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Solvability of a boundary value problem of self-reference functional differential equation with infinite point and integral conditions

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Abstract

The existence of solutions of a boundary value problem of self-reference functional differential equation with infinite point and integral conditions will be studied. Some properties of solution will be given. Two examples to illustrate main results.

Keywords: Self-reference, infinite point, nonlocal problem, continuous dependence.

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1. Introduction

Many investigators have interested in studying nonlocal problem of the functional differential equations with infinite point conditions, in order to achieve various goals; see [6–8, 10, 12, 15, 20, 21].

The differential and integral equations with deviating arguments that appear in recent literature, the deviation of the argument usually involves only the time itself, see [1, 2]. However, another case, in which the deviating arguments depend on both the state variable x and the time t, is of importance in theory and practice. Several papers have appeared recently that are devoted to such kind of differential equations, see for example [3–5, 9, 11, 16–19] and the references cited therein.

Here we are concerning with the nonlocal problem of functional differential equation with self-dependence on a nonlinear integral operator

$$\frac{dx}{dt} = f\left(t, x(\int_0^t g(s, x(s))ds)\right), \quad \text{a.e.,} \quad t \in (0, T], \tag{1.1}$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \quad \tau_k \in (0, T). \tag{1.2}$$

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The existence of solutions $x \in AC[0,T]$ will be studied. The continuous dependence of the unique solution on x_0, on the functional g and on the nonlocal parameter a_k will be proved.

As applications, the nonlocal problem of equation (1.1) with the integral condition

$$\int_0^T x(s)dg(s) = x_0 \tag{1.3}$$

and the infinite-point boundary condition

$$\sum_{k=1}^{\infty} a_k x(\tau_k) = x_0 \tag{1.4}$$

will be studied.

2. Integral representation

Consider the nonlocal problem of functional differential equation (1.1)-(1.2) with the assumptions:

1. $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition, i.e., f is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in [0,T]$. There exist a bounded measurable function $c(t)$ and a positive constant $b > 0$, such that
$$|f(t,x)| \leqslant c(t) + b|x|, \qquad |c(t)| \leqslant M;$$
2. $g : [0,T] \times \mathbb{R} \to \mathbb{R}^+$ satisfies Carathéodory condition, i.e., g is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in [0,T]$ such that $|g(t,x)| \leqslant 1$;
3. $2bT < 1$.

Definition 2.1. By a solution of the nonlocal problem (1.1)-(1.2) we mean an absolutely continuous function $x \in AC[0,T]$ that satisfies the nonlocal problem itself.

Theorem 2.2. *Let $B^{-1} = \sum_{k=1}^{m} a_k \neq 0$ and the assumptions 1-3 be satisfied, then the nonlocal problem (1.1)-(1.2) and the functional integral equation*

$$x(t) = B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right) ds\right] + \int_0^t f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right) ds, \quad t \in [0,T], \tag{2.1}$$

are equivalent.

Proof. Let the solution of the nonlocal problem (1.1)-(1.2) exists. Integrating both sides of (1.1) we obtain

$$x(t) = x(0) + \int_0^t f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right) ds. \tag{2.2}$$

Using the nonlocal condition (1.2), we obtain

$$\sum_{k=1}^{m} a_k x(\tau_k) = x(0) \sum_{k=1}^{m} a_k + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right) ds,$$

then we can deduce that

$$x(0) = B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right) ds\right]. \tag{2.3}$$

Using (2.2) and (2.3), we obtain

$$x(t) = B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds\right] + \int_0^t f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds.$$

Conversely, let $x \in AC[0,T]$ be a solution of the functional integral equation (2.1). Differentiation (2.1) we obtain

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}\left\{B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds\right] + \int_0^t f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds\right\}\\ &= 0 + \frac{d}{dt}\int_0^t f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds = f\left(t, x(\int_0^t g(\theta, x(\theta))d\theta)\right) \in L^1[0,T].\end{aligned}$$

Also, from the integral equation (2.1), we obtain

$$x(\tau_k) = B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds\right] + \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds,$$

and

$$\begin{aligned}\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{m} a_k B^{-1}&\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds\right]\\ &+ \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds.\end{aligned}$$

Then

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0. \qquad \square$$

3. Existence of solution

Theorem 3.1. *Let the assumptions* 1-3 *be satisfied, then the nonlocal problem* (1.1)-(1.2) *has at least one solution* $x \in AC[0,T]$.

