S\). Let \(G\) and \(H\) be two filters of \(S\) such that \(G \not\subseteq A_\alpha\) and \(H \not\subseteq A_\alpha\). Then, \(\alpha \not\subseteq A(a)\) and \(\alpha \not\subseteq A(b)\) for some \(a, b \in S\). By (3), \(A(a) \subseteq A(b)\) or \(A(b) \subseteq A(a)\). Hence,
\[
\alpha \not\subseteq \max(A(a), A(b)) = A(a) \lor A(b).
\]
Also, by (3),
\[
\alpha \not\subseteq \bigwedge \{A(x) : x \in [a] \cap [b]\}.
\]
Hence \(\alpha \not\subseteq A(x)\) for some \(x \in [a] \cap [b]\). This implies \(G \cap H \not\subseteq A_\alpha\). Hence \(A_\alpha\) is a prime filter of \(S\). Thus \(A\) is a WPLF of \(S\).

Now, we slightly generalize an \(\alpha\)-level \(L\)-fuzzy filter \(A^F_\alpha\) corresponding to a filter \(F\) (see Result 6).

**Definition 3.6.** For any filter \(F\) of \(S\) and \(\alpha, \beta \in L\), define an \(L\)-fuzzy subset \(A^F_{\alpha, \beta}\) of \(S\) as follows:
\[
A^F_{\alpha, \beta}(x) = \begin{cases} 
1 & \text{if } x = 1, \\
\alpha & \text{if } 1 \not\in F, \\
\beta & \text{if } x \not\in F.
\end{cases}
\]
Note that \(A^F_{1,1} = A^F_1\) and \(A^F_{1,0} = \chi_r\), the characteristic function corresponding to \(F\).

The following is straightforward verification.

**Lemma 3.7.** Let \(F\) be a proper filter of \(S\) and \(\alpha, \beta \in L\). Then
\[
A^F_{\alpha, \beta}\text{ is an } L\text{-fuzzy filter of } S \iff \beta \leq \alpha,
\]
and, in the case,
\[
A^F_{\alpha, \beta}\text{ is proper } \iff \beta < 1.
\]

**Theorem 3.8.** For any proper filter \(P\) of \(S\), the following are equivalent:

1. \(P\) is a prime filter of \(S\).
2. \(A^P_{1,\beta}\) is a WPLF of \(S\) for each \(\beta < 1\).
3. \(\chi_r\) is a WPLF of \(S\).

**Proof.** (1) \(\Rightarrow\) (2): Suppose \(P\) is prime and let \(\beta < 1\) in \(L\). Put \(\Lambda = A^P_{1,\beta}\). Then,
\[
\Lambda(x) = \begin{cases} 
1 & \text{if } x \in P, \\
\beta & \text{if } x \not\in P.
\end{cases}
\]
Let \(a\) and \(b\) \(\in S\). Then,
\[
a \in P \text{ or } b \in P \Rightarrow \Lambda(a) = 1 \text{ or } \Lambda(b) = 1 \text{ and } [a] \cap [b] \subseteq P
\Rightarrow \Lambda(a) = 1 \text{ or } \Lambda(b) = 1 \text{ and } \Lambda(x) = 1 \text{ for all } x \in [a] \cap [b]
\Rightarrow \bigwedge \{\Lambda(x) : x \in [a] \cap [b]\} = 1 = \Lambda(a) \text{ or } \Lambda(b).
\]
\[
a \not\in P \text{ and } b \not\in P \Rightarrow \Lambda(a) = \beta = \Lambda(b) \text{ and } [a] \cap [b] \not\subseteq P
\Rightarrow \Lambda(a) = \beta = \Lambda(b) \text{ and } x \not\in P \text{ for some } x \in [a] \cap [b]
\Rightarrow \bigwedge \{\Lambda(x) : x \in [a] \cap [b]\} = \beta = \Lambda(a) = \Lambda(b).
\]
Therefore
\[ \bigwedge \{ A(x) : x \in [a] \cap [b] \} = A(a) \text{ or } A(b). \]

Thus A is WPLF.

(2) \implies (3): It is clear by the fact that \( \chi_p = A_{1,0}^P \).

