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Norm of matrix operator on Orlicz-binomial spaces and related operator ideal

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Abstract

The purpose of this article is to introduce Orlicz extension of binomial sequence spaces $\mathbf{b}_{\varphi}^{r,s}$ and investigate some topological and inclusion properties of the new spaces. We give an upper estimation of $\|\mathbf{A}\|_{\ell_{\varphi},\mathbf{b}_{\varphi}^{r,s}}$, where $\mathbf{A}$ is the Hausdorff matrix operator or Nörlund matrix operator. A Hardy type formula is established in the case of Hausdorff matrix operator. Finally we introduce operator ideal using the space $\mathbf{b}_{\varphi}^{r,s}$ and the sequence of s-number function and prove its completeness under certain assumptions.

Keywords: Binomial sequence space, upper bounds, Hausdorff Matrix, Nörlund Matrix, Orlicz function, s-number, operator ideal.

2020 MSC: 26D15, 46A45, 47A30, 40G05, 47L20, 47B06.

1. Introduction

Throughout this paper $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and ω denotes the linear space of all real sequences. Any vector subspace of ω is called a sequence space. Also by ℓ_p, we mean the space of absolutely p-summable series, where $1 \leqslant p < \infty$. The space ℓ_p is a Banach space according to the ℓ_p norm given by

$$\|\mathbf{x}\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{\frac{1}{p}}. \tag{1.1}$$

For some recent papers on sequence spaces, one may refer to [5, 8, 16, 20, 36, 39, 40]. Let X and Y be two sequence spaces and let $\mathbf{A} = (a_{nk})$ be an infinite matrix of real entries. We write $\mathbf{A}_n$ to denote the sequences in the nth row of the matrix $\mathbf{A}$. We recall that $\mathbf{A}$ defines a matrix mapping from X to Y if for

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every sequence $\mathbf{x} = (x_k)$, the $\mathbf{A}$-transform of $\mathbf{x}$, i.e., $\mathbf{Ax} = \{\mathbf{A}_n\mathbf{x}\}_{n=0}^{\infty} \in Y$, where

$$\mathbf{A}_n\mathbf{x} = \sum_{k=0}^{\infty} a_{nk}x_k, \quad n \in \mathbb{N}_0.$$

An Orlicz function φ is a function from $(0,\infty)$ to $(0,\infty)$ which is continuous, increasing and convex with $\varphi(0+) = 0$ and so has a unique inverse $\varphi^{-1} : (0,\infty) \to (0,\infty)$. As usual in the Orlicz theory the domain of φ is extended to the real line by $\varphi(x) = \varphi(|x|)$ and $\varphi(0) = 0$ (for details on Orlicz functions see [12, 13, 27]).

Let $\mathbf{x} = (x_n)$ be a sequence of real numbers with $x_n > 0$ for all $n \in \mathbb{N}_0$. The Orlicz sequence space is defined as

$$\ell_\varphi = \left\{ \mathbf{x} \in \omega : \sum_{n=0}^{\infty} \varphi\left(\frac{x_n}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space ℓ_φ is a Banach space equipped with the Orlicz-Luxemborg norm $\|\cdot\|_\varphi$, defined by

$$\|x\|_\varphi = \inf\left\{ \rho > 0 : \sum_{n=0}^{\infty} \varphi\left(\frac{x_n}{\rho}\right) \leqslant 1 \right\}. \tag{1.2}$$

Clearly $|x_n| \leqslant |y_n|$ for all $n \in \mathbb{N}_0$, then $\|\mathbf{x}\|_\varphi \leqslant \|\mathbf{y}\|_\varphi$. Further if $0 < \|\mathbf{x}\|_\varphi < \infty$, then $\sum\limits_{n=0}^{\infty} \varphi\left(\frac{x_n}{\|\mathbf{x}\|_\varphi}\right) \leqslant 1$ [25, Lemma 1].

In particular, if $\varphi(t) = |t|^p$, $p \geqslant 1$, then the space ℓ_φ reduces to the ℓ_p space and the norm $\|\mathbf{x}\|_\varphi$ given by (1.2) reduces to the norm $\|\mathbf{x}\|_p$ given by (1.1).

By supermultiplicative function, we shall mean any function $\varphi : (0,\infty) \to (0,\infty)$ such that for all positive a and b

$$\varphi(ab) \geqslant \varphi(a)\varphi(b).$$

An immediate example of supermultiplicative function is $\varphi(t) = t^p$, $p \geqslant 1$. Throughout the article, we consider this supermultiplicative Orlicz function φ which satisfies $\varphi(1) = 1$.

We recall that an upper bound for a matrix operator $\mathcal{T}$ from a sequence space X into another sequence space Y is the value of U satisfying the inequality

$$\|\mathcal{T}x\|_Y \leqslant U\|x\|_X,$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms on the spaces X and Y, respectively. Here, U does not depend on x. The best possible value of U is regarded as the operator norm of $\mathcal{T}$.

The Euler mean matrix $\mathbf{E}^r$ of order r is defined by the matrix $\mathbf{E}^r = (e^r_{nk})$, where $0 < r < 1$ and

$$e^r_{nk} = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k, & (0 \leqslant k \leqslant n), \\ 0, & (k > n). \end{cases}$$

The Euler sequence spaces $\mathbf{e}^r_p$ and $\mathbf{e}^r_\infty$ were introduced by Altay et al. [7] as follows (also see Altay and Başar [6]):

$$\mathbf{e}^r_p = \left\{ \mathbf{x} = (x_k) \in \omega : \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \binom{n}{k}(1-r)^{n-k}r^k x_k \right|^p < \infty \right\}$$

and

$$\mathbf{e}^r_\infty = \left\{ \mathbf{x} = (x_k) \in \omega : \sup_{n\in\mathbb{N}_0} \left| \sum_{k=0}^{n} \binom{n}{k}(1-r)^{n-k}r^k x_k \right| < \infty \right\}.$$

Let $r, s \in \mathbb{R}$ and $r + s \neq 0$, then the binomial matrix $\mathbf{B}^{r,s} = (b^{r,s}_{nk})$ is defined by:

$$b^{r,s}_{nk} = \begin{cases} \frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k, & (0 \leqslant k \leqslant n), \\ 0, & (k > n). \end{cases}$$

Bişgin [10, 11] obtained another generalization of Euler sequence spaces by introducing the binomial sequence spaces $\mathbf{b}_p^{r,s}$ and $\mathbf{b}_\infty^{r,s}$ as follows:

$$\mathbf{b}_p^{r,s} = \left\{ \mathbf{x} = (x_k) \in \omega : \sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^p < \infty \right\}$$

and

$$\mathbf{b}_\infty^{r,s} = \left\{ \mathbf{x} = (x_k) \in \omega : \sup_{n \in \mathbb{N}_0} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right| < \infty \right\}.$$

It is clear that, if we take $r + s = 1$, then the binomial matrix $\mathbf{B}^{r,s}$ reduces to the Euler mean matrix $\mathbf{E}^r$ of order r. Thus binomial matrix generalizes the Euler mean matrix.