Proof.

$$\begin{aligned}|x(0)| &= \left|B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds\right]\right|\\ &\leqslant B^{-1}\left[|x_0| + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \left|f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)\right|ds\right]\\ &\leqslant B^{-1}\left[|x_0| + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \left(c(s) + b|x(\int_0^s g(\theta, x(\theta))d\theta) - x(0)| + b|x(0)|\right)ds\right]\\ &\leqslant B^{-1}\left[|x_0| + \sum_{k=1}^{m} a_k \int_0^{\tau_k} (c(s) + bL\int_0^s |g(\theta, x(\theta))|d\theta + b|x(0)|)ds\right]\\ &\leqslant B^{-1}|x_0| + TM + bLT^2 + bT|x(0)|.\end{aligned}$$

Hence

$$|x(0)| \leqslant \frac{B^{-1}|x_0| + TM + bLT^2}{1 - bT}. \tag{3.1}$$

Define the operator A associated with the integral equation (2.1). As

$$Ax(t) = B^{-1}\Big[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Big(s, x(\int_0^s g(\theta, x(\theta))d\theta)\Big)ds\Big] + \int_0^t f\Big(s, x(\int_0^s g(\theta, x(\theta))d\theta)\Big)ds,$$

and the set S_L by $S_L = \{x \in \mathbb{R} : |x(t) - x(s)| \leqslant L|t-s|, \forall t, s \in [0,T]\}$, $L = \frac{bB^{-1}|x_0|+M}{1-2bT}$. Then we have, for $x \in S_L$

$$\begin{aligned}
|Ax(t)| &= \Big|B^{-1}\Big[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Big(s, x(\int_0^s g(\theta, x(\theta))d\theta)\Big)ds\Big] + \int_0^t f\Big(s, x(\int_0^s g(\theta, x(\theta))d\theta)\Big)ds\Big| \\
&\leqslant B^{-1}\Big[|x_0| + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \Big|f\Big(s, x(\int_0^s g(\theta, x(\theta))d\theta)\Big)\Big|ds\Big] + \int_0^t \Big|f\Big(s, x(\int_0^s g(\theta, x(\theta))d\theta)\Big)\Big|ds \\
&\leqslant B^{-1}\Big[|x_0| + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \Big(c(s) + b|x(\int_0^s g(\theta, x(\theta))d\theta)|\Big)ds\Big] \\
&\quad + \int_0^t \Big(c(s) + b|x(\int_0^s g(\theta, x(\theta))d\theta)|\Big)ds \\
&\leqslant B^{-1}\Big[|x_0| + \sum_{k=1}^{m} a_k \int_0^{\tau_k} (c(s) + b|x(\int_0^s g(\theta, x(\theta))d\theta) - x(0)| + b|x(0)|)ds\Big] \\
&\quad + \int_0^t \Big(c(s) + b|x(\int_0^s g(\theta, x(\theta))d\theta) - x(0)| + b|x(0)|\Big)ds \\
&\leqslant B^{-1}\Big[|x_0| + \sum_{k=1}^{m} a_k \int_0^{\tau_k} (c(s) + bL\int_0^s |g(\theta, x(\theta))|d\theta + b|x(0)|)ds\Big] \\
&\quad + \int_0^t \Big(c(s) + bL\int_0^s |g(\theta, x(\theta))|d\theta + b|x(0)|\Big)ds \\
&\leqslant B^{-1}|x_0| + TM + bLT^2 + bT|x(0)| + TM + bLT^2 + bT|x(0)| \\
&= B^{-1}|x_0| + 2TM + 2bLT^2 + 2bT|x(0)|.
\end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we get

$$\begin{aligned}
|Ax(t)| &\leqslant B^{-1}|x_0| + 2TM + 2bLT^2 + 2bT\Big(\frac{B^{-1}|x_0| + TM + bLT^2}{1-bT}\Big) \\
&= \frac{B^{-1}|x_0| + bTB^{-1}|x_0| + 2TM + 2bLT^2}{1-bT} = \frac{B^{-1}|x_0| + TM + bLT^2}{1-bT} + LT.
\end{aligned}$$

