



Fourier expansions for Genocchi polynomials of higher order



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Abstract

In this paper, Fourier expansions and integral representations for Genocchi polynomials of higher order are established. Using the Fourier expansion, the explicit formula for Genocchi polynomials at rational arguments in terms of Hurwitz zeta function is also obtained.

Keywords: Genocchi polynomials, Bernoulli polynomials, Euler polynomials, Fourier series, Hurwitz zeta function.

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1. Introduction

The main objective in the study of combinatorics is to develop tools for counting. One of the most powerful tools frequently used in counting is the notion of generating function. This notion has been used to solve certain type of recurrence relations and to construct asymptotic expansion for some combinatorial and special numbers. The present study involves Genocchi polynomials that are known to have application in automata theory, number theory and combinatorics. Some applications may be drawn from certain expressions of Genocchi polynomials like those expressions in a form of Fourier series expansion. It is known that Fourier series and transforms have become an integral part of the toolboxes of mathematicians and scientists. Nowadays, these concepts have diverse applications such as file compression (e.g. the JPEG image format), signal processing in communications and astronomy, acoustics, optics, and cryptography (see [16]).

Genocchi polynomials were named after the Italian mathematician Angelo Genocchi (1817–1899). These polynomials were first defined by means of the following generating function

$$\sum_{n=0}^{\infty} G_n(x) \frac{z^n}{n!} = \frac{2z}{e^z + 1} e^{xz}, \quad |z| < \pi,$$

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and were studied by several researchers in different areas of mathematics (see [1–4, 6, 7, 9, 11–13, 15]. One of those was the work of Luo [12]. In his paper, Luo [12] used Lipschitz summation to obtain the Fourier series expansion of Genocchi polynomials given below:

$$\begin{aligned} G_n(x) &= \frac{2n!}{(\pi i)^n} \sum_{k \in \mathbb{Z}} \frac{e^{(2k-1)\pi i x}}{(2x-1)^n} \\ &= \frac{4n!}{\pi^n} \sum_{k=0}^{\infty} \frac{\cos((2k+1)\pi x - n\pi/2)}{(2x+1)^n}. \end{aligned}$$

Consequently, Luo [12] obtained the integral representation and explicit formula at rational arguments in terms of Hurwitz zeta function for Genocchi polynomials as follows:

$$\begin{aligned} G_n(x) &= 2n \int_0^\infty \frac{e^{\pi t} \cos(\pi x - n\pi/2) - e^{-\pi t} \cos(\pi x + n\pi/2)}{\cosh(2\pi t) - \cosh(2\pi x)} t^{n-1} dt, \\ G_n\left(\frac{p}{q}\right) &= \frac{4n!}{(2\pi q)^n} \sum_{j=1}^q \zeta\left(n, \frac{2j-1}{2q}\right) \cos\left[\frac{(2j-1)\pi p}{q} - \frac{l\pi}{2}\right], \end{aligned}$$

where $n, p, q \in \mathbb{N}$ and $0 \leq \Re(x) \leq 1$. Lou extended his research (see [13]) and established the following Fourier expansion of Apostol Genocchi polynomials and its integral representation using the same method in [12]:

$$\begin{aligned} G_n(x; \lambda) &= \frac{2n!}{\lambda^x} \sum_{k \in \mathbb{Z}} \frac{e^{(2k-1)\pi i x}}{[(2x-1)\pi i - \log \lambda]^n} \\ &= \frac{2n!i^n}{\lambda^x} \left[\sum_{k=0}^{\infty} \frac{e^{[n\pi/2-(2k+1)\pi x]i}}{[(2x-1)\pi i + \log \lambda]^n} + \sum_{k=0}^{\infty} \frac{e^{[-n\pi/2+(2k+1)\pi x]i}}{[(2x-1)\pi i - \log \lambda]^n} \right], \\ G_n(x; e^{2\pi i \xi}) &= \frac{2n}{e^{2\pi i x \xi}} \int_0^\infty \frac{M(n; x, t) \cosh(2\pi \xi t) + iN(n; x, t) \sinh(2\pi \xi t)}{\cosh(2\pi t) - \cosh(2\pi x)} t^{n-1} dt, \end{aligned}$$

where

$$\begin{aligned} M(n; x, t) &= e^{\pi t} \cos\left(\pi x - \frac{n\pi}{2}\right) - e^{-\pi t} \cos\left(\pi x + \frac{n\pi}{2}\right), \\ N(n; x, t) &= e^{\pi t} \sin\left(\pi x - \frac{n\pi}{2}\right) - e^{-\pi t} \sin\left(\pi x + \frac{n\pi}{2}\right). \end{aligned}$$

Furthermore, Lou [13] gave the formula at rational arguments in terms of the Hurwitz zeta function and explicit relationship between the Apostol-Genocchi polynomials and Gaussian hypergeometric functions as follows:

$$\begin{aligned} G_n\left(\frac{p}{q}; e^{2\pi i \xi}\right) &= \frac{2n!}{(2\pi q)^n} \left\{ \sum_{j=1}^q \zeta\left(n, \frac{2j+2\xi-1}{2q}\right) \exp\left[\left(\frac{n}{2} - \frac{(2j+2\xi-1)p}{q}\right)\pi i\right] \right. \\ &\quad \left. + \sum_{j=1}^q \zeta\left(n, \frac{2j-2\xi-1}{2q}\right) \exp\left[\left(-\frac{n}{2} + \frac{(2j-2\xi-1)p}{q}\right)\pi i\right] \right\}. \end{aligned}$$

Recently, Araci and Acikgoz [2] established Fourier expansion of Apostol Frobenius-Euler polynomials using the Cauchy residue theorem and a complex integral over a contour. As the Lipschitz summation fails to apply, in this paper, the method of Araci and Acikgoz [2] is used to establish the Fourier series of higher order Genocchi polynomials. But the same method in [12, 13] is used to obtain the integral representations and explicit formula at rational arguments in terms of Hurwitz zeta function for Genocchi polynomials of higher order.