Euler weighted sequence space $\mathbf{e}_{w,p}^{\theta}$ has been studied recently by Talebi and Dehgan [37] as follows:

$$\mathbf{e}_{w,p}^{\theta} = \left\{ \mathbf{x} = (x_k) \in \omega : \sum_{n=0}^{\infty} w_n \left| \sum_{k=0}^{n} \binom{n}{k} (1-\theta)^{n-k} \theta^k x_k \right|^p < \infty \right\},$$

where $1 \leqslant p < \infty$, $0 < \theta < 1$ and $w = (w_n)$ is a decreasing non-negative sequences of real numbers with $\sum\limits_{n=0}^{\infty} \frac{w_n}{n+1} = \infty$.

More recently, Manna [26] has studied Orlicz extension of weighted Euler sequence space and obtained norm inequalities involving generalized Hausdorff and Nörlund matrix operators which strengthen the results of Talebi and Dehgan [38]. The lower bounds of operators on different sequence spaces were studied in [18, 21–24]. Recently Roopaei and Foroutannia [34, 35] and Ilkhan [15] discussed the norms of matrix operators on different sequence spaces. Following Bişgin [10], Manna [26], Talebi and Dehgan [37], we introduce Orlicz extension of binomial sequence spaces $\mathbf{b}_\varphi^{r,s}$.

The paper is organized as follows. In the Section 2, we introduce Orlicz-binomial sequence space $\mathbf{b}_\varphi^{r,s}$, investigate topological properties and inclusion relations. In the Section 3, we give an upper bound estimation for the norm of Hausdorff matrix as an operator from ℓ_φ to $\mathbf{b}_\varphi^{r,s}$ and provide some immediate corollaries. In the Section 4, we give an upper bound estimation for $\|\mathbf{N}\|_{\ell_\varphi, \mathbf{b}_\varphi^{r,s}}$ where $\mathbf{N} = \mathbf{N}(x_n)$ is the Nörlund matrix associated with the sequence $\mathbf{x} = (x_n)$. In the final section, we introduce operator ideal $\mathcal{L}_{\mathbf{b}_\varphi^{r,s}}^{(s)}$ using the space $\mathbf{b}_\varphi^{r,s}$ and the sequence of s-number functions and prove its completeness under certain assumption.

2. Orlicz-binomial sequence spaces $\mathbf{b}_\varphi^{r,s}$

Let φ be an Orlicz function. Then the Orlicz-binomial sequence space $\mathbf{b}_\varphi^{r,s}$ can be defined as the set of all sequences whose $\mathbf{B}^{r,s}$-transform is in ℓ_φ. That is

$$\mathbf{b}_\varphi^{r,s} = \{\mathbf{x} \in \omega : \mathbf{B}^{r,s}\mathbf{x} \in \ell_\varphi\} = \left\{ \mathbf{x} \in \omega : \sum_{n=0}^{\infty} \varphi\left(\frac{1}{\rho} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k x_k \right) < \infty \text{ for some } \rho > 0 \right\}.$$

One can observe that the sequence space $\mathbf{b}_\varphi^{r,s}$ reduces to $\mathbf{b}_p^{r,s}$ when $\varphi(t) = t^p$, $p \geqslant 1$ as studied by Bişgin [11] which further reduces to the space $\mathbf{b}_p^r$ as studied by Altay et al. [7] when $r + s = 1$. Also if $r + s = 1$, the sequence space $\mathbf{b}_\varphi^{r,s}$ reduces to the Orlicz-Euler sequence space $\mathbf{e}_\varphi^r$ studied by Manna [26].

Clearly the space $\mathbf{b}_\varphi^{r,s}$ is a normed linear space equipped with the norm $\|\mathbf{x}\|_\varphi^{r,s} = \|\mathbf{B}^{r,s}\mathbf{x}\|_\varphi$. We begin with the following theorem.

Theorem 2.1. *The sequence space $\mathbf{b}_\varphi^{r,s}$ is a Banach space equipped with the norm $\|\cdot\|_\varphi^{r,s}$.*

Proof. Let (x^i) be a Cauchy sequence in $\mathbf{b}_\varphi^{r,s}$. Then for any $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}_0$ such that

$$\left\|x^i - x^j\right\|_\varphi^{r,s} < \varepsilon \text{ for each } i, j \geqslant n_0.$$

Choose $) < \rho_\varepsilon < \varepsilon$ such that

$$\sum_{n=0}^{\infty} \varphi\left(\frac{1}{\rho_\varepsilon}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k\left(x_k^i - x_k^j\right)\right) \leqslant 1 \text{ holds for each } i, j \geqslant n_0. \tag{2.1}$$

Using the assumption $\varphi(1) = 1$, we obtain

$$\frac{1}{\rho_\varepsilon}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k\left(x_k^i - x_k^j\right) \leqslant 1 \text{ holds for each } i, j \geqslant n_0.$$

Thus it is clear that the sequence $\left(x_k^i\right)$ is a Cauchy sequence of real numbers and hence converges. Let $\left(x_k^i\right) \to x_k$ as $i \to \infty$ for each $k \geqslant 0$. Since φ is continuous, we obtain from (2.1)

$$\sum_{n=0}^{\infty} \varphi\left(\frac{1}{\rho_\varepsilon}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k\left(x_k^i - x_k\right)\right) \leqslant 1 \text{ holds for each } i \geqslant n_0.$$

Thus $\mathbf{x} \in \mathbf{b}_\varphi^{r,s}$ and $\left\|x^i - x\right\|_\varphi \leqslant \rho_\varepsilon < \varepsilon$ for $i \geqslant n_0$. So $(\mathbf{b}_\varphi^{r,s}, \|\cdot\|_\varphi^{r,s})$ is a Banach space. □

Theorem 2.2. *The sequence space $\mathbf{b}_\varphi^{r,s}$ is linearly isomorphic to ℓ_φ.*

