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A parabolic transform and averaging methods for integro-partial differential equations

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Abstract

Averaging methods of the integro-partial differential equation is studied, without any restrictions on the characteristic form of the partial differential operators. By using the parabolic transform and the averaging methods, the integro-partial differential equation can be solved.

Keywords: Averaging method, integro-partial differential equation, parabolic transform, existence and uniqueness of solutions.

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1. Introduction

Consider the following integro-partial differential equation

$$\frac{\partial u(x,t)}{\partial t} = \psi(x,t) + \varepsilon \int_0^t L(x,t,\theta,D)u(x,\theta)d\theta, \tag{1.1}$$

$$u(x,0) = \varphi(x), \tag{1.2}$$

where

$$L(x,t,\theta,D) = \sum_{|q|\leqslant m} a_q(x,t,\theta)D^q,$$

$\varepsilon > 0$, $q = (q_1, \cdots, q_n)$ is an n-dimensional multi index, $|q| = q_1 + \cdots + q_n$, $D^q = D_1^{q_1} \cdots D_n^{q_n}$, $D_j = \frac{\partial}{\partial x_j}$, $j = 1, \cdots, n$, $x = (x_1, \cdots, x_n) \in \mathfrak{R}^n$, $\mathfrak{R}^n$ is the n-dimensional Euclidean space, $0 \leqslant \theta \leqslant t \leqslant T$.

Let

$$S = \{(x,t,\theta) : x \in \mathfrak{R}^n, 0 \leqslant \theta \leqslant t \leqslant T\},$$

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$C_b(S)$ is the set of all bounded continuous functions on S and the coefficients $a_q \in C_b(S)$ for all q, $|q| \leqslant m$.

Consider the following Cauchy problem [8]

$$\frac{\partial u(x,t)}{\partial t} = (D_1^2 + \cdots + D_n^2)^{2M+1} u(x,t), \tag{1.3}$$

$$u(x,0) = \varphi(x) \in C_b(\mathfrak{R}^n), \tag{1.4}$$

where M is a sufficiently large positive integer.

Define the operator $P(t)$ by

$$P(t)\varphi = \int_{\mathfrak{R}^n} G(x-y,t)\varphi(y)dy.$$

The solution of the Cauchy problem (1.3), (1.4) is given by

$$u(x,t) = G(t)\varphi = \int_{\mathfrak{R}^n} G(x-y,t)\varphi(y)dy,$$

where the function G is the fundamental solution of the Cauchy problem (1.3), (1.4) and $dy = dy_1 \cdots dy_n$. For sufficiently large M, we find $\gamma \in (0,1)$ and a constant $N > 0$ such that

$$\max_x |D^q u(x,t)| \leqslant \frac{N}{t^\gamma} \max_x |\varphi(x)|,$$

for all $|q| \leqslant m$, $m < M$, $t > 0$.

A parabolic transform of a function Q is a function $\tilde{Q}$ defined by [8]

$$\tilde{Q}(x, t_1, \cdots, t_r, c_1 t + c_2) = \int_{\mathfrak{R}^n} G(x-y, c_1 t + c_2) Q(y, t_1, \cdots, t_r) dy,$$

where $c_1 \geqslant 0$, $c_2 \geqslant 0$, $t_j, t \in [0,T]$, $j = 1, \cdots, r$ and $Q(y, t_1, \cdots, t_r) \in C_b(\mathfrak{R}^n \times [0,T]^\gamma)$.

From (1.1), we have

$$u(x,t) = \varphi(x) + \int_0^t \psi(x,\theta)d\theta + \varepsilon \int_0^t \sum_{|q| \leqslant m} b_q(x,t,\theta) D^q u(x,\theta) d\theta,$$

where

$$b_q(x,t,\theta) = \int_\theta^t a_q(x,s,\theta)ds,$$

let

$$L_1(x,t,\theta,D) = \sum_{|q| \leqslant m} b_q(x,t,\theta) D^q.$$

In Section 2, we study the averaging of the linear operator of the integro-partial differential equation (1.1), (1.2) by using the parabolic transform where we generalize some known results due to Krol [14]. Compare also [1–13, 16].