2. Fourier expansion for Genocchi Polynomials of higher order

The following lemma which contains the convergence of certain integral expression is needed to establish the Fourier expansion of the Genocchi polynomials of higher order.

Lemma 2.1. *Let C_N be a circle about the origin of radius $(2N - 1)\pi$, $N \in \mathbb{Z}$. Then as $N \rightarrow \infty$, for either $n = 0$ and $0 < x < 1$ or $n > 0$ and $0 \leq x \leq 1$,*

$$\int_{C_N} \frac{(2z)^m e^{xz}}{(e^z + 1)^m z^{n+1}} dz \rightarrow 0.$$

Proof. Using the basic property of integration,

$$\left| \int_{C_N} \frac{(2z)^m e^{xz}}{(e^z + 1)^m z^{n+1}} dz \right| \leq \int_{C_N} \frac{|e^{xz}|}{|(e^z + 1)^m|} \cdot \frac{|2^m||dz|}{|z^{n-m+1}|}.$$

For $0 \leq x \leq 1$, $|(e^z + 1)^m| < |(e^z)^m|$ and $z = a + bi$,

$$\begin{aligned} \frac{|e^{xz}|}{|(e^z + 1)^m|} &= \frac{|e^{x(a+bi)}|}{|(e^z + 1)^m|} = \frac{e^{xa} \cdot 1}{|(e^z + 1)^m|} = \frac{e^{x\Re(z)}}{|(e^z + 1)^m|} \\ &\leq \frac{e^{x\Re(z)}}{|e^{mz}|} = \frac{e^{x\Re(z)}}{e^{m\Re(z)}} = \frac{1}{e^{(m-x)\Re(z)}} \leq 1. \end{aligned}$$

Thus,

$$\begin{aligned} \left| \int_{C_N} \frac{(2z)^m e^{xz}}{(e^z + 1)^m z^{n+1}} dz \right| &\leq \int_{C_N} 1 \cdot \frac{|2^m|}{|z^{n-m+1}|} |dz| \\ &\leq 2^m \int_{C_N} \frac{1}{|z^{n-m+1}|} |dz| \\ &\leq \frac{2^m}{[(2N-1)\pi]^{n-m+1}} \int_{C_N} |dz| \\ &\leq \frac{2^m (2N-1)\pi}{[(2k-1)\pi]^{n-m+1}} \\ &\leq \frac{2^m}{[(2N-1)\pi]^{n-m}}. \end{aligned}$$

As $N \rightarrow \infty$, the last expression goes to 0. Hence, as $N \rightarrow \infty$

$$\int_{C_N} \frac{(2z)^m e^{xz}}{(e^z + 1)^m z^{n+1}} dz,$$

goes to 0. □

Theorem 2.2. *For either $n = 0$ and $0 < x < 1$ or $n > 0$ and $0 \leq x \leq 1$,*

$$G_n^m(x) = \frac{(-1)^{m-1} 2^m n!}{(\pi i)^n} \sum_{k \in \mathbb{Z}} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{(2k-1)\pi i x}}{(2k-1)^{n-v}} \quad (2.1)$$

$$\begin{aligned} &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{k=0}^{\infty} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \\ &\quad \times \left[\frac{e^{((2k+1)\pi x - n\pi/2)i}}{(2k+1)^{n-v}} + \frac{(-1)^v e^{(-(2k+1)\pi x + n\pi/2)i}}{(2k+1)^{n-v}} \right], \end{aligned} \quad (2.2)$$

where $B_v^m(x)$ is the Bernoulli polynomial of higher order

Proof. Consider the integral $\int_C f_n(z) dz$ and the function

$$f_n(z) = \frac{(2z)^m e^{xz}}{(e^z + 1)^m z^{n+1}},$$

over the circle $C = \{z \mid |z| \leq (2N - 1 + \epsilon)\pi; \epsilon \in \mathbb{R}, (\epsilon\pi i \neq 0 \pmod{\pi i})\}$. Clearly, the function $f_n(z)$ has the poles $z = 0$ of order $n + 1$ and $z_k = (2k - 1)\pi i$ ($k \in \mathbb{Z}$). Hence, using Cauchy residue theorem,

$$\int_C f_n(z) dz = 2\pi i \left(\text{Res}(f_n(z), z = 0) + \sum_{k \in \mathbb{Z}} \text{Res}(f_n(z), z = z_k) \right).$$

Now, the first residue $\text{Res}(f_n(z), z = 0)$ is equal to

$$\begin{aligned} \text{Res}(f_n(z), z = 0) &= \lim_{z \rightarrow 0} \frac{1}{n!} \frac{d^n}{dz^n} (z - 0)^{n+1} \frac{(2z)^m e^{xz}}{(e^z + 1)^m z^{n+1}} \\ &= \lim_{z \rightarrow 0} \frac{1}{n!} \frac{d^n}{dz^n} \frac{(2z)^m e^{xz}}{(e^z + 1)^m} = \lim_{z \rightarrow 0} \frac{1}{n!} \frac{d^n}{dz^n} \sum_{l=0}^{\infty} G_l^m(x) \frac{z^l}{l!} \\ &= \lim_{z \rightarrow 0} \frac{1}{n!} \sum_{l=n}^{\infty} G_l^m(x) \frac{l!}{(l-n)!} \frac{z^{l-n}}{l!} \\ &= \lim_{z \rightarrow 0} \frac{1}{n!} \sum_{l=n}^{\infty} G_l^m(x) \frac{z^{l-n}}{(l-n)!}. \end{aligned}$$