(3) \implies (1): Suppose \( \chi_p \) is WPLF. Let \( a \) and \( b \) be \s
such that \( a \not\in P \) and \( b \not\in P \). Then \( \chi_p(a) = 0 = \chi_p(b) \).
By supposition and hence by Theorem 3.5,
\[ \bigwedge \{ \chi_p(x) : x \in [a] \cap [b] \} = \chi_p(a) \text{ or } \chi_p(b). \]

So that
\[ \bigwedge \{ \chi_p(x) : x \in [a] \cap [b] \} = 0. \]

Hence \( \chi_p(x) = 0 \) for some \( x \in [a] \cap [b] \). (for, \( \chi_p(x) = 1 \) for all \( x \in [a] \cap [b] \) \( \implies \chi_p(a) = 1 \) or \( \chi_p(b) = 1 \); a contradiction). Therefore \( x \not\in P \). So that \( [a] \cap [b] \not\subseteq P \). Thus P is prime.

Lemma 3.9. For any bounded semilattice \( S \), the following are equivalent:

1. \( [a] \) is a meet-prime element in the lattice \( \mathcal{F}(S) \) of all filters of \( S \).

2. For any \( 1 \neq a \) and \( 1 \neq b \) in \( S \), there exists \( 1 \neq c \in S \) such that \( c \geq a \) and \( b \), i.e., \( c \in [a] \cap [b] \).

Theorem 3.10. Let \( P \) be a proper filter of \( S \) and suppose that \( [1] \) is a meet-prime element in the lattice \( \mathcal{F}(S) \) of filters of \( S \). Then \( P \) is prime iff \( A_{\alpha,\beta}^P \) is WPLF for all \( 1 \neq \beta \leq \alpha \) in \( L \).

Proof. Suppose \( P \) is prime and \( 1 \neq \beta \leq \alpha \in L \). Put \( A = A_{\alpha,\beta}^P \). Then \( A \) is a proper \( L \)-fuzzy filter of \( S \) (by Lemma 3.7). Let \( a \) and \( b \) be \s
Then \( A(a) \) and \( A(b) \leq A(x) \) for all \( x \in [a] \cap [b] \). Let \( \gamma \in L \) such that \( \gamma \leq A(x) \) for all \( x \in [a] \cap [b] \). Now,
\[ a = 1 \text{ or } b = 1 \implies A(a) = 1 \text{ or } A(b) \text{ and hence } V \subseteq A(a) = 1 \text{ or } A(b) \]
\[ \implies \bigwedge \{ A(x) : x \in [a] \cap [b] \} = A(a) \text{ or } A(b) \]
\[ a \not\in P \text{ and } b \not\in P \implies A(a) = \beta = A(b) \text{ and } [a] \cap [b] \not\subseteq P \]
\[ \implies A(a) = \beta = A(b) \text{ and } A(x) = \beta \text{ for some } x \in [a] \cap [b] \]
\[ \implies \gamma \leq A(x) = \beta = A(a) = A(b) \]
\[ \implies \bigwedge \{ A(x) : x \in [a] \cap [b] \} = A(a) = A(b), \]

and
\[ 1 \neq a \in P, 1 \neq b \in P \implies A(a) = \alpha = A(b) \text{ and there exists } 1 \neq c \in S \text{ such that } c \in [a] \cap [b] \subseteq P \]
\[ \implies \gamma \leq A(c) = \alpha = A(a) = A(b) \]
\[ \implies \bigwedge \{ A(x) : x \in [a] \cap [b] \} = A(a) = A(b). \]

Thus, by Theorem 3.5, \( A \) is WPLF.

Finally in this section we discuss an inter-relationship between \( L \)-fuzzy filters (refer Result 6) and WPLFs.

Theorem 3.11. Every prime \( L \)-fuzzy filter of \( S \) is WPLF.

Proof. Let \( B \) be a Prime \( L \)-fuzzy filter of \( S \). Then, \( B = A_{\alpha}^P \) for some \( \alpha \) in \( L \) and \( B \) is a meet-prime element in \( L \). Since \( P \) is prime and \( \alpha < 1 \), we have \( A_{\alpha}^P \) is a WPLF of \( S \) (by Theorem 3.8). Thus \( B \) is WPLF.
The converse of the above theorem is true. For, consider the example given in the following.