2. The averaging of the linear operator

Consider the following equation [8]

$$w(x,t) = \tilde{\varphi}(x, c_1 t) + \int_0^t \tilde{\psi}(x,\theta,c_1 t)d\theta + \varepsilon \int_0^t \sum_{|q| \leqslant m} \tilde{b_q}(x,t,\theta,c_1 t) D^q \tilde{w}(x,\theta,c_2 - c_1\theta) d\theta, \tag{2.1}$$

where c_1, c_2 are positive constants and $c_2 \geqslant c_1 T$.

If $\psi, b_q \in C_b(\Re^n \times [0,T])$, then (2.1) can be solved [8].

Suppose that $w(x,t,\frac{1}{nT},\frac{1}{n})$ is the solution of (2.1) with $c_1 = \frac{1}{nT}$ and $c_2 = \frac{1}{n}$. Consider the sequence

$$u_n(x,t) = \int_{\Re^n} G(x-y, \frac{1}{n} - \frac{t}{nT}) w(y,t,\frac{1}{nT},\frac{1}{n}) dy. \tag{2.2}$$

The sequence $\{u_n(x,t)\}$ satisfies the equation

$$u_n(x,t) = G(\frac{1}{n})\varphi + G(\frac{1}{n})\psi + G(\frac{1}{n}) \int_0^t \sum_{|q|\leqslant m} b_q(x,t,\theta) D^q u_n(x,\theta) d\theta.$$

Let

$$\tilde{L}(x,t,\theta,c_1t,D) = \sum_{|q|\leqslant m} \tilde{a_q}(x,t,\theta,c_1t) D^q,$$

$$\tilde{L_1}(x,t,\theta,c_1t,D) = \sum_{|q|\leqslant m} \tilde{b_q}(x,t,\theta,c_1t) D^q,$$

$$F(x,t) = \int_0^t \tilde{\psi}(x,\theta,c_1t) d\theta,$$

$$W(x,t) = \varepsilon \int_0^t \sum_{|q|\leqslant m} \tilde{b_q}(x,t,\theta,c_1t) D^q \tilde{w}(x,\theta,c_2 - c_1\theta) d\theta.$$

We have

$$w(x,t) = \tilde{\varphi}(x,c_1t) + F(x,t) + W(x,t),$$
$$w(x,0) = \tilde{\varphi}(x,0).$$

By averaging the coefficients $b_q(x,t,\theta)$ and $\tilde{b_q}(x,t,\theta,c_1t)$ over t, we can average the operators $L_1(x,t,\theta,D)$ and $\tilde{L_1}(x,t,\theta,c_1t,D)$,

$$\bar{b_q}(x,\theta) = \frac{1}{T}\int_0^T b_q(x,t,\theta) dt,$$

$$\bar{\bar{b_q}}(x,\theta) = \frac{1}{T}\int_0^T \tilde{b_q}(x,t,\theta,c_1t) dt,$$

for all (x,t,θ), $x \in \Re^n$ the averaged operators $\bar{L_1}(x,\theta,D)$, $\bar{\bar{L_1}}(x,\theta,D)$ can be produced.

As an approximating problem for (1.1), (1.2), we consider the following equation

$$\frac{\partial u^*(x,t)}{\partial t} = \bar{\psi}(x) + \varepsilon \int_0^t \bar{L}(x,\theta,D) u^*(x,\theta) d\theta. \tag{2.3}$$

With the initial condition

$$u^*(x,0) = \varphi(x), \tag{2.4}$$

we get,

$$u^*(x,t) = \varphi(x) + t\bar{\psi}(x) + \varepsilon \int_0^t \bar{L_1}(x,\theta,D) u^*(x,\theta) d\theta.$$