Now, let $t_1, t_2 \in (0,T]$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
|Ax(t_2) - Ax(t_1)| &\leqslant \int_{t_1}^{t_2} \Big|f\Big(s, x(\int_0^s g(\theta, x(\theta))d\theta)\Big)\Big|ds \\
&\leqslant \int_{t_1}^{t_2} \Big(c(s) + b|x(\int_0^s g(\theta, x(\theta))d\theta) - x(0)| + b|x(0)|\Big)ds \\
&\leqslant \int_{t_1}^{t_2} \Big(c(s) + bL\int_0^s |g(\theta, x(\theta))|d\theta + b|x(0)|\Big)ds \\
&\leqslant (t_2 - t_1)M + (t_2 - t_1)bLT + (t_2 - t_1)b|x(0)| \\
&= (t_2 - t_1)(M + bLT + b|x(0)|).
\end{aligned} \tag{3.3}$$

From (3.1) and (3.3)

$$\begin{aligned}|Ax(t_2)-Ax(t_1)| &\leqslant (t_2-t_1)\left(M+bLT+b(\frac{B^{-1}|x_0|+TM+bLT^2}{1-bT})\right)\\ &\leqslant (t_2-t_1)\frac{bB^{-1}|x_0|+M+bLT}{1-bT}=(t_2-t_1)L.\end{aligned}$$

This proves that $A:S_L\to S_L$, the class of functions $\{Ax\}$ is uniformly bounded and equi-continuous in S_L. Let $x_n\in S_L$, $x_n\to x$ $(n\to\infty)$, then from continuity of the functions f and g, we obtain $f(t,x_n(t),y_n(t))\to f(t,x(t),y(t))$ and $g(t,x_n(t))\to g(t,x(t))$ as $n\to\infty$. Also

$$\begin{aligned}\lim_{n\to\infty}Ax_n(t)=\lim_{n\to\infty}\Bigg(&B^{-1}\left[x_0-\sum_{k=1}^m a_k\int_0^{\tau_k}f\left(s,x_n(\int_0^s g(\theta,x_n(\theta))d\theta\right)ds\right]\\ &+\int_0^t f\left(s,x_n(\int_0^s g(\theta,x_n(\theta))d\theta\right)ds\Bigg).\end{aligned}$$

Now

$$\begin{aligned}|x_n(\int_0^s g(\theta,x_n(\theta))d\theta)-x(\int_0^s g(\theta,x(\theta))d\theta)| &\leqslant |x_n(\int_0^s g(\theta,x_n(\theta))d\theta)-x_n(\int_0^s g(\theta,x(\theta))d\theta)|\\ &\quad+|x_n(\int_0^s g(\theta,x(\theta))d\theta)-x(\int_0^s g(\theta,x(\theta))d\theta)|\\ &\leqslant L\int_0^s|g(\theta,x_n(\theta))-g(\theta,x(\theta))|d\theta+\frac{\epsilon}{2}\leqslant\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Using assumptions (1)-(2) and Lebesgue dominated convergence Theorem [14] we obtain

$$\begin{aligned}\lim_{n\to\infty}Ax_n(t)=B^{-1}&\left[x_0-\sum_{k=1}^m a_k\int_0^{\tau_k}\lim_{n\to\infty}f\left(s,x_n(\int_0^s g(\theta,x_n(\theta))d\theta\right)ds\right]\\ &+\int_0^t\lim_{n\to\infty}f\left(s,x_n(\int_0^s g(\theta,x_n(\theta))d\theta\right)ds.\end{aligned}$$