Proof. Define a mapping $\mathcal{T} : \mathbf{b}_\varphi^{r,s} \to \ell_\varphi$ by $\mathbf{x} \mapsto \mathbf{y} = \mathcal{T}\mathbf{x}$, where the sequence $\mathbf{y} = (y_k)$ is the $\mathbf{B}^{r,s}$-transform of the sequence $\mathbf{x} = (x_k)$, i.e.,

$$y_k = \sum_{j=0}^{k}\frac{1}{(s+r)^k}\binom{k}{j}s^{k-j}r^j x_j.$$

Clearly, $\mathcal{T}$ is linear and injective. Let $\mathbf{y} \in \ell_\varphi$ and define the sequence $\mathbf{x} = (x_k)$ by

$$x_k = \sum_{i=0}^{k}(s+r)^i\binom{k}{i}(-s)^{k-i}r^{-k}y_i.$$

Then, one obtains

$$\begin{aligned}
\|\mathbf{x}\|_\varphi^{r,s} = \|\mathbf{B}^{r,s}\mathbf{x}\|_\varphi &= \sum_{k=0}^{\infty}\varphi\left(\frac{\mathbf{B}^{r,s}\mathbf{x}}{\rho}\right)\\
&= \sum_{k=0}^{\infty}\varphi\left(\frac{1}{\rho}\sum_{j=0}^{k}\frac{1}{(s+r)^k}\binom{k}{j}s^{k-j}r^j x_j\right)\\
&= \sum_{k=0}^{\infty}\varphi\left(\frac{1}{\rho}\sum_{j=0}^{k}\frac{1}{(s+r)^k}\binom{k}{j}s^{k-j}r^j\sum_{i=0}^{j}(s+r)^i\binom{j}{i}(-s)^{j-i}r^{-j}y_i\right)\\
&= \sum_{k=0}^{\infty}\varphi\left(\frac{\sum_{j=0}^{k}\delta_{kj}y_j}{\rho}\right) = \sum_{k=0}^{\infty}\varphi\left(\frac{y_k}{\rho}\right) = \|\mathbf{y}\|_\varphi \leqslant \rho.
\end{aligned}$$

Thus we conclude that $\mathbf{x} \in \mathbf{b}_\varphi^{r,s}$ and $\mathcal{T}$ is norm preserving. Consequently, $\mathcal{T}$ is surjective. Thus $\mathbf{b}_\varphi^{r,s} \cong \ell_\varphi$. □

Now we establish certain inclusion properties concerning Orlicz-binomial sequence space. We start with the following result.

Theorem 2.3. *Let φ be an Orlicz and supermultiplicative function. Then the inclusion $\ell_\varphi \subset \mathbf{b}_\varphi^{r,s}$ holds.*

Proof. Let $\mathbf{x} = (x_k) \in \ell_\varphi$ with $x \neq 0$. Applying Jensen's inequality, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\varphi\left(\frac{\mathbf{B}^{r,s}\mathbf{x}}{\rho}\right) &\leqslant \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k\varphi\left(\frac{x_k}{\rho}\right)\\ &= \sum_{k=0}^{\infty}\varphi\left(\frac{x_k}{\rho}\right)\sum_{n=k}^{\infty}\binom{n}{k}\left(\frac{s}{s+r}\right)^{n-k}\left(\frac{r}{s+r}\right)^{k}\\ &= \frac{s+r}{r}\sum_{k=0}^{\infty}\varphi\left(\frac{x_k}{\rho}\right)\\ &= \sum_{k=0}^{\infty}\varphi\left(\frac{x_k}{\rho}\right)\varphi\left(\varphi^{-1}\left(\frac{s+r}{r}\right)\right) \leqslant \sum_{k=0}^{\infty}\varphi\left(\frac{x_k}{\rho}\varphi^{-1}\left(\frac{s+r}{r}\right)\right).\end{aligned}$$

Let us put $\rho = \|\mathbf{x}\|_\varphi \varphi^{-1}\left(\frac{s+r}{r}\right)$. Then the above inequality implies that

$$\sum_{n=0}^{\infty}\varphi\left(\frac{\mathbf{B}^{r,s}\mathbf{x}}{\rho}\right) \leqslant \sum_{k=0}^{\infty}\varphi\left(\frac{x_k}{\rho}\varphi^{-1}\left(\frac{s+r}{r}\right)\right) = \sum_{k=0}^{\infty}\varphi\left(\frac{x_k}{\|\mathbf{x}\|_\varphi}\right) \leqslant 1.$$

Hence, by the definition of Orlicz-Luxemborg norm, we obtain that

$$\|\mathbf{x}\|_\varphi^{r,s} \leqslant \rho = \varphi^{-1}\left(\frac{s+r}{r}\right)\|\mathbf{x}\|_\varphi .$$

Therefore, $\ell_\varphi \subset \mathbf{b}_\varphi^{r,s}$. To establish the strictness part, we consider $\varphi(t) = t^p$, $p \geqslant 1$. Then the sequence $\mathbf{x} = (x_k) = (-1)^k \in \mathbf{b}_\varphi^{r,s}$ but $\mathbf{x} \notin \ell_\varphi$. This completes the proof. □

Theorem 2.4. *Let φ be an Orlicz and supermultiplicative function. Then the inclusion $\mathbf{e}_\varphi^r \subset \mathbf{b}_\varphi^{r,s}$ is strict.*

Proof. The inclusion part is obvious since the sequence space $\mathbf{b}_\varphi^{r,s}$ reduces to $\mathbf{e}_\varphi^r$ when $r + s = 1$. To establish the strictness part, we consider $\varphi(t) = t^p$, $p \geqslant 1$ and a sequence $\mathbf{x} = (x_k) = (\frac{-2}{r})^k$ and let $0 < r < 1$ and $s = 4$. Then one can easily deduce that $(x_k) = (\frac{-2}{r})^k \notin \ell_\varphi$, $\mathbf{E}^r(x) = (-2-r)^k \notin \ell_\varphi$ and $\mathbf{B}^{r,s}x = (\frac{1}{s+r})^k \in \ell_\varphi$. Thus there exists at least one sequence $\mathbf{x} = (x_k) \in \mathbf{b}_\varphi^{r,s} \setminus \mathbf{e}_\varphi^r$. This establishes the result. □