Note that the limit of each term of the expansion is to 0 as $z \rightarrow 0$ except the term when $l = n$. Then

$$\text{Res}(f_n(z), z = 0) = \frac{1}{n!} G_n^m(x) \frac{z^{n-n}}{(n-n)!} = \frac{1}{n!} G_n^m(x).$$

On the other hand, the residue $\text{Res}(f_n(z), z = z_k)$ is equal to

$$\begin{aligned} \text{Res}(f_n(z), z = z_k) &= \frac{1}{(m-1)!} \lim_{z \rightarrow z_k} \frac{d^{m-1}}{dz^{m-1}} (z - z_k)^m \frac{(2z)^m e^{xz}}{(e^z + 1)^m z^{n+1}} \\ &= \frac{1}{(m-1)!} \lim_{z \rightarrow z_k} \frac{d^{m-1}}{dz^{m-1}} 2^m e^{xz} \frac{(z - z_k)^m}{(e^z + 1)^m} z^{m-n-1} \\ &= \frac{1}{(m-1)!} \lim_{z \rightarrow z_k} \frac{d^{m-1}}{dz^{m-1}} 2^m e^{xz} \frac{(z - z_k)^m}{(-1)^m (-e^z - 1)^m} z^{m-n-1} \\ &= \frac{1}{(m-1)!} \lim_{z \rightarrow z_k} \frac{d^{m-1}}{dz^{m-1}} (-1)^m 2^m e^{xz} \frac{(z - z_k)^m}{(-e^z - 1)^m} z^{m-n-1}. \end{aligned} \tag{2.3}$$

If $z_k = (2k - 1)\pi i$ where $k \in \mathbb{Z}$ and $e^{-(2k-1)\pi i} = \cos(2k-1)\pi - \sin(2k-1)\pi = -1$, then

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^m \frac{(z - z_k)^n}{n!} &= \frac{(z - z_k)^m}{(e^{z-z_k} - 1)^m} = \frac{(z - z_k)^m}{(e^z e^{-z_k} - 1)^m} \\ &= \frac{(z - z_k)^m}{(e^z (-1) - 1)^m} \\ &= \frac{(z - z_k)^m}{(-e^z - 1)^m}. \end{aligned} \tag{2.4}$$

By using (2.4), (2.3) becomes

$$\begin{aligned}\text{Res}(f_n(z), z = z_k) &= \frac{1}{(m-1)!} \lim_{z \rightarrow z_k} \frac{d^{m-1}}{dz^{m-1}} (-1)^m 2^m e^{xz} \frac{(z - z_k)^m}{(-e^z - 1)^m} z^{m-n-1} \\ &= \frac{1}{(m-1)!} \lim_{z \rightarrow z_k} \frac{d^{m-1}}{dz^{m-1}} (-1)^m 2^m e^{xz} \sum_{n=0}^{\infty} B_n^m \frac{(z - z_k)^n}{n!} z^{m-n-1}.\end{aligned}$$

Applying the general Leibniz rule,

$$\begin{aligned}\text{Res}(f_n(z), z = z_k) &= \frac{1}{(m-1)!} \lim_{z \rightarrow z_k} (-1)^m 2^m \sum_{v=0}^{m-1} \binom{m-1}{v} \frac{d^{m-1-v}}{dz^{m-1-v}} z^{m-n-1} \\ &\quad \times \frac{d^v}{dz^v} \left(e^{xz} \sum_{n=0}^{\infty} B_n^m \frac{(z - z_k)^n}{n!} \right) \\ &= \frac{1}{(m-1)!} \lim_{z \rightarrow z_k} (-1)^m 2^m \sum_{v=0}^{m-1} \binom{m-1}{v} \\ &\quad \times \frac{(m-n-1)!}{(m-n-1-(m-1-v))!} z^{m-n-1-(m-1-v)} \\ &\quad \times \sum_{l=0}^v \binom{v}{l} \frac{d^{v-l}}{dz^{v-l}} e^{xz} \frac{d^l}{dz^l} \sum_{n=0}^{\infty} B_n^m \frac{(z - z_k)^n}{n!} \\ &= \frac{1}{(m-1)!} \lim_{z \rightarrow z_k} (-1)^m 2^m \sum_{v=0}^{m-1} \binom{m-1}{v} \frac{(m-n-1)!}{(v-n)!} z^{v-n} \\ &\quad \times \sum_{l=0}^v \binom{v}{l} x^{v-l} e^{xz} \sum_{n=l}^{\infty} B_n^m \frac{n!}{(n-l)!} \frac{(z - z_k)^{n-l}}{n!} \\ &= \frac{1}{(m-1)!} \lim_{z \rightarrow z_k} (-1)^m 2^m \sum_{v=0}^{m-1} \frac{(m-1)!}{v!(m-1-v)!} \frac{(m-n-1)!}{(v-n)!} z^{v-n} \\ &\quad \times e^{xz} \sum_{l=0}^v \binom{v}{l} x^{v-l} \sum_{n=l}^{\infty} B_n^m \frac{(z - z_k)^{n-l}}{(n-l)!} \\ &= \frac{1}{(m-1)!} (-1)^m 2^m \lim_{z \rightarrow z_k} \sum_{v=0}^{m-1} \frac{(m-1)!}{v!(m-1-v)!} \frac{(m-n-1)!}{(v-n)!} z^{v-n} \\ &\quad \times \lim_{z \rightarrow z_k} e^{xz} \sum_{l=0}^v \binom{v}{l} x^{v-l} \sum_{n=l}^{\infty} B_n^m \frac{(z - z_k)^{n-l}}{(n-l)!} \\ &= (-1)^m 2^m \lim_{z \rightarrow z_k} \sum_{v=0}^{m-1} \frac{1}{v!(m-1-v)!} \frac{(m-n-1)!}{(v-n)!} z^{v-n} \\ &\quad \times \lim_{z \rightarrow z_k} e^{xz} \sum_{l=0}^v \binom{v}{l} x^{v-l} \sum_{n=l}^{\infty} B_n^m \frac{(z - z_k)^{n-l}}{(n-l)!}.\end{aligned}$$