As an approximating problem for (2.1), we consider also the following equation

$$w^*(x,t) = \bar{\bar{\varphi}}(x) + \int_0^t \bar{\bar{\psi}}(x,\theta) d\theta + \varepsilon \int_0^t \bar{\bar{L_1}}(x,\theta,D) \tilde{w}^*(x,\theta,c_2 - c_1\theta) d\theta, \tag{2.5}$$

where

$$\bar{\bar{\varphi}}(x) = \frac{1}{T}\int_0^T \tilde{\varphi}(x,c_1t) dt,$$

$$\bar{\bar{\tilde{\psi}}}(x,\theta) = \frac{1}{T}\int_0^T \tilde{\psi}(x,\theta,c_1 t)dt.$$

If $\bar{\psi}, \bar{b_q} \in C_b(\mathfrak{R}^n)$, then (2.5) can be solved and it is clear that all the derivatives $D^q w^* \in C_b(\mathfrak{R}^n \times [0,T])$, for all $|q| \leqslant m$ [8].

Suppose that $w^*(x,t,\frac{1}{nT},\frac{1}{n})$ is the solution of (2.5) with $c_1 = \frac{1}{nT}$ and $c_2 = \frac{1}{n}$. Let the sequence

$$u_n^*(x,t) = \int_{\mathfrak{R}^n} G(x-y,\frac{1}{n}-\frac{t}{nT})w^*(y,t,\frac{1}{nT},\frac{1}{n})dy. \tag{2.6}$$

The sequence $\{u_n^*(x,t)\}$ satisfies the equation

$$u_n^*(x,t) = G(\frac{1}{n})\bar{\varphi} + G(\frac{1}{n})\bar{\psi} + G(\frac{1}{n})\int_0^t \sum_{|q|\leqslant m} \bar{b_q}(x,\theta)D^q u_n^*(x,\theta)d\theta.$$

Let

$$F_1(x,t) = \int_0^t \bar{\bar{\tilde{\psi}}}(x,\theta)d\theta,$$

$$W^*(x,t) = \varepsilon\int_0^t \bar{\bar{\tilde{L_1}}}(x,\theta,D)\tilde{w}^*(x,\theta,c_2-c_1\theta)d\theta.$$

We have

$$w^*(x,t) = \tilde{\varphi}(x,c_1 t) + F_1(x,t) + W^*(x,t), \tag{2.7}$$
$$w^*(x,0) = \bar{\bar{\tilde{\varphi}}}(x),$$

another straightforward analysis displays the existence and uniqueness of the solutions of the problems (1.1), (1.2), (2.1), (2.3), (2.4) and (2.5) on the time-scale $\frac{1}{\varepsilon}$.

We consider the domain $B = \mathfrak{R}^n \times [0,T]$. The norm $\|\ .\ \|_\infty$ is defined by the supremum norm on B and denoted by $\| u(x,t) \|_\infty = \sup_B |u(x,t)|$.

Theorem 2.1. *There exist two sequences $\{u_n(x,t)\}$ and $\{u_n^*(x,t)\}$ with the initial conditions $u_n(x,0) = \varphi_n(x)$, $u_n^*(x,0) = \varphi_n(x)$. If the sequence $\{\varphi_n(x)\}$ converges to $\varphi(x)$, then we have the estimate*

$$\| u_n(x,t) - u_n^*(x,t) \|_\infty = O(\varepsilon),$$

on the time-scale $\frac{1}{\varepsilon}$.

Proof. We consider the following near-identity transformation

$$\hat{w}(x,t) = w^*(x,t) + \varepsilon\int_0^t (\tilde{L_1}(x,t,\theta,c_1 t,D) - \bar{\bar{\tilde{L_1}}}(x,\theta,D))d\theta w^*(x,t). \tag{2.8}$$

We get

$$\begin{aligned}\| \hat{w}(x,t) - w^*(x,t) \|_\infty &= \varepsilon \,\| \int_0^t (\tilde{L_1}(x,t,\theta,c_1 t,D) - \bar{\bar{\tilde{L_1}}}(x,\theta,D))d\theta w^*(x,t) \|_\infty \\ &= O(\varepsilon), \quad \text{on the time-scale } \frac{1}{\varepsilon}.\end{aligned}$$