3. Upper bound for Hausdorff matrix operators

In this section, we establish a Hardy type formula as an upper estimate for $\|\mathbf{H}_\mu\|_{\ell_\varphi,\mathbf{b}_\varphi^{r,s}}$, where $d\mu$ is a Borel probability measure on $[0,1]$ and $\mathbf{H}_\mu$ is the generalized Hausdorff matrix associated with $d\mu$. Let $a > -1$ and $c > 0$, then the generalized Hausdorff matrix, $\mathbf{H}_\mu = \mathbf{H}_\mu(\theta) = (h_{nk}(\theta))_{n,k\geqslant 0}$ is defined by (see [9, 17])

$$h_{nk} = \begin{cases}\binom{n+a}{n-k}\Delta^{n-k}\mu_k, & (k \leqslant n),\\ 0, & (k > n).\end{cases}$$

where $\Delta\mu_k = \mu_k - \mu_{k+1}$ and $\mu = (\mu_n)_{n\in\mathbb{N}}$ is a sequence of real numbers, normalized so that $\mu_0 = 1$ and

$$\mu_k = \int_0^1 \theta^k d\mu(\theta), \quad k = 0, 1, 2, \ldots .$$

An equivalent expression for the generalized Hausdorff matrix $\mathbf{H}_\mu = (h_{nk})$ is given by

$$h_{nk} = \begin{cases} \binom{n+a}{n-k} \int_0^1 \theta^{c(k+a)}(1-\theta^c)^{n-k} d\mu(\theta), & (k \leqslant n), \\ 0, & (k > n). \end{cases}$$

When $a = 0$ and $c = 1$, then $\mathbf{H}_\mu$ reduces to the ordinary Hausdorff matrix (see [9]) which generalizes various classes of matrices. These classes are:

(a) taking $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$ gives the Cesàro matix of order α;
(b) taking $d\mu(\theta) =$ point evaluation at $\theta = \alpha$ gives the Euler matrix of order α;
(c) taking $d\mu(\theta) = |\log\theta|^{\alpha-1}/\Gamma(\alpha)d\theta$ gives the Hölder matrix of order α;
(d) taking $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$ gives the Gamma matrix of order α.

The Cesàro, Hölder, and Gamma matrices have non negative entries when $\alpha > 0$, and also the Euler matrices, when $0 < \alpha < 1$. Now we consider the following hypothesis related to Orlicz function and Hausdorff matrix:

'Hypothesis OH': Let φ be an Orlicz and supermultiplicative function, φ^{-1} be its inverse, and $\|\cdot\|_\varphi$ be the Orlicz-Luxemborg norm. Denote $(x)_q = \frac{\Gamma(x+q)}{x}$ for $x \geqslant 0$ and $H_\mu = (h_{nk})$, $h_{nk} \geqslant 0$. Further, let $a > -1, c > 0, q > -a-1$ and $\frac{1}{(n+a+1)_q}$ be non-increasing for $n \in \mathbb{N}_0$.

Now we state a lemma due to Love [25] which is essential for deducing our results.

Lemma 3.1 ([25, Theorem 2]). *Suppose that the* **'Hypothesis OH'** *holds. Then for any non-negative sequence* $\boldsymbol{x} = (x_k)$ *and* $\mu = (\mu_k)$ *of real numbers normalized so that* $\mu_0 = 1$*, the following inequality holds:*

$$\|\boldsymbol{H}_\mu x\|_\varphi \leqslant \tilde{C}\,\|\boldsymbol{x}\|_\varphi, \tag{3.1}$$

where

$$\tilde{C} = \int_0^1 \varphi^{-1}\left(\theta^{-(q+1)c}\right) d\mu(\theta). \tag{3.2}$$

Theorem 3.2. *Suppose that the* **'Hypothesis OH'** *holds. Then the Hausdorff matrix* $\boldsymbol{H}_\mu$ *maps* ℓ_φ *to* $\boldsymbol{b}_\varphi^{r,s}$ *and*

$$\|\boldsymbol{H}_\mu\|_{\ell_\varphi, \boldsymbol{b}_\varphi^{r,s}} \leqslant \tilde{C}\varphi^{-1}\left(\frac{s+r}{r}\right),$$

where $\tilde{C}$ *is given by* (3.2).

Proof. Let $\mathbf{x} = (x_n)$ be a non negative sequence of real numbers in ℓ_φ. Let $\rho > 0$ be a real number, then using Jensen's inequality, we have

$$\begin{aligned}
\|\mathbf{H}_\mu\|_{\ell_\varphi, b_\varphi^{r,s}} &= \sum_{n=0}^{\infty} \varphi\left(\frac{1}{\rho}\sum_{k=0}^{n} \frac{1}{(s+r)^n}\binom{n}{k} s^{n-k} r^k \sum_{i=0}^{k} h_{ki}x_i\right) \\
&\leqslant \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{(s+r)^n}\binom{n}{k} s^{n-k}r^k \varphi\left(\frac{1}{\rho}\sum_{i=0}^{k} h_{ki}x_i\right) \\
&\leqslant \sum_{k=0}^{\infty} \varphi\left(\frac{1}{\rho}\sum_{i=0}^{k} h_{ki}x_i\right) \sum_{n=k}^{\infty}\binom{n}{k}\left(\frac{s}{s+r}\right)^{n-k}\left(\frac{r}{s+r}\right)^k \\
&= \frac{s+r}{r}\sum_{k=0}^{\infty} \varphi\left(\frac{1}{\rho}\sum_{i=0}^{k} h_{ki}x_i\right) \\
&= \sum_{k=0}^{\infty} \varphi\left(\frac{1}{\rho}\sum_{i=0}^{k} h_{ki}x_i\right) \varphi\left(\varphi^{-1}\left(\frac{s+r}{r}\right)\right)
\end{aligned}$$

$$\leqslant \sum_{k=0}^{\infty} \varphi\left(\frac{1}{\rho}\sum_{i=0}^{k} h_{ki}x_i\varphi^{-1}\left(\frac{s+r}{r}\right)\right).$$

Let $\rho = \|\mathbf{H}_\mu x\|_\varphi \varphi^{-1}\left(\frac{s+r}{r}\right)$. Then the above inequality implies that

$$\|\mathbf{H}_\mu\|_{\ell_\varphi, \mathbf{b}_\varphi^{r,s}} \leqslant \sum_{k=0}^{\infty} \varphi\left(\frac{1}{\rho}\sum_{i=0}^{k} h_{ki}x_i\varphi^{-1}\left(\frac{s+r}{r}\right)\right) = \sum_{k=0}^{\infty} \varphi\left(\frac{\mathbf{H}_\mu \mathbf{x}}{\|\mathbf{H}_\mu \mathbf{x}\|_\varphi}\right) \leqslant 1.$$

Now using the definition of Orlicz-Luxemborg norm and equation (3.1), we get

$$\|\mathbf{H}_\mu \mathbf{x}\|_\varphi^{r,s} \leqslant \rho = \|\mathbf{H}_\mu \mathbf{x}\|_\varphi \varphi^{-1}\left(\frac{s+r}{r}\right) \leqslant \tilde{C}\varphi^{-1}\left(\frac{s+r}{r}\right)\|\mathbf{x}\|_\varphi.$$