**Example 3.12.** Let $S$ be the 5-element lattice $\{0, b, c, a, 1\}$ represented by the Hasse-diagram given below Figure 4 and $L$ be the 3-element chain $\{0, s, 1\}$ with $0 < s < 1$.

![Figure 4: Hasse-diagram of 5-element lattice $S$.](image)

Define $A : S \rightarrow L$ by $A = \{(0,0), (b,s), (c,0), (a,s), (1,1)\}$. Then $A$ is a proper $L$–fuzzy filter of $S$. Here the $\alpha$-cuts of $A$ are $A_0 = S$, $A_s = \{b, a, 1\}$ and $A_1 = \{1\}$, which are prime filters of $S$. Hence $A$ is WPLF. But $A$ is not Prime $L$–fuzzy filter since $A$ is not two-valued.

4. Minimal WPLF

By a minimal prime filter $M$ of $S$, we mean that there is no prime filter $Q$ of $S$ such that $Q \subset M$ and analogously, a minimal WPLF is a minimal element in the set of all WPLFs under the point-wise partial ordering.

**Theorem 4.1.** Let $A$ be a WPLF of $S$. If $A$ is a minimal WPLF of $S$, then $A_1$, i.e., 1-cut of $A$ is a minimal prime filter of $S$.

**Proof.** Suppose that $A$ is a minimal WPLF of $S$. Then $A_1 = \{x \in S : A(x) = 1\}$ is a prime filter of $S$. To prove $A_1$ is minimal, let $Q$ be a prime ideal of $S$ such that $Q \subset A_1$. Then, choose $x \in A_1$ such that $x \notin Q$. Since $Q$ is prime and hence by Theorem 3.8, $\chi_Q$ is a WPLF of $S$ and $\chi_Q(x) < A(x)$. Therefore $\chi_Q \not\leq A$. This shows that $A$ is not minimal; a contradiction. Thus $A_1$ is a minimal prime filter of $S$.

Converse of above theorem is not true. For example, in Example 3.2, $A$ is an WPLF and $A_1 = \{a\}$ which is a minimal prime filter of $S$. But $A$ is not minimal. If we define $B : S \rightarrow L$ by $B = \{(0,0), (c,0.25), (b,0.25), (a,1)\}$, then $B$ is a WPLF of $S$ and $B \not\leq A$.

**Theorem 4.2.** Let $A$ be a WPLF of $S$ and $(1)$ is a meet-prime element in the lattice $\mathcal{F}(S)$ filters of $S$. Then, $A$ is a minimal WPLF of $S$ iff, $A_\alpha$ is a minimal prime filter of $S$, for each $\alpha \in L$.

**Proof.** Assume that $A$ is a minimal WPLF of $S$. If $A_\beta$ is not a minimal prime filter of $S$ for some $0 < \beta < 1$. Then, there exists a prime filter $P$ of $S$ such that $P \subset A_\beta$. Now, define $B : S \rightarrow L$ by

$$B(x) = \begin{cases} 1 & \text{if } x = 1, \\ \beta & \text{if } 1 \neq x \in P, \\ 0 & \text{if } x \notin P. \end{cases}$$

Clearly $B = A^P_{\beta,0}$. By Theorem 3.5, $B$ is a WPLF of $S$. As $P \subset A_\beta$, choose $y \in A_\beta$ such that $y \notin P$. Then,$

$$\beta \leq A(y), \quad \text{and} \quad B(y) < A(y).$$
Also $B(x) \leq A(x)$ for all $x \neq y \in S$. Therefore $B \not\subseteq A$; a contradiction to our assumption. Thus $A_\alpha$ is a minimal prime filter of $S$ for all $\alpha \in L$.

Conversely, assume that $A_\alpha$ is a minimal prime filter of $S$ for all $\alpha \in L$. If $B$ is a WPLF of $S$ such that $B \leq A$. Then, $B_\alpha \subseteq A_\alpha$ for all $\alpha \in L$. By assumption, $A_\alpha = B_\alpha$ for all $\alpha \in L$. Hence $B = A$. Thus $A$ is a minimal WPLF of $S$. $\square$

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