Since $\sum_{n=l}^{\infty} B_n^m \frac{(z - z_k)^{n-l}}{(n-l)!} \rightarrow 0$ as $z \rightarrow z_k$ except when $n = l$,

$$\begin{aligned}\text{Res}(f_n(z), z = z_k) &= (-1)^m 2^m \sum_{v=0}^{m-1} \frac{1}{v!(m-1-v)!} \frac{(m-n-1)!}{(v-n)!} z_k^{v-n} \\ &\quad \times e^{xz_k} \sum_{l=0}^v \binom{v}{l} x^{v-l} B_l^m \frac{(z - z_k)^{l-l}}{(l-l)!}\end{aligned}$$

$$= (-1)^m 2^m \sum_{v=0}^{m-1} \frac{(m-n-1)!}{v!(m-1-v)!(v-n)!} z_k^{v-n} \\ \times e^{xz_k} \sum_{l=0}^v \binom{v}{l} x^{v-l} B_l^m.$$

From the property $B_v^m(x) = \sum_{l=0}^v \binom{v}{l} x^{v-l} B_l^m$, the above equation is equal to

$$\text{Res}(f_n(z), z = z_k) = (-1)^m 2^m \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{z_k^{v-n}}{v!} e^{xz_k} B_v^m(x) \\ = (-1)^m 2^m \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{B_v^m(x)}{v!} \frac{e^{xz_k}}{z_k^{n-v}}.$$

Substituting the value of $z_k = (2k-1)\pi i$ gives,

$$\text{Res}(f_n(z), z = z_k) = (-1)^m 2^m \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{B_v^m(x)}{v!} \frac{e^{(2k-1)\pi ix}}{((2k-1)\pi i)^{n-v}} \\ = \frac{(-1)^m 2^m}{(\pi i)^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{(2k-1)\pi ix}}{((2k-1))^{n-v}}.$$

Combining these residues gives,

$$\int_C f_n(z) dz = 2\pi i \left(\text{Res}(f_n(z), z = 0) + \sum_{k \in \mathbb{Z}} \text{Res}(f_n(z), z = z_k) \right) \\ = 2\pi i \left(\frac{1}{n!} G_n^m(x) + \frac{(-1)^m 2^m}{(\pi i)^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{(2k-1)\pi ix}}{((2k-1))^{n-v}} \right).$$

Taking $N \rightarrow \infty$, and by Lemma 2.1,

$$\int_C f_n(z) dz = 0.$$

Hence,

$$G_n^m(x) = \frac{(-1)^{m-1} 2^m n!}{(\pi i)^n} \sum_{k \in \mathbb{Z}} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{(2k-1)\pi ix}}{(2k-1)^{n-v}}.$$

This completes the proof of (2.1). Furthermore, by replacing k with $k+1$ and using the fact that

$$i^{-n} = e^{-n\pi i/2},$$

$$G_n^m(x) = \frac{(-1)^{m-1} 2^m n!}{(\pi i)^n} \sum_{k \in \mathbb{Z}} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{(2k+2-1)\pi ix}}{(2k+2-1)^{n-v}} \\ = \frac{(-1)^{m-1} 2^m n!}{(\pi)^n} \sum_{k \in \mathbb{Z}} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{i^{-n} e^{(2k+1)\pi ix}}{(2k+1)^{n-v}} \\ = \frac{(-1)^{m-1} 2^m n!}{(\pi)^n} \sum_{k \in \mathbb{Z}} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{[(2k+1)\pi x - n\pi/2]i}}{(2k+1)^{n-v}}$$

$$\begin{aligned}
&= \frac{(-1)^{m-1} 2^m n!}{(\pi)^n} \left[\sum_{k=0}^{\infty} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{[(2k+1)\pi x - n\pi/2]i}}{(2k+1)^{n-v}} \right. \\
&\quad \left. + \sum_{k=-\infty}^{-1} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{[(2k+1)\pi x - n\pi/2]i}}{(2k+1)^{n-v}} \right] \\
&= \frac{(-1)^{m-1} 2^m n!}{(\pi)^n} \left[\sum_{k=0}^{\infty} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{[(2k+1)\pi x - n\pi/2]i}}{(2k+1)^{n-v}} \right. \\
&\quad \left. + \sum_{k=0}^{\infty} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{[-(2k+1)\pi x - n\pi/2]i}}{[-(2k+1)]^{n-v}} \right] \\
&= \frac{(-1)^{m-1} 2^m n!}{(\pi)^n} \left[\sum_{k=0}^{\infty} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{[(2k+1)\pi x - n\pi/2]i}}{(2k+1)^{n-v}} \right. \\
&\quad \left. + \sum_{k=0}^{\infty} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{(-1)^v e^{n\pi i} e^{[-(2k+1)\pi x - n\pi/2]i}}{(2k+1)^{n-v}} \right] \\
&= \frac{(-1)^{m-1} 2^m n!}{(\pi)^n} \left[\sum_{k=0}^{\infty} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{e^{[(2k+1)\pi x - n\pi/2]i}}{(2k+1)^{n-v}} \right. \\
&\quad \left. + \sum_{k=0}^{\infty} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{(-1)^v e^{[-(2k+1)\pi x + n\pi/2]i}}{(2k+1)^{n-v}} \right].
\end{aligned}$$

Combining the two double summation yields

$$\begin{aligned}
G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{k=0}^{\infty} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \\
&\quad \times \left[\frac{e^{((2k+1)\pi x - n\pi/2)i}}{(2k+1)^{n-v}} + \frac{(-1)^v e^{(-(2k+1)\pi x + n\pi/2)i}}{(2k+1)^{n-v}} \right].
\end{aligned}$$

This is exactly Equation (2.2). \square

Remark 2.3. The proof of Corcino and Corcino [6] for Theorem 2.2 used the method of Lopez and Temme [10] different from the method of Araci and Acikgoz [2] that we used in this paper.