This gives

$$\|\mathbf{H}_\mu\|_{\ell_\varphi, \mathbf{b}_\varphi^{r,s}} \leqslant \tilde{C}\varphi^{-1}\left(\frac{s+r}{r}\right).$$

□

Corollary 3.3. *Choose* $c = 1$ *and* $a = 0$. *Then* $\tilde{C} = \int_0^1 \varphi^{-1}\left(\theta^{-(q+1)}\right) d\mu(\theta)$ *and Cesàro, Hölder, Euler, and Gamma matrices map* ℓ_φ *into* $\boldsymbol{b}_\varphi^{r,s}$ *and*

$$\|\boldsymbol{C}(\alpha)\|_{\ell_\varphi, b_\varphi^{r,s}} \leqslant \alpha\varphi^{-1}\left(\frac{s+r}{r}\right)\int_0^1 \varphi^{-1}\left(\theta^{-(q+1)}\right)(1-\theta)^{\alpha-1} d\theta, \ \alpha > 0;$$

$$\|\boldsymbol{H}(\alpha)\|_{\ell_\varphi, b_\varphi^{r,s}} \leqslant \frac{1}{\Gamma(\alpha)}\varphi^{-1}\left(\frac{s+r}{r}\right)\int_0^1 \varphi^{-1}\left(\theta^{-(q+1)}\right)|\log\theta|^{\alpha-1} d\theta, \ \alpha > 0;$$

$$\|\boldsymbol{E}(\alpha)\|_{\ell_\varphi, b_\varphi^{r,s}} \leqslant \varphi^{-1}\left(\frac{s+r}{r}\right)\varphi^{-1}\left(\alpha^{-(q+1)}\right), \ 0 < \alpha < 1;$$

$$\|\Gamma(\alpha)\|_{\ell_\varphi, b_\varphi^{r,s}} \leqslant \alpha\varphi^{-1}\left(\frac{s+r}{r}\right)\int_0^1 \varphi^{-1}\left(\theta^{-(q+1)}\right)\theta^{\alpha-1} d\theta.$$

Corollary 3.4. *Choose* $c = 1$, $a = 0$, $q = 0$ *and* $\varphi(t) = t^p$, $p \geqslant 1$, *and denote* $p^* = \frac{p}{p-1}$. *Then Cesàro, Hölder, Euler and Gamma matrices map* ℓ_φ *to* $\boldsymbol{b}_\varphi^{r,s}$ *and*

$$\|\boldsymbol{C}(\alpha)\|_{\ell_\varphi, b_\varphi^{r,s}} \leqslant \left(\frac{s+r}{r}\right)^{\frac{1}{p}} \frac{\Gamma(\alpha+1)\Gamma\left(\frac{1}{p^*}\right)}{\Gamma\left(\alpha+\frac{1}{p^*}\right)}, \quad \alpha > 0;$$

$$\|\boldsymbol{H}(\alpha)\|_{\ell_\varphi, b_\varphi^{r,s}} \leqslant \left(\frac{s+r}{r}\right)^{\frac{1}{p}} \frac{1}{\Gamma(\alpha)}\int_0^1 \theta^{\frac{-1}{p}}|\log\theta|^{\alpha-1} d\theta, \quad \alpha > 0;$$

$$\|\boldsymbol{E}(\alpha)\|_{\ell_\varphi, b_\varphi^{r,s}} \leqslant \left(\frac{s+r}{r}\right)^{\frac{1}{p}} \alpha^{\frac{-1}{p}}, \quad 0 < \alpha < 1;$$

$$\|\Gamma(\alpha)\|_{\ell_\varphi, b_\varphi^{r,s}} \leqslant \left(\frac{s+r}{r}\right)^{\frac{1}{p}} \frac{\alpha p}{\alpha p - 1}, \quad \alpha p > 1.$$

4. Upper bound for Nörlund matrix operator

In this section, we give an upper bound estimation for the norm of Nörlund matrix as an operator from ℓ_φ to $\mathbf{b}^{r,s}_\varphi$. Let $u = (u_n)$ be a sequence of non-negative numbers with $u_0 > 0$. We write $U_n = \sum_{k=0}^{n} u_k$, $n \geqslant 0$. Then the Nörlund mean with respect to the sequence $u = (u_n)$ is defined by the matrix $\mathbf{N} = \mathbf{N}(u_n) = (a^u_{nk})$ given by

$$a^u_{nk} = \begin{cases} \frac{u_{n-k}}{U_n}, & (0 \leqslant k \leqslant n) \\ 0, & k > n. \end{cases}$$

In the case when $u_n = e$, Nörlund matrix reduces to Cesàro matrix. Note that one can assume $u_0 = 1$ because $\mathbf{N}(u_n) = \mathbf{N}(cu_n)$ for any $c > 0$.

Theorem 4.1. *Let $u = (u_n)$ be a sequence of non-negative real numbers with $u_0 = 1$. Then*

$$\|\boldsymbol{N}\|_{\ell_\varphi, \boldsymbol{b}^{r,s}_\varphi} \leqslant \varphi^{-1}\left(\frac{s+r}{r}\sum_{n=0}^{\infty}\frac{u_n}{U_n}\right).$$

Proof. Let $x \in \ell_\varphi$ be any non-negative sequence of real numbers and $\rho > 0$. Applying Jensen's inequality, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\varphi\left(\frac{1}{\rho}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k\sum_{i=0}^{k}\frac{u_{k-i}}{U_k}x_i\right) &\leqslant \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k\varphi\left(\frac{1}{\rho}\sum_{i=0}^{k}\frac{u_{k-i}}{U_k}x_i\right) \\
&= \sum_{k=0}^{\infty}\varphi\left(\frac{1}{\rho}\sum_{i=0}^{k}\frac{u_{k-i}}{U_k}x_i\right)\sum_{n=k}^{\infty}\binom{n}{k}\left(\frac{s}{s+r}\right)^{n-k}\left(\frac{r}{s+r}\right)^{k} \\
&\leqslant \frac{s+r}{r}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{u_{k-i}}{U_k}\varphi\left(\frac{x_i}{\rho}\right) \\
&= \frac{s+r}{r}\sum_{i=0}^{\infty}\varphi\left(\frac{x_i}{\rho}\right)\sum_{k=i}^{\infty}\frac{u_{k-i}}{U_k} \\
&\leqslant \frac{s+r}{r}\sum_{k=0}^{\infty}\frac{u_k}{U_k}\sum_{i=0}^{\infty}\varphi\left(\frac{x_i}{\rho}\right) \\
&= \sum_{i=0}^{\infty}\varphi\left(\frac{x_i}{\rho}\right)\varphi\left(\varphi^{-1}\left(\frac{s+r}{r}\sum_{k=0}^{\infty}\frac{u_k}{U_k}\right)\right) \\
&\leqslant \sum_{i=0}^{\infty}\varphi\left(\frac{x_i}{\rho}\varphi^{-1}\left(\frac{s+r}{r}\sum_{k=0}^{\infty}\frac{u_k}{U_k}\right)\right).
\end{aligned}$$