By differentiating of the near-identity transformation (2.8) and using (2.7), (2.8), we obtain

$$\frac{\partial\hat{w}(x,t)}{\partial t} = \frac{\partial w^*(x,t)}{\partial t} + \varepsilon\int_0^t (\tilde{L_1}(x,t,\theta,c_1 t,D) - \bar{\bar{\tilde{L_1}}}(x,\theta,D))d\theta \frac{\partial w^*(x,t)}{\partial t}$$

$$
\begin{aligned}
&+\varepsilon\int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta w^*(x,t)\\
&=\varepsilon\int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta\hat{w}(x,t)+\frac{\partial W^*(x,t)}{\partial t}+\frac{\partial F_1(x,t)}{\partial t}\\
&+\varepsilon[\int_0^t(\tilde{L_1}(x,t,\theta,c_1t,D)-\bar{\bar{\tilde{L}}}_1(x,\theta,D))d\theta\frac{\partial W^*(x,t)}{\partial t}\\
&-\varepsilon\int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta\int_0^t(\tilde{L_1}(x,t,\theta,c_1t,D)-\bar{\bar{\tilde{L}}}_1(x,\theta,D))d\theta w^*(x,t)]\\
&+\varepsilon\int_0^t(\tilde{L_1}(x,t,\theta,c_1t,D)-\bar{\bar{\tilde{L}}}_1(x,\theta,D))d\theta\frac{\partial F_1(x,t)}{\partial t},
\end{aligned}
$$

with initial value $\hat{w}(x,0)=\bar{\bar{\varphi}}(x)$. Let

$$\frac{\partial}{\partial t}-\varepsilon\int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta=\mathcal{L}.$$

We have

$$\mathcal{L}(\hat{w}-w^*)=O(\varepsilon)\quad \text{on the time-scale } \frac{1}{\varepsilon}.$$

Moreover $\hat{w}(x,0)-w^*(x,0)=0$.

We use the barrier functions see [15]. Let the barrier function

$$
\begin{aligned}
B(x,t)=\varepsilon &\parallel M(x,t)\parallel_\infty t+\parallel J(x,t)\parallel_\infty t\\
&+\parallel \frac{\partial F_1(x,t)}{\partial t}-\frac{\partial F(x,t)}{\partial t}\parallel_\infty t\\
&+\frac{1}{2}\varepsilon\parallel \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta(\frac{\partial F_1(x,t)}{\partial t}-\frac{\partial F(x,t)}{\partial t})\parallel_\infty t^2\\
&+\frac{1}{2}\varepsilon\parallel \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta J(x,t)\parallel_\infty t^2\\
&+\frac{1}{6}\varepsilon^2\parallel (\int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta)^2(\frac{\partial F_1(x,t)}{\partial t}-\frac{\partial F(x,t)}{\partial t})\parallel_\infty t^3,
\end{aligned}
$$

where

$$
\begin{aligned}
M(x,t)=&\int_0^t(\tilde{L_1}(x,t,\theta,c_1t,D)-\bar{\bar{\tilde{L}}}_1(x,\theta,D))d\theta\frac{\partial W^*(x,t)}{\partial t}\\
&-\varepsilon\int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta\int_0^t(\tilde{L_1}(x,t,\theta,c_1t,D)-\bar{\bar{\tilde{L}}}_1(x,\theta,D))d\theta w^*(x,t),
\end{aligned}
$$

and

$$
\begin{aligned}
J(x,t)=&\frac{\partial W^*(x,t)}{\partial t}-\frac{\partial W(x,t)}{\partial t}+\varepsilon\int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta w\\
&+\varepsilon\int_0^t(\tilde{L_1}(x,t,\theta,c_1t,D)-\bar{\bar{\tilde{L}}}_1(x,\theta,D))d\theta\frac{\partial F_1(x,t)}{\partial t},
\end{aligned}
$$

and the functions (we omit the arguments)