Then $Ax_n\to Ax$ as $n\to\infty$. This mean that the operator A is continuous. Hence by Schauder fixed point Theorem [13] there exist at least one solution $x\in AC[0,T]$ of the nonlocal problem (1.1)-(1.2). □

4. Nonlocal integral condition

Let $x\in C[0,1]$ be the solution of the nonlocal problem (1.1)-(1.2). Let $a_k=h(t_k)-h(t_{k-1})$, h is increasing function, $\tau_k\in(t_{k-1},t_k)$, $0=t_0<t_1<t_2,\ldots<t_m=1$ then, as $m\to\infty$ the nonlocal condition (1.2) will be

$$\sum_{k=1}^m(h(t_k)-h(t_{k-1}))x(\tau_k)=x_0.$$

And

$$\lim_{m\to\infty}\sum_{k=1}^m(h(t_k)-h(t_{k-1}))x(\tau_k)=\int_0^T x(s)dh(s)=x_0.$$

Theorem 4.1. *Let the assumptions* 1-3 *be satisfied, then the nonlocal problem of* (1.1), (1.3) *has at least one solution.*

Proof. As $m \to \infty$, the solution of the nonlocal problem (1.1)-(1.2) will be

$$\begin{aligned} x(t) &= \lim_{m\to\infty} \frac{1}{\sum_{k=1}^{m} a_k}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds\right] + \int_0^t f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds \\ &= \frac{1}{h(T)-h(0)}\left[x_0 - \lim_{m\to\infty}\sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds(h(t_k)-h(t_{k-1}))\right] \\ &\quad + \int_0^t f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds \\ &= \frac{1}{h(T)-h(0)}\left[x_0 - \int_0^T\int_0^t f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds.dh(t)\right] + \int_0^t f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds. \end{aligned}$$

□

5. Infinite-point boundary condition

Theorem 5.1. *Let the assumptions* 1-3 *be satisfied, then the nonlocal problem of* (1.1), (1.4) *has at least one solution.*

Proof. Let the assumptions of Theorem 3.1 be satisfied. Let $\sum_{k=1}^{m} a_k$ be convergent, then

$$\begin{aligned} x_m(t) = &\frac{1}{\sum_{k=1}^{m} a_k}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds\right] \\ &+ \int_0^t f\left(s, x_m(\int_0^s g(\theta, x_m(\theta))d\theta)\right)ds. \end{aligned} \tag{5.1}$$

Take the limit to (5.1), as $m \to \infty$, we have

$$\begin{aligned} \lim_{m\to\infty} x_m(t) &= \lim_{m\to\infty}\left[\frac{1}{\sum_{k=1}^{n} a_k}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds\right]\right. \\ &\quad \left. + \int_0^t f\left(s, x_m(\int_0^s g(\theta, x_m(\theta))d\theta)\right)ds\right] \\ &= \lim_{m\to\infty}\frac{1}{\sum_{k=1}^{n} a_k}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds\right] \\ &\quad + \lim_{m\to\infty}\int_0^t f\left(s, x_m(\int_0^s g(\theta, x_m(\theta))d\theta)\right)ds. \end{aligned} \tag{5.2}$$

Now, $|a_k x(\tau_k)| \leqslant |a_k|\|x\|$, then by comparison test $\sum_{k=1}^{\infty} a_k x(\tau_k)$ is convergent. Also

$$|\int_0^{\tau_k} f(s, x(\int_0^s g(\theta, x(\theta))d\theta))ds| \leqslant \int_0^{\tau_k}(c(s) + b|x(\int_0^s g(\theta, x(\theta))d\theta)|)ds \leqslant TM + bLT^2 + bT|x(0)|,$$

then $|a_k \int_0^{\tau_k} f(s, x(s), \int_0^s g(\theta, x(\theta))d\theta)ds| \leqslant |a_k|.M_1$ and by the comparison test $\sum_{k=1}^{\infty} a_k \int_0^{\tau_k} f(s, x(s), \int_0^s g(\theta, x(\theta))d\theta)ds$ is convergent. Using assumptions 1-2 and Lebesgue Dominated convergence Theorem [14], from (5.2) we obtain

$$x(t) = \frac{1}{\sum_{k=1}^{\infty} a_k}\left[x_0 - \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds\right] + \int_0^t f\left(s, x(\int_0^s g(\theta, x(\theta))d\theta)\right)ds.$$