Put $\rho = \|x\|_{\ell_\varphi}\varphi^{-1}\left(\frac{s+r}{r}\sum_{k=0}^{\infty}\frac{u_k}{U_k}\right)$, then the above inequality becomes

$$\|\mathbf{N}\mathbf{x}\|_{\ell_\varphi, \mathbf{b}^{r,s}_\varphi} \leqslant \sum_{i=0}^{\infty}\varphi\left(\frac{x_i}{\|x\|_{\ell_\varphi}}\right) \leqslant 1.$$

Now using the definition of Orlicz-Luxemborg norm, we get

$$\|\mathbf{N}\mathbf{x}\|_{\ell_\varphi, \mathbf{b}^{r,s}_\varphi} \leqslant \rho = \varphi^{-1}\left(\frac{s+r}{r}\sum_{k=0}^{\infty}\frac{u_k}{U_k}\right)\|\mathbf{x}\|_{\ell_\varphi}.$$

This establishes the result. ∎

Corollary 4.2. *Let $u = (u_n)$ be a non-negative sequence of real numbers such that $\frac{u_n}{U_n} = \frac{1}{(n+1)^2}$, $n = 0, 1, 2, \ldots$. Then the Nörlund matrix maps ℓ_φ into $\boldsymbol{b}_\varphi^{r,s}$ and*

$$\|\boldsymbol{N}\|_{\ell_\varphi, \boldsymbol{b}_\varphi^{r,s}} \leqslant \varphi^{-1}\left(\frac{(s+r)\pi^2}{6r}\right).$$

Corollary 4.3. *Let $\varphi(t) = t^p$ $p \geqslant 1$ and $u = (u_n)$ be a non-negative sequence of real numbers such that $\frac{u_n}{U_n} = \frac{1}{(n+1)^2}$, $n = 0, 1, 2, \ldots$. Then the Nörlund matrix maps ℓ_φ into $\boldsymbol{b}_\varphi^{r,s}$ and*

$$\|\boldsymbol{N}\|_{\ell_\varphi, \boldsymbol{b}_\varphi^{r,s}} \leqslant \left(\frac{(s+r)\pi^2}{6r}\right)^{\frac{1}{p}}.$$

5. The operator ideals $\mathcal{L}^{(s)}_{\mathbf{b}_\varphi^{r,s}}$

Throughout this section, we denote by X and Y, the Banach spaces over the complex field $\mathbb{C}$ and by $\mathbf{L}(X, Y)$, the class of all bounded linear maps from X to Y. Let $\mathbf{L}$ be the class of all bounded linear operators between any pair of Banach spaces.

A map $s : \mathbf{L} \to \omega^+$, where ω^+ is the class of sequences of non-negative real numbers, is called an s-number function if it satisfies the following conditions:

(i) $\|s\| = s_0(\mathcal{S}) \geqslant s_1(\mathcal{S}) \geqslant \cdots \geqslant 0,\ s(\mathcal{S}) = \{s_n(\mathcal{S})\}, \mathcal{S} \in \mathbf{L}$;
(ii) $s_n(\mathcal{S} + \mathcal{T}) \leqslant s_n(\mathcal{S}) + \|\mathcal{T}\|$ for $\mathcal{S}, \mathcal{T} \in \mathbf{L}(X, Y)$ and $n \in \mathbb{N}_0$;
(iii) $s_n(\mathcal{R}\mathcal{S}\mathcal{T}) \leqslant \|\mathcal{R}\|\, s_n(\mathcal{S})\, \|\mathcal{T}\|$ for $\mathcal{T} \in \mathbf{L}(X_0, X), \mathcal{S} \in \mathbf{L}(X, Y), \mathcal{R} \in \mathbf{L}(Y, Y_0)$, and $n \in \mathbb{N}_0$;
(iv) if $\operatorname{rank}(\mathcal{S}) < n$, then $s_n(\mathcal{S}) = 0$;
(v) if $\dim X \geqslant n$, then $s_n(\mathcal{I}_X) = 1$, where $\mathcal{I}_X$ denotes the identity map of X.

An s-number function is called additive if the condition (ii) is replaced by

(ii) $s_{m+n-1}(\mathcal{S} + \mathcal{T}) \leqslant s_m(\mathcal{S}) + s_n(\mathcal{T})$ for $\mathcal{S}, \mathcal{T} \in \mathbf{L}(X, Y)$ and $m, n \in \mathbb{N}_0$.

If the condition (iii) is replaced by

(iii) $s_{m+n-1}(RT) \leqslant s_m(\mathcal{R}) s_n(\mathcal{T})$ for $\mathcal{R} \in \mathbf{L}(Y_0, Y)$ and $\mathcal{T} \in \mathbf{L}(X, Y_0), m, n \in \mathbb{N}_0$,

then the s-number function is called multiplicative, where the negative subscript is consider to be naught.

For a subset $\mathbf{A}$ of $\mathbf{L}$, we write $\mathbf{A}(X, Y) = \mathbf{A} \cap \mathbf{L}(X, Y)$ where X and Y are Banach spaces. The collection $\mathbf{A}$ is said to be an operator ideal if it satisfies the following conditions:

(i) $\mathbf{A}$ contains all finite rank operators;
(ii) $\mathcal{T} + \mathcal{S} \in \mathbf{A}(X, Y)$ for $\mathcal{S}, \mathcal{T} \in \mathbf{A}(X, Y)$;
(iii) if $\mathcal{T} \in \mathbf{A}(X, Y)$ and $\mathcal{S} \in \mathbf{L}(Y, Z)$, then $\mathcal{S}\mathcal{T} \in \mathbf{A}(X, Z)$ and also if $\mathcal{T} \in \mathbf{L}(X, Y)$ and $\mathcal{S} \in \mathbf{A}(Y, Z)$, then $\mathcal{S}\mathcal{T} \in \mathbf{A}(X, Z)$.