3. Integral representations and an explicit formula for Genocchi polynomials of higher order

In this section, an integral representation for the Genocchi polynomials of higher order using the Fourier expansion is obtained. Moreover, an explicit formula at rational arguments is derived.

Theorem 3.1. For either $n = 0$ and $0 < x < 1$ or $n > 0$ and $0 \leq x \leq 1$,

$$\begin{aligned}
G_n^m(x) &= (-1)^{m-1} 2^{m-1} n! \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{i^v}{v!(n-v-1)!} B_v^m(x) \\
&\quad \times \int_0^\infty \left[\frac{(1 + (-1)^v)[e^{\pi t} \cos(\pi x - n\pi/2) - e^{-\pi t} \cos(\pi x + n\pi/2)]}{\cosh(2\pi t) - \cos(2\pi x)} \right. \\
&\quad \left. + i \frac{(1 - (-1)^v)[e^{\pi t} \sin(\pi x - n\pi/2) + e^{-\pi t} \sin(\pi x + n\pi/2)]}{\cosh(2\pi t) - \cos(2\pi x)} \right] t^{n-v-1} dt.
\end{aligned}$$

Proof. Using (2.2),

$$\begin{aligned}
 G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \\
 &\quad \times \left[\sum_{k=0}^{\infty} \frac{e^{((2k+1)\pi x - n\pi/2)i}}{(2k+1)^{n-v}} + \sum_{k=0}^{\infty} \frac{(-1)^v e^{(-(2k+1)\pi x + n\pi/2)i}}{(2k+1)^{n-v}} \right] \\
 &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \cdot \frac{(n-v-1)!}{(n-v-1)!} \\
 &\quad \times \left[\sum_{k=0}^{\infty} \frac{e^{((2k+1)\pi x - n\pi/2)i}}{(2k+1)^{n-v}} + \sum_{k=0}^{\infty} \frac{(-1)^v e^{(-(2k+1)\pi x + n\pi/2)i}}{(2k+1)^{n-v}} \right] \\
 &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \cdot \frac{1}{(n-v-1)!} \\
 &\quad \times \left[\sum_{k=0}^{\infty} \frac{e^{((2k+1)\pi x - n\pi/2)i} (n-v-1)!}{(2k+1)^{n-v}} \right. \\
 &\quad \left. + \sum_{k=0}^{\infty} \frac{(-1)^v e^{(-(2k+1)\pi x + n\pi/2)i} (n-v-1)!}{(2k+1)^{n-v}} \right] \\
 &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!(n-v-1)!} B_v^m(x) \\
 &\quad \times \left[\sum_{k=0}^{\infty} e^{((2k+1)\pi x - n\pi/2)i} \cdot \frac{(n-v-1)!}{(2k+1)^{n-v}} \right. \\
 &\quad \left. + \sum_{k=0}^{\infty} (-1)^v e^{(-(2k+1)\pi x + n\pi/2)i} \cdot \frac{(n-v-1)!}{(2k+1)^{n-v}} \right]. \tag{3.1}
 \end{aligned}$$

Applying the integral formula

$$\int_0^\infty t^n e^{-at} dt = \frac{n!}{a^{n+1}} \quad (n = 0, 1, \dots; \Re(a) > 0),$$

in (3.1),

$$\begin{aligned}
 G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!(n-v-1)!} B_v^m(x) \\
 &\quad \times \left[\sum_{k=0}^{\infty} e^{((2k+1)\pi x - n\pi/2)i} \int_0^\infty t^{n-v-1} e^{-(2k+1)t} dt \right. \\
 &\quad \left. + (-1)^v \sum_{k=0}^{\infty} e^{(-(2k+1)\pi x + n\pi/2)i} \int_0^\infty t^{n-v-1} e^{-(2k+1)t} dt \right] \\
 &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!(n-v-1)!} B_v^m(x) \\
 &\quad \times \left[\sum_{k=0}^{\infty} e^{2\pi ixk} e^{(\pi x - n\pi/2)i} \int_0^\infty t^{n-v-1} e^{-2tk} e^{-t} dt \right]
 \end{aligned}$$