Then we have proved. □

6. Uniqueness of the solution

Let f and g satisfy the following assumptions

4. $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable in t for any $x \in \mathbb{R}$ and satisfies the Lipschitz condition

$$|f(t,x) - f(t,u)| \leqslant b|x-u|;$$

5. $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable in t for any $x \in \mathbb{R}$ and satisfies the Lipschitz condition

$$|g(t,x) - g(t,u)| \leqslant b_1|x-u|;$$

6. $(2bb_1LT^2 + 2bT) < 1$.

Theorem 6.1. *Let the assumptions* 4-6 *be satisfied, then the solution of the nonlocal problem* (1.1)-(1.2) *is unique.*

Proof. From assumption 4 we have f is measurable in t for any $x, y \in \mathbb{R}$ and satisfies the lipschitz condition, then it is continuous in $x \in \mathbb{R}$ $\forall t \in [0, T]$, and

$$|f(t,x)| \leqslant b|x| + |f(t,0)|.$$

Then condition 1 is satisfied. Also by the same way we can show that assumption 2 satisfied by assumption 5. Now, from Theorem 3.1, the solution of the nonlocal problem (1.1)-(1.2) exists. Let x, y be two solutions of (1.1)-(1.2), then

$$\begin{aligned}
|x(t)-y(t)| \leqslant{} & B^{-1}\Bigg(\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left|f\left(t, x(\int_0^s g(\theta,x(\theta))d\theta)\right) - f\left(t, x(\int_0^s g(\theta,x(\theta))d\theta)\right)\right| ds\Bigg)\\
&+ \int_0^t \left|f\left(s, x(\int_0^s g(\theta,x(\theta))d\theta)\right) - f\left(s, y(\int_0^s g(\theta,y(\theta))d\theta)\right)\right| ds\\
\leqslant{} & bB^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} |x(\int_0^s g(\theta,x(\theta))d\theta) - y(\int_0^s g(\theta,y(\theta))d\theta)| ds\\
&+ \int_0^t |x(\int_0^s g(\theta,x(\theta))d\theta) - y(\int_0^s g(\theta,y(\theta))d\theta)| ds\\
\leqslant{} & bB^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} |x(\int_0^s g(\theta,x(\theta))d\theta) - x(\int_0^s g(\theta,y(\theta))d\theta)| ds\\
&+ bB^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} |x(\int_0^s g(\theta,y(\theta))d\theta) - y(\int_0^s g(\theta,y(\theta))d\theta)| ds\\
&+ b\int_0^t |x(\int_0^s g(\theta,x(\theta))d\theta) - x(\int_0^s g(\theta,y(\theta))d\theta)| ds\\
&+ b\int_0^t |x(\int_0^s g(\theta,y(\theta))d\theta) - y(\int_0^s g(\theta,y(\theta))d\theta)| ds\\
\leqslant{} & bLB^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\int_0^s |g(\theta,x(\theta)) - g(\theta,y(\theta))| d\theta ds + bT\|x-y\|\\
&+ bL\int_0^t\int_0^s |g(\theta,x(\theta)) - g(\theta,y(\theta))| d\theta ds + bT\|x-y\|\\
\leqslant{} & bb_1LB^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\int_0^s |x(\theta)-y(\theta)| d\theta ds + bT\|x-y\|
\end{aligned}$$

$$
\begin{aligned}
&+ bb_1L\int_0^t\int_0^s |x(\theta)-y(\theta)|d\theta ds + bT\|x-y\| \\
&\leqslant 2bb_1LT^2\|x-y\| + 2bT\|x-y\| = (2bb_1LT^2+2bT)\|x-y\|.
\end{aligned}
$$