$$Q_1(x,t)=\hat{w}(x,t)-w(x,t)-B(x,t),\ Q_2(x,t)=\hat{w}(x,t)-w(x,t)+B(x,t).$$

We get

$$\begin{aligned}
\mathcal{L}Q_1(x,t) &= (\frac{\partial}{\partial t} - \varepsilon\frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta)[\hat{w}(x,t) - w(x,t) - B(x,t)] \\
&= J(x,t) - \| J(x,t) \|_\infty + \varepsilon M(x,t) - \varepsilon \| M(x,t) \|_\infty \\
&\quad + \frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t} - \| \frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t} \|_\infty \\
&\quad + \varepsilon \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta \| \frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t} \|_\infty t \\
&\quad - \varepsilon \| \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta(\frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t}) \|_\infty t \\
&\quad + \frac{1}{2}\varepsilon^2 \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta \| \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta(\frac{\partial F_1(x,t)}{\partial t} \\
&\quad - \frac{\partial F(x,t)}{\partial t}) \|_\infty t^2 \\
&\quad - \frac{1}{2}\varepsilon^2 \| (\int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta)^2(\frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t}) \|_\infty t^2 \\
&\quad + \varepsilon \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta \| J(x,t) \|_\infty t \\
&\quad - \varepsilon \| \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta J(x,t) \|_\infty t \\
&\quad + \varepsilon^2 \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta \| M(x,t) \|_\infty t \\
&\quad + \frac{1}{6}\varepsilon^3 \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta \| (\int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta)^2 \\
&\quad \times (\frac{\partial F_1(x,t)}{\partial t} - \frac{\partial F(x,t)}{\partial t}) \|_\infty t^3 \\
&\quad + \frac{1}{2}\varepsilon^2 \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta \| \int_0^t \frac{\partial}{\partial t}\tilde{L_1}(x,t,\theta,c_1t,D)d\theta \\
&\quad \times J(x,t) \|_\infty t^2 \\
&\leqslant 0,
\end{aligned}$$

$Q_1(x,0) = \bar{\tilde{\varphi}}(x) - \tilde{\varphi}(x,0)$, similarly, $\mathcal{L}\ Q_2(x,t) \geqslant 0$, $Q_2(x,0) = \bar{\tilde{\varphi}}(x) - \tilde{\varphi}(x,0)$. $Q_1(x,t)$ and $Q_2(x,t)$ are bounded, resulting in $Q_1(x,t) \leqslant 0$ and $Q_2(x,t) \geqslant 0$, we have

$$-B(x,t) \leqslant \hat{w}(x,t) - w(x,t) \leqslant B(x,t),$$

so we can estimate

$$\| \hat{w}(x,t) - w(x,t) \|_\infty \leqslant \| B(x,t) \|_\infty = O(\varepsilon),$$

on the time-scale $\frac{1}{\varepsilon}$. We apply the triangle inequality to have

$$\begin{aligned}
\| w(x,t) - w^*(x,t) \|_\infty &\leqslant \| \hat{w}(x,t) - w^*(x,t) \|_\infty + \| \hat{w}(x,t) - w(x,t) \|_\infty \\
&= O(\varepsilon), \text{ on the time-scale } \frac{1}{\varepsilon}.
\end{aligned} \tag{2.9}$$

From (2.2), (2.6) and (2.9), we have

$$\| u_n(x,t) - u_n^*(x,t) \|_\infty \leqslant \int_{\mathfrak{R}^n} |G(x-y, \frac{1}{n} - \frac{t}{nT})| \| w(y,t,\frac{1}{nT},\frac{1}{n})$$

$$
\begin{aligned}
& - w^*(y, t, \frac{1}{nT}, \frac{1}{n}) \|_\infty \, dy \\
& = O(\varepsilon), \text{ on the time-scale } \frac{1}{\varepsilon}. \qquad \square
\end{aligned}
$$

3. Conclusion

The integro-partial differential equation can be solved without any restrictions on the characteristic form by using the parabolic transform and the averaging methods.

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