$$\begin{aligned}
& +(-1)^v \sum_{k=0}^{\infty} e^{-2\pi i x k} e^{(-\pi x + n\pi/2)i} \int_0^{\infty} t^{n-v-1} e^{-2tk} e^{-t} dt \Big] \\
& = \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!(n-v-1)!} B_v^m(x) \\
& \quad \times \left[\int_0^{\infty} t^{n-v-1} e^{-t} e^{(\pi x - n\pi/2)i} \sum_{k=0}^{\infty} e^{2\pi i x k} e^{-2tk} dt \right. \\
& \quad \left. + (-1)^v \int_0^{\infty} t^{n-v-1} e^{-t} e^{(-\pi x + n\pi/2)i} \sum_{k=0}^{\infty} e^{-2\pi i x k} e^{-2tk} dt \right] \\
& = \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!(n-v-1)!} B_v^m(x) \\
& \quad \times \left[\int_0^{\infty} t^{n-v-1} e^{-t} e^{(\pi x - n\pi/2)i} \sum_{k=0}^{\infty} e^{2(\pi i x - t)k} dt \right. \\
& \quad \left. + (-1)^v \int_0^{\infty} t^{n-v-1} e^{-t} e^{(-\pi x + n\pi/2)i} \sum_{k=0}^{\infty} e^{-2(\pi i x + t)k} dt \right].
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{k=0}^{\infty} e^{2(\pi i x - t)k} &= \frac{1}{1 - e^{2(\pi i x - t)}} = \frac{e^{2t}}{e^{2t} - e^{2\pi i x}}, \\
\sum_{k=0}^{\infty} e^{-2(\pi i x + t)k} &= \frac{1}{1 - e^{-2(\pi i x + t)}} = \frac{e^{2t}}{e^{2t} - e^{-2\pi i x}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!(n-v-1)!} B_v^m(x) \\
&\quad \times \left[\int_0^{\infty} t^{n-v-1} e^{-t} e^{(\pi x - n\pi/2)i} \cdot \frac{e^{2t}}{e^{2t} - e^{2\pi i x}} dt \right. \\
&\quad \left. + (-1)^v \int_0^{\infty} t^{n-v-1} e^{-t} e^{(-\pi x + n\pi/2)i} \cdot \frac{e^{2t}}{e^{2t} - e^{-2\pi i x}} dt \right] \\
&= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!(n-v-1)!} B_v^m(x) \\
&\quad \times \left[\int_0^{\infty} \frac{e^t e^{(\pi x - n\pi/2)i}}{e^{2t} - e^{2\pi i x}} t^{n-v-1} dt \right. \\
&\quad \left. + (-1)^v \int_0^{\infty} \frac{e^t e^{(-\pi x + n\pi/2)i}}{e^{2t} - e^{-2\pi i x}} t^{n-v-1} dt \right] \\
&= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!(n-v-1)!} B_v^m(x) \\
&\quad \times \left[\int_0^{\infty} \frac{e^{-n\pi i/2} e^{\pi i x}}{e^{2t} - e^{2\pi i x}} e^t t^{n-v-1} dt \right. \\
&\quad \left. + (-1)^v \int_0^{\infty} \frac{e^{n\pi i/2} e^{-\pi i x}}{e^{2t} - e^{-2\pi i x}} e^t t^{n-v-1} dt \right].
\end{aligned}$$

Using the fact that

$$\begin{aligned}\frac{e^{\pi i x}}{e^{2t} - e^{2\pi i x}} &= \frac{e^{\pi i x}}{e^{2t} - e^{2\pi i x}} \cdot \frac{1 - e^{-2t} e^{-2\pi i x}}{1 - e^{-2t} e^{-2\pi i x}} = \frac{\frac{1}{2} e^{-\pi i x} (e^{2\pi i x} - e^{-2t})}{\cosh(2t) - \cos(2\pi x)}, \\ \frac{e^{-\pi i x}}{e^{2t} - e^{-2\pi i x}} &= \frac{e^{-\pi i x}}{e^{2t} - e^{-2\pi i x}} \cdot \frac{1 - e^{-2t} e^{2\pi i x}}{1 - e^{-2t} e^{2\pi i x}} = \frac{\frac{1}{2} e^{\pi i x} (e^{-2\pi i x} - e^{-2t})}{\cosh(2t) - \cos(2\pi x)},\end{aligned}$$

$$\begin{aligned}G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!(n-v-1)!} B_v^m(x) \\ &\quad \times \left[\int_0^\infty \frac{e^{-n\pi i/2} \frac{1}{2} e^{-\pi i x} (e^{2\pi i x} - e^{-2t})}{\cosh(2t) - \cos(2\pi x)} e^t t^{n-v-1} dt \right. \\ &\quad \left. + (-1)^v \int_0^\infty \frac{e^{n\pi i/2} \frac{1}{2} e^{\pi i x} (e^{-2\pi i x} - e^{-2t})}{\cosh(2t) - \cos(2\pi x)} e^t t^{n-v-1} dt \right] \\ &= \frac{(-1)^{m-1} 2^m n!}{2\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!(n-v-1)!} B_v^m(x) \\ &\quad \times \left[\int_0^\infty \frac{e^{-n\pi i/2} e^{-\pi i x} (e^{2\pi i x} - e^{-2t})}{\cosh(2t) - \cos(2\pi x)} e^t t^{n-v-1} dt \right. \\ &\quad \left. + (-1)^v \int_0^\infty \frac{e^{n\pi i/2} e^{\pi i x} (e^{-2\pi i x} - e^{-2t})}{\cosh(2t) - \cos(2\pi x)} e^t t^{n-v-1} dt \right].\end{aligned}$$

Using the transformation $t \rightarrow \pi t$, above equation gives

$$\begin{aligned}G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{2\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!(n-v-1)!} B_v^m(x) \\ &\quad \times \left[\int_0^\infty \frac{e^{-n\pi i/2} e^{-\pi i x} (e^{2\pi i x} - e^{-2\pi t})}{\cosh(2\pi t) - \cos(2\pi x)} e^{\pi t} t^{n-v-1} (\pi)^{n-v} dt \right. \\ &\quad \left. + (-1)^v \int_0^\infty \frac{e^{n\pi i/2} e^{\pi i x} (e^{-2\pi i x} - e^{-2\pi t})}{\cosh(2\pi t) - \cos(2\pi x)} e^{\pi t} t^{n-v-1} (\pi)^{n-v} dt \right] \\ &= (-1)^{m-1} 2^{m-1} n! \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{i^v}{v!(n-v-1)!} B_v^m(x) \\ &\quad \times \left[\int_0^\infty \frac{e^{-n\pi i/2} e^{-\pi i x} (e^{2\pi i x} - e^{-2\pi t}) e^{\pi t}}{\cosh(2\pi t) - \cos(2\pi x)} t^{n-v-1} dt \right. \\ &\quad \left. + (-1)^v \int_0^\infty \frac{e^{n\pi i/2} e^{\pi i x} (e^{-2\pi i x} - e^{-2\pi t}) e^{\pi t}}{\cosh(2\pi t) - \cos(2\pi x)} t^{n-v-1} dt \right].\end{aligned}$$