Hence

$$(1-(2bb_1LT^2+2bT))\|x-y\| \leqslant 0.$$

Since $(2bb_1LT^2+2bT) < 1$, then $x(t) = y(t)$ and the solution of the integral equation (2.1) is unique. □

7. Continuous dependence

7.1. Continuous dependence on x_0

Definition 7.1. The solution $x \in AC[0,T]$ of the nonlocal problem (1.1)-(1.2) depends continuously on x_0, if

$$\forall \epsilon > 0, \quad \exists\delta(\epsilon), \quad \text{s.t.}, \quad |x - x_0^*| < \delta \Rightarrow \|x - x^*\| < \epsilon,$$

where x^* is the solution of the nonlocal problem

$$\frac{dx^*}{dt} = f\Big(t, x^*(\int_0^t g(s, x^*(s))ds)\Big), \quad \text{a.e.,} \quad t \in (0,T], \tag{7.1}$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k x^*(\tau_k) = x_0^*, \quad \tau_k \in (0,T). \tag{7.2}$$

Theorem 7.2. *Let the assumptions of Theorem 6.1 be satisfied, then the solution of the nonlocal problem (1.1)-(1.2) depends continuously on x_0.*

Proof. Let $x,\ x^*$ be two solutions of the nonlocal problem (1.1)-(1.2) and (7.1)-(7.2), respectively, then

$$
\begin{aligned}
|x(t)-y(t)| &\leqslant B^{-1}|x_0 - x_0^*| + B^{-1}\Big(\sum_{k=1}^m a_k\int_0^{\tau_k}\Big|f\Big(t, x(\int_0^s g(\theta,x(\theta))d\theta)\Big) \\
&\quad - f\Big(t, x(\int_0^s g(\theta,x(\theta))d\theta)\Big)\Big|ds\Big) + \int_0^t\Big|f\Big(s, x(\int_0^s g(\theta,x(\theta))d\theta)\Big) - f\Big(s, y(\int_0^s g(\theta,y(\theta))d\theta)\Big)\Big|ds \\
&\leqslant B^{-1}\delta + bB^{-1}\sum_{k=1}^m a_k\int_0^{\tau_k}|x(\int_0^s g(\theta,x(\theta))d\theta) - y(\int_0^s g(\theta,y(\theta))d\theta)|ds \\
&\quad + \int_0^t |x(\int_0^s g(\theta,x(\theta))d\theta) - y(\int_0^s g(\theta,y(\theta))d\theta)|ds \\
&\leqslant B^{-1}\delta + bB^{-1}\sum_{k=1}^m a_k\int_0^{\tau_k}|x(\int_0^s g(\theta,x(\theta))d\theta) - x(\int_0^s g(\theta,y(\theta))d\theta)|ds \\
&\quad + bB^{-1}\sum_{k=1}^m a_k\int_0^{\tau_k}|x(\int_0^s g(\theta,y(\theta))d\theta) - y(\int_0^s g(\theta,y(\theta))d\theta)|ds \\
&\quad + b\int_0^t |x(\int_0^s g(\theta,x(\theta))d\theta) - x(\int_0^s g(\theta,y(\theta))d\theta)|ds \\
&\quad + b\int_0^t |x(\int_0^s g(\theta,y(\theta))d\theta) - y(\int_0^s g(\theta,y(\theta))d\theta)|ds \\
&\leqslant B^{-1}\delta + bLB^{-1}\sum_{k=1}^m a_k\int_0^{\tau_k}\int_0^s |g(\theta,x(\theta)) - g(\theta,y(\theta))|d\theta ds + bT\|x-y\|
\end{aligned}
$$

$$
\begin{aligned}
&+ bL\int_0^t\int_0^s |g(\theta,x(\theta))-g(\theta,y(\theta))|d\theta ds + bT\|x-y\| \\
&\leqslant B^{-1}\delta + bb_1LB^{-1}\sum_{k=1}^{m} a_k\int_0^{\tau_k}\int_0^s |x(\theta)-y(\theta)|d\theta ds + bT\|x-y\| \\
&+ bb_1L\int_0^t\int_0^s |x(\theta)-y(\theta)|d\theta ds + bT\|x-y\| \\
&\leqslant B^{-1}\delta + 2bb_1LT^2\|x-y\| + 2bT\|x-y\| \\
&= B^{-1}\delta + (2bb_1LT^2+2bT)\|x-y\|.
\end{aligned}
$$