The collection $\mathbf{A}(X, Y)$, for a given pair of Banach spaces X and Y, is called a component of $\mathbf{A}$. For more details on s-number and operator ideal, we strictly refer to [1–4, 14, 19, 28–33] and the references cited therein.

An ideal quasi norm is a real valued function f defined on an operator ideal $\mathbf{A}$, which satisfies the following properties:

(i) $0 \leqslant f(\mathcal{T}) < \infty$, for each $\mathcal{T} \in \mathbf{A}$ and $f(\mathcal{T}) = 0$ if and only if $\mathcal{T} = 0$;
(ii) there exists a constant $N \geqslant 1$ such that $f(\mathcal{S} + \mathcal{T}) \leqslant N[f(\mathcal{S}) + f(\mathcal{T})]$ for $\mathcal{S}, \mathcal{T} \in \mathbf{A}(X, Y)$, where $\mathbf{A}(X, Y)$ is any component of $\mathbf{A}$;
(iii) (a) $f(\mathcal{R}\mathcal{S}) \leqslant \|\mathcal{R}\|\, f(\mathcal{S})$ for $\mathcal{S} \in \mathbf{A}(X, Z)$, $\mathcal{R} \in \mathbf{L}(Z, Y)$; and
(b) $f(\mathcal{R}\mathcal{S}) \leqslant \|\mathcal{S}\|\, f(\mathcal{R})$ for $\mathcal{S} \in \mathbf{L}(X, Z)$, $\mathcal{R} \in \mathbf{A}(Z, Y)$.

We start with the following definition.

Definition 5.1. An operator $\mathcal{T} \in \mathbf{L}(X, Y)$ is said to be of type $\mathbf{b}_{\varphi}^{r,s}$ if $\{s_n(\mathcal{T})\} \in \mathbf{b}_{\varphi}^{r,s}$.

Let $\mathcal{L}^{(s)}_{\mathbf{b}_{\varphi}^{r,s}}$ denotes the collection of all such mappings, i.e.,

$$\mathcal{L}^{(s)}_{\mathbf{b}_{\varphi}^{r,s}} = \{\mathcal{T} \in \mathbf{L}(X, Y) : \{s_n(\mathcal{T})\} \in \mathbf{b}_{\varphi}^{r,s}\}.$$

For $\mathcal{T} \in \mathcal{L}^{(s)}_{\mathbf{b}_{\varphi}^{r,s}}$, we define

$$\|\mathcal{T}\|^{(s)}_{\mathbf{b}_{\varphi}^{r,s}} = \inf\left\{\rho > 0 : \sum_{n=0}^{\infty} \varphi\left(\frac{1}{\rho}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k s_k(\mathcal{T})\right) \leqslant 1\right\}.$$

Theorem 5.2. *The class $\mathcal{L}^{(s)}_{\mathbf{b}_{\varphi}^{r,s}}$ is an operator ideal equipped with the norm $\|\cdot\|^{(s)}_{\mathbf{b}_{\varphi}^{r,s}}$.*

Proof. Note that all the finite rank operators are contained in $\mathcal{L}^{(s)}_{\mathbf{b}_{\varphi}^{r,s}}$, since $s_n(\mathcal{T}) = 0$ for $n \geqslant n_0$ if $\operatorname{rank}(\mathcal{T}) < n_0$. Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{L}^{(s)}_{\mathbf{b}_{\varphi}^{r,s}}$, then

$$\sum_{n=0}^{\infty} \varphi\left(\frac{1}{\rho_1}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k s_k(\mathcal{T}_1)\right) < \infty \text{ for some } \rho_1 > 0, \text{ and}$$
$$\sum_{n=0}^{\infty} \varphi\left(\frac{1}{\rho_2}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k s_k(\mathcal{T}_2)\right) < \infty \text{ for some } \rho_2 > 0.$$

Now

$$\begin{aligned}
&\sum_{n=0}^{\infty} \varphi\left(\frac{1}{\rho_1+\rho_2}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k s_k(\mathcal{T}_1+\mathcal{T}_2)\right)\\
&\quad\leqslant \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k\varphi\left(\frac{1}{\rho_1+\rho_2}s_k(\mathcal{T}_1+\mathcal{T}_2)\right) \quad \text{(using Jensen's inequality)}\\
&\quad= \sum_{k=0}^{\infty}\varphi\left(\frac{1}{\rho_1+\rho_2}s_k(\mathcal{T}_1+\mathcal{T}_2)\right)\sum_{n=k}^{\infty}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k\\
&\quad= \frac{s+r}{r}\sum_{k=0}^{\infty}\varphi\left(\frac{1}{\rho_1+\rho_2}s_k(\mathcal{T}_1+\mathcal{T}_2)\right)\\
&\quad\leqslant \frac{s+r}{r}\left(\sum_{k=0}^{\infty}\frac{\rho_1}{\rho_1+\rho_2}\varphi\left(\frac{s_k(\mathcal{T}_1)}{\rho_1}\right)+\sum_{k=0}^{\infty}\frac{\rho_1}{\rho_1+\rho_2}\varphi\left(\frac{s_k(\mathcal{T}_2)}{\rho_2}\right)\right) < \infty.
\end{aligned}$$

Thus $\mathcal{T}_1 + \mathcal{T}_2 \in \mathcal{L}^{(s)}_{\mathbf{b}_{\varphi}^{r,s}}$. Let $\mathcal{T} \in \mathcal{L}^{(s)}_{\mathbf{b}_{\varphi}^{r,s}}(X_0, Y_0)$, $\mathcal{R} \in \mathcal{L}^{(s)}_{\mathbf{b}_{\varphi}^{r,s}}(X, X_0)$, $\mathcal{S} \in \mathcal{L}^{(s)}_{\mathbf{b}_{\varphi}^{r,s}}(Y_0, Y)$. Using the property (iii) of s-number function, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \varphi\left(\frac{1}{\rho}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k s_k(\mathcal{R}\mathcal{T}\mathcal{S})\right)\\
&\quad\leqslant \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k\varphi\left(\frac{1}{\rho}s_k(\mathcal{R}\mathcal{T}\mathcal{S})\right) \quad \text{(using Jensen's inequality)}\\
&\quad\leqslant \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k\varphi\left(\frac{\|\mathcal{R}\|\,\|\mathcal{S}\|}{\rho}s_k(\mathcal{T})\right) \quad \text{(using property (iii) of } s\text{-number function)}
\end{aligned}$$

$$= \frac{s+r}{r} \sum_{k=0}^{\infty} \varphi \left(\frac{\|\mathcal{R}\| \, s_k(\mathcal{T}) \, \|\mathcal{S}\|}{\rho} \right) < \infty.$$