One can easily verify the following identities:

$$\begin{aligned}e^{-n\pi i/2} e^{-\pi i x} (e^{2\pi i x} - e^{-2\pi t}) e^{\pi t} &= [\cos(\pi x - n\pi/2) + i \sin(\pi x - n\pi/2)] e^{\pi t} \\ &\quad - [\cos(\pi x + n\pi/2) - i \sin(\pi x + n\pi/2)] e^{-\pi t}, \\ e^{n\pi i/2} e^{\pi i x} (e^{-2\pi i x} - e^{-2\pi t}) e^{\pi t} &= [\cos(\pi x - n\pi/2) - i \sin(\pi x - n\pi/2)] e^{\pi t} \\ &\quad - [\cos(\pi x + n\pi/2) + i \sin(\pi x + n\pi/2)] e^{-\pi t}.\end{aligned}$$

By using these identities, we obtain

$$\begin{aligned}
G_n^m(x) &= (-1)^{m-1} 2^{m-1} n! \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{i^v}{v!(n-v-1)!} B_v^m(x) \\
&\quad \times \left\{ \int_0^\infty \left[\frac{[\cos(\pi x - n\pi/2) + i \sin(\pi x - n\pi/2)] e^{\pi t}}{\cosh(2\pi t) - \cos(2\pi x)} \right. \right. \\
&\quad \left. \left. - \frac{[\cos(\pi x + n\pi/2) - i \sin(\pi x + n\pi/2)] e^{-\pi t}}{\cosh(2\pi t) - \cos(2\pi x)} \right] t^{n-v-1} dt \right. \\
&\quad \left. + (-1)^v \int_0^\infty \left[\frac{[\cos(\pi x - n\pi/2) - i \sin(\pi x - n\pi/2)] e^{\pi t}}{\cosh(2\pi t) - \cos(2\pi x)} \right. \right. \\
&\quad \left. \left. - \frac{[\cos(\pi x + n\pi/2) + i \sin(\pi x + n\pi/2)] e^{-\pi t}}{\cosh(2\pi t) - \cos(2\pi x)} \right] t^{n-v-1} dt \right. \\
&= (-1)^{m-1} 2^{m-1} n! \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{i^v}{v!(n-v-1)!} B_v^m(x) \\
&\quad \times \int_0^\infty \left[\frac{(1 + (-1)^v)[e^{\pi t} \cos(\pi x - n\pi/2) - e^{-\pi t} \cos(\pi x + n\pi/2)]}{\cosh(2\pi t) - \cos(2\pi x)} \right. \\
&\quad \left. + i \frac{(1 - (-1)^v)[e^{\pi t} \sin(\pi x - n\pi/2) + e^{-\pi t} \sin(\pi x + n\pi/2)]}{\cosh(2\pi t) - \cos(2\pi x)} \right] t^{n-v-1} dt.
\end{aligned}$$

This is the desired integral representation of Genocchi polynomials of higher order. \square

Theorem 3.2. For $n, q \in \mathbb{N}$ and $p \in \mathbb{Z}$, the following formula of higher order Genocchi polynomials at rational arguments in terms of Hurwitz zeta function

$$\begin{aligned}
G_n^m\left(\frac{p}{q}\right) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m\left(\frac{p}{q}\right) \frac{1}{(2q)^{n-v}} \\
&\quad \times \left[\sum_{j=1}^q \zeta\left(n-v, \frac{2j-1}{2q}\right) \cdot e^{\left[\frac{(2j-1)p\pi}{q} - \frac{n\pi}{2}\right]i} \right. \\
&\quad \left. + (-1)^v \sum_{j=1}^q \zeta\left(n-v, \frac{2j-1}{2q}\right) \cdot e^{\left[-\frac{(2j-1)p\pi}{q} + \frac{n\pi}{2}\right]i} \right].
\end{aligned}$$

Proof. From (2.2)

$$\begin{aligned}
G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{k=0}^\infty \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \\
&\quad \times \left[\frac{e^{((2k+1)\pi x - n\pi/2)i}}{(2k+1)^{n-v}} + \frac{(-1)^v e^{(-(2k+1)\pi x + n\pi/2)i}}{(2k+1)^{n-v}} \right] \\
&= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \\
&\quad \times \sum_{k=0}^\infty \left[\frac{e^{((2k+1)\pi x - n\pi/2)i}}{(2k+1)^{n-v}} + \frac{(-1)^v e^{(-(2k+1)\pi x + n\pi/2)i}}{(2k+1)^{n-v}} \right] \\
&= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x)
\end{aligned}$$

$$\times \left[\sum_{k=0}^{\infty} \frac{e^{((2k+1)\pi x - n\pi/2)i}}{(2k+1)^{n-v}} + \sum_{k=0}^{\infty} \frac{(-1)^v e^{(-(2k+1)\pi x + n\pi/2)i}}{(2k+1)^{n-v}} \right].$$

Replacing k by $k-1$, above equation gives

$$\begin{aligned} G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \\ &\quad \times \left[\sum_{k=1}^{\infty} \frac{e^{((2k-2+1)\pi x - n\pi/2)i}}{(2k-2+1)^{n-v}} + \sum_{k=1}^{\infty} \frac{(-1)^v e^{(-(2k-2+1)\pi x + n\pi/2)i}}{(2k-2+1)^{n-v}} \right] \\ &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \\ &\quad \times \left[\sum_{k=1}^{\infty} \frac{e^{((2k-1)\pi x - n\pi/2)i}}{(2k-1)^{n-v}} + \sum_{k=1}^{\infty} \frac{(-1)^v e^{(-(2k-1)\pi x + n\pi/2)i}}{(2k-1)^{n-v}} \right]. \end{aligned}$$