Hence

$$\|x-x^*\| \leqslant \frac{B^{-1}\delta}{[1-(2bb_1LT^2+2bT)]} = \epsilon.$$

Then the solution of the nonlocal problem (1.1)-(1.2) depends continuously on x_0. □

7.2. Continuous dependence on a_k

Definition 7.3. The solution $x \in AC[0,T]$ of the the nonlocal problem (1.1)-(1.2) depends continuously on a_k, if

$$\forall\epsilon>0, \quad \exists\,\delta(\epsilon), \quad \text{s.t.,} \quad |a_k-a_k^*|<\delta \Rightarrow \|x-x^*\|<\epsilon,$$

where x^* is the solution of the nonlocal problem

$$\frac{dx^*}{dt} = f\Big(t, x^*(\int_0^t g(s,x^*(s))ds)\Big), \quad \text{a.e.,} \quad t\in(0,T], \tag{7.3}$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k^* x^*(\tau_k) = x_0, \quad \tau_k\in(0,T). \tag{7.4}$$

Theorem 7.4. *Let the assumptions of Theorem 6.1 be satisfied and* $\sup_{t\in[0,T]}\int_0^T\Big|f(s,0)\Big|ds \leqslant L_1$, *then the solution of the the nonlocal problem* (1.1)-(1.2) *depends continuously on* a_k.

Proof. Let $B^* = \sum_{k=1}^n a_k^* \neq 0$ and x and x^* be two solutions of the nonlocal problem (1.1)-(1.2) and (7.3)-(7.4), respectively, then

$$
\begin{aligned}
|x(t)-x^*(t)| &= \Big|B^{-1}\Big[x_0-\sum_{k=1}^{m} a_k\int_0^{\tau_k} f\Big(s,x(\int_0^s g(\theta,x(\theta))d\theta\Big)ds\Big] + \int_0^t f\Big(s,x(\int_0^s g(\theta,x(\theta))d\theta\Big)ds \\
&- B^{*-1}\Big[x_0-\sum_{k=1}^{m} a_k^*\int_0^{\tau_k} f\Big(s,x^*(\int_0^s g(\theta,x^*(\theta))d\theta))ds\Big] - \int_0^t f\Big(s,x^*(\int_0^s g(s,x^*(\theta))d\theta\Big)ds\Big| \\
&\leqslant B^{-1}B^{*-1}m\delta x_0 + B^{*-1}\sum_{k=1}^{m} a_k^*\int_0^{\tau_k}\Big|f\Big(s,x^*(\int_0^s g(\theta,x^*(\theta))d\theta\Big)-f\Big(s,x(\int_0^s g(\theta,x(\theta))d\theta\Big)\Big|ds \\
&+ B^{*-1}(\sum_{k=1}^{m}|a_k^*-a_k|)\int_0^{\tau_k}\Big|f\Big(s,x(\int_0^s g(\theta,x(\theta))d\theta\Big)ds\Big| \\
&+ B^{-1}B^{*-1}\sum_{k=1}^{m}|a_k-a_k^*|\sum_{k=1}^{m} a_k\int_0^{\tau_k}\Big|f\Big(s,x(\int_0^s g(\theta,x(\theta))d\theta\Big)\Big|ds \\
&+ \int_0^t\Big|f\Big(s,x(s),\int_0^s g(\theta,x(\theta))d\theta\Big)-f\Big(s,x^*(s),\int_0^s g(s,x^*(\theta))d\theta\Big)\Big|ds,
\end{aligned}
$$