Thus $\mathcal{RTS} \in \mathcal{L}^{(s)}_{\mathbf{b}^{r,s}_{\varphi}}$. Thus $\mathcal{L}^{(s)}_{\mathbf{b}^{r,s}_{\varphi}}$ is an operator ideal. □

Theorem 5.3. *The operator ideal $\mathcal{L}^{(s)}_{\mathbf{b}^{r,s}_{\varphi}}$ is complete under the quasi-norm $\|\cdot\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}}$.*

Proof. First we shall show that $\|\cdot\|^{r,s}_{b_\varphi}$ is a quasi-norm on $\mathcal{L}^{(s)}_{\mathbf{b}^{r,s}_{\varphi}}$. Note that $\|\mathcal{T}\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} \geqslant 0$ for each $\mathcal{T} \in \mathcal{L}^{(s)}_{\mathbf{b}^{r,s}_{\varphi}}$ and $\|\mathcal{T}\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} = 0$ for $\mathcal{T} = 0$. Now, let $\mathcal{T} \in \mathcal{L}^{(s)}_{\mathbf{b}^{r,s}_{\varphi}}$ such that $\|\mathcal{T}\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} = 0$. Then for $\varepsilon > 0$, we can find $0 < \rho < \varepsilon$ and

$$\sum_{n=0}^{\infty} \varphi \left(\frac{1}{\rho} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k s_k(\mathcal{T}) \right) \leqslant 1.$$

Using the assumption $\varphi(1) = 1$, one obtains

$$\frac{1}{\varepsilon} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k s_k(\mathcal{T}) \leqslant \frac{1}{\rho} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k s_k(\mathcal{T}) \leqslant 1.$$

Now

$$\frac{1}{\varepsilon} \left(\frac{s}{s+r} \right)^n s_0(\mathcal{T}) \leqslant \frac{1}{\varepsilon} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k s_k(\mathcal{T}) \leqslant 1.$$

Since ε is arbitrary, we get

$$\|\mathcal{T}\| = s_0(\mathcal{T}) = 0 \implies \mathcal{T} = 0.$$

Next we establish the triangular inequality. Let $\mathcal{T}_1$, $\mathcal{T}_2 \in \mathcal{L}^{(s)}_{\mathbf{b}^{r,s}_{\varphi}}$ and $\varepsilon > 0$ arbitrary. Choose $\rho_1 > 0$, $\rho_2 > 0$ such that

$$\frac{1}{\rho_1} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k s_k(\mathcal{T}_1) \leqslant 1, \quad \rho_1 \leqslant \|\mathcal{T}_1\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} + \frac{\varepsilon}{2}, \text{ and}$$
$$\frac{1}{\rho_2} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k s_k(\mathcal{T}_2) \leqslant 1, \quad \rho_2 \leqslant \|\mathcal{T}_2\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} + \frac{\varepsilon}{2}.$$

We choose $N > 1$. Then

$$\sum_{n=0}^{\infty} \varphi \left(\frac{1}{N(\rho_1 + \rho_2)} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k s_k(\mathcal{T}_1 + \mathcal{T}_2) \right) \leqslant 1.$$

From which one can deduce that

$$\|\mathcal{T}_1 + \mathcal{T}_2\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} \leqslant N(\rho_1 + \rho_2) \leqslant N \left(\|\mathcal{T}_1\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} + \|\mathcal{T}_2\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} + \varepsilon \right).$$

Since $\varepsilon > 0$ is arbitrary, therefore

$$\|\mathcal{T}_1 + \mathcal{T}_2\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} \leqslant N \left(\|\mathcal{T}_1\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} + \|\mathcal{T}_2\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} \right).$$

Now we shall establish the completeness of $\mathcal{L}^{(s)}_{\mathbf{b}^{r,s}_{\varphi}}$. Let $(\mathcal{T}^{(i)})$ be a Cauchy sequence in $\mathcal{L}^{(s)}_{\mathbf{b}^{r,s}_{\varphi}}$, then for $\varepsilon > 0$ there exists a positive integer n_0 such that $\left\|\mathcal{T}^{(i)} - \mathcal{T}^{(j)}\right\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} < \varepsilon$ for each $i, j \geqslant n_0$. We choose $0 < \rho < \varepsilon$ and

$$\sum_{n=0}^{\infty} \varphi \left(\frac{1}{\rho} \sum_{k=0}^{n} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k s_k(\mathcal{T}^{(i)} - \mathcal{T}^{(j)}) \right) \leqslant 1 \tag{5.1}$$

for $i, j \geqslant n_0$. Using the assumption $\varphi(1) = 1$ and the same argument above, one can deduce that $\left\|\mathcal{T}^{(i)} - \mathcal{T}^{(j)}\right\| \to 0$ as $i, j \to \infty$. Hence $(\mathcal{T}^{(i)})$ is a Cauchy squence in $\mathbf{L}(X, Y)$ and hence converges, say to $\mathcal{T}$, i.e., $\left\|\mathcal{T}^{(i)} - \mathcal{T}\right\| \to 0$ as $i \to \infty$. Since φ is continuous, therefore using equation (5.1) as $i \to \infty$,

$$\sum_{n=0}^{\infty} \varphi\left(\frac{1}{\rho}\sum_{k=0}^{n}\frac{1}{(s+r)^n}\binom{n}{k}s^{n-k}r^k s_k(\mathcal{T}^{(i)} - \mathcal{T})\right) \leqslant 1.$$

Thus $\mathcal{T} \in \mathcal{L}^{(s)}_{\mathbf{b}^{r,s}_{\varphi}}$ and $\left\|\mathcal{T}^{(i)} - \mathcal{T}\right\|^{(s)}_{\mathbf{b}^{r,s}_{\varphi}} \leqslant \rho < \varepsilon$ as $i \to \infty$. This establishes the result. □

6. Conclusion

In this article, we give an upper bound estimation for the norms of Hausdorff matrix and Nörlund matrix as operators from ℓ_φ to $\mathbf{b}^{r,s}_{\varphi}$, thereby obtaining a Hardy type formulae in the case of Hausdorff matrix. We have used Jensen's inequality to prove all the results. Note that by ignoring the weighted version, i.e., by taking $\lambda_n = 1$ and $\nu_n = 1$ for all $n \in \mathbb{N}_0$ in the results of Manna [26] and Talebi and Dehgan [37], respectively, then our investigated results in this paper intend to generalize the results obtained by the authors in [26, 37]. We also defined operator ideal for Orlicz-binomial sequence space and proved its completeness. We expect that the results obtained in this paper might be a reference for further studies in this field.

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