By applying the elementary series identity

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^q \sum_{k=0}^{\infty} f(qk+j),$$

which was used by Lou in his papers [12, 13],

$$\begin{aligned} G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \\ &\quad \times \left[\sum_{j=1}^q \sum_{k=0}^{\infty} \frac{e^{((2qk+2j-1)\pi x - n\pi/2)i}}{(2qk+2j-1)^{n-v}} + \sum_{j=1}^q \sum_{k=0}^{\infty} \frac{(-1)^v e^{(-(2qk+2j-1)\pi x + n\pi/2)i}}{(2qk+2j-1)^{n-v}} \right]. \end{aligned}$$

Dividing both numerator and denominator by $(2q)^{n-v}$ yields

$$\begin{aligned} G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \\ &\quad \times \left[\frac{1}{(2q)^{n-v}} \sum_{j=1}^q \sum_{k=0}^{\infty} \frac{(e^{2q\pi xi})^k e^{((2j-1)\pi x - n\pi/2)i}}{\left(k + \frac{2j-1}{2q}\right)^{n-v}} \right. \\ &\quad \left. + \frac{1}{(2q)^{n-v}} \sum_{j=1}^q \sum_{k=0}^{\infty} \frac{(-1)^v (e^{-2q\pi xi})^k e^{(-(2j-1)\pi x + n\pi/2)i}}{\left(k + \frac{2j-1}{2q}\right)^{n-v}} \right]. \end{aligned}$$

Simplifying further gives

$$\begin{aligned} G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{1}{(2q)^{n-v}} \\ &\quad \times \left[\sum_{j=1}^q \left(\sum_{k=0}^{\infty} \frac{(e^{2q\pi xi})^k}{\left(k + \frac{2j-1}{2q}\right)^{n-v}} \right) \cdot e^{((2j-1)\pi x - n\pi/2)i} \right. \\ &\quad \left. + (-1)^v \sum_{j=1}^q \left(\sum_{k=0}^{\infty} \frac{(e^{-2q\pi xi})^k}{\left(k + \frac{2j-1}{2q}\right)^{n-v}} \right) \cdot e^{(-(2j-1)\pi x + n\pi/2)i} \right]. \end{aligned} \tag{3.2}$$

Recall the Hurwitz-Lerch zeta function defined by

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s},$$

for $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $s \in \mathbb{C}$ when $|z| < 1$. When $|z| = 1$,

$$\zeta(s, a) := \Phi(1, s, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}.$$

Then, (3.2) becomes

$$\begin{aligned} G_n^m(x) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m(x) \frac{1}{(2q)^{n-v}} \\ &\quad \times \left[\sum_{j=1}^q \Phi \left(e^{2q\pi x i}, n-v, \frac{2j-1}{2q} \right) \cdot e^{((2j-1)\pi x - n\pi/2)i} \right. \\ &\quad \left. + (-1)^v \sum_{j=1}^q \Phi \left(e^{-2q\pi x i}, n-v, \frac{2j-1}{2q} \right) \cdot e^{(-(2j-1)\pi x + n\pi/2)i} \right]. \end{aligned}$$

By letting $x = \frac{p}{q}$ in the above equation,

$$\begin{aligned} G_n^m \left(\frac{p}{q} \right) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m \left(\frac{p}{q} \right) \frac{1}{(2q)^{n-v}} \\ &\quad \times \left[\sum_{j=1}^q \Phi \left(e^{2\pi p i}, n-v, \frac{2j-1}{2q} \right) \cdot e^{\left[\frac{(2j-1)p\pi}{q} - \frac{n\pi}{2} \right] i} \right. \\ &\quad \left. + (-1)^v \sum_{j=1}^q \Phi \left(e^{-2\pi p i}, n-v, \frac{2j-1}{2q} \right) \cdot e^{\left[\frac{-(2j-1)p\pi}{q} + \frac{n\pi}{2} \right] i} \right]. \end{aligned} \tag{3.3}$$

Since p is an integer, $e^{2\pi p i} = \cos(2p\pi) + i \sin(2p\pi) = 1$ and $e^{-2\pi p i} = \cos(2p\pi) - i \sin(2p\pi) = 1$, Equation (3.3) yields

$$\begin{aligned} G_n^m \left(\frac{p}{q} \right) &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m \left(\frac{p}{q} \right) \frac{1}{(2q)^{n-v}} \\ &\quad \times \left[\sum_{j=1}^q \Phi \left(1, n-v, \frac{2j-1}{2q} \right) \cdot e^{\left[\frac{(2j-1)p\pi}{q} - \frac{n\pi}{2} \right] i} \right. \\ &\quad \left. + (-1)^v \sum_{j=1}^q \Phi \left(1, n-v, \frac{2j-1}{2q} \right) \cdot e^{\left[\frac{-(2j-1)p\pi}{q} + \frac{n\pi}{2} \right] i} \right] \\ &= \frac{(-1)^{m-1} 2^m n!}{\pi^n} \sum_{v=0}^{m-1} \binom{m-n-1}{m-v-1} \frac{(\pi i)^v}{v!} B_v^m \left(\frac{p}{q} \right) \frac{1}{(2q)^{n-v}} \\ &\quad \times \left[\sum_{j=1}^q \zeta \left(n-v, \frac{2j-1}{2q} \right) \cdot e^{\left[\frac{(2j-1)p\pi}{q} - \frac{n\pi}{2} \right] i} \right] \end{aligned}$$

$$+(-1)^v \sum_{j=1}^q \zeta\left(n-v, \frac{2j-1}{2q}\right) \cdot e^{\left[\frac{-(2j-1)p\pi}{q} + \frac{n\pi}{2}\right]i},$$

which is exactly the desired explicit formula. \square

4. Conclusion and recommendation

The Fourier expansion of Genocchi polynomials of higher order is obtained using the method of Araci and Acikgoz [3], while the integral representation and explicit formula at rational arguments in terms of Hurwitz zeta function of Genocchi polynomials of higher order are derived using the method of Luo [12, 13]. With these, it is interesting to investigate if the asymptotic approximation of Genocchi polynomials of higher order is attainable.

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