using the Lipschitz condition, we get

$$\begin{aligned}|x(t)-x^*(t)| \leqslant\ & B^{-1}B^{*-1}m\delta x_0 + B^{*-1}\sum_{k=1}^{m} a_k^* \int_0^{\tau_k} b[|x^*(\int_0^s g(\theta,x^*(\theta))d\theta) \\ & - x^*(\int_0^s g(\theta,x(\theta))d\theta)| + |x^*(\int_0^s g(\theta,x(\theta))d\theta) - x(\int_0^s g(\theta,x(\theta))d\theta)|]ds \\ & + B^{*-1}m\delta(Tb\|x\|+L_1) + B^{*-1}m\delta(Tb\|x\|+L_1) \\ & + b\int_0^t [|x(\int_0^s g(\theta,x(\theta))d\theta) - x(\int_0^s g(s,x^*(\theta))d\theta)| \\ & + |x(\int_0^s g(s,x^*(\theta))d\theta) - x^*(\int_0^s g(s,x^*(\theta))d\theta)|]ds \\ \leqslant\ & B^{-1}B^{*-1}m\delta x_0 + 2bLb_1T^2\|x-x^*\| + 2bT\|x^*-x\| \\ & + B^{*-1}m\delta(Tb\|x\|+L_1) + B^{*-1}m\delta(Tb\|x\|+L_1).\end{aligned}$$

Hence

$$\|x-x^*\| \leqslant \frac{mx_0 + 2Bm(Tb\|x\|+L_1)}{[1-(2bLb_1T^2+2bT)]B.B^*}.\delta = \epsilon.$$

Therefore the solution of the nonlocal problem (1.1)-(1.2) depends continuously on a_k. □

7.3. Continuous dependence on the functional g

Definition 7.5. The solution $x \in AC[0,T]$ of the nonlocal problem (1.1)-(1.2) depends continuously on the functional g, if

$$\forall \epsilon > 0, \quad \exists\ \delta(\epsilon), \quad \text{s.t.,} \quad |g-g^*| < \delta \Rightarrow \|x-x^*\| < \epsilon,$$

where x^* is the solution of the nonlocal problem

$$\frac{dx^*}{dt} = f\left(t, x^*(\int_0^t g^*(s,x^*(s))ds\right), \quad \text{a.e.,} \quad t \in (0,T], \tag{7.5}$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k x^*(\tau_k) = x_0, \quad \tau_k \in (0,T). \tag{7.6}$$

Theorem 7.6. *Let the assumptions of Theorem 6.1 be satisfied, then the solution of the nonlocal problem (1.1)-(1.2) depends continuously on the function* g.

Proof. Let x and x^* be two solutions of the nonlocal problem (1.1)-(1.2) and (7.5)-(7.6), respectively, then

$$\begin{aligned}|x(t)-x^*(t)| =\ & \Big|B^{-1}[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(t, x(\int_0^s g(\theta,x(\theta))d\theta\right)ds\Big] + \int_0^t f\left(t, x(\int_0^s g(\theta,x(\theta))d\theta\right)ds \\ & - B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(t, x^*(\int_0^s g^*(\theta,x^*(\theta))d\theta\right)ds\right] + \int_0^t f\left(t, x^*(\int_0^s g^*(\theta,x^*(\theta))d\theta\right)ds \\ \leqslant\ & \Big|B^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \Big|f\left(t, x(\int_0^s g(\theta,x(\theta))d\theta\right) - f\left(t, x^*(\int_0^s g^*(\theta,x^*(\theta))d\theta\right)ds\Big| \\ & + \int_0^t \Big|f\left(t, x(\int_0^s g(\theta,x(\theta))d\theta\right) - f\left(t, x^*(\int_0^s g^*(\theta,x^*(\theta))d\theta\right)\Big|ds \\ \leqslant\ & bB^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} |x(\int_0^s g(\theta,x(\theta))d\theta)) - x^*(\int_0^s g(\theta,x^*(\theta))d\theta))|ds\end{aligned}$$

$$
\begin{aligned}
&+ b\int_0^t |x(\int_0^s g(\theta,x(\theta))d\theta) - x^*(\int_0^s g^*(\theta,x^*(\theta)d\theta))|ds\\
&\leqslant bB^{-1}\sum_{k=1}^m a_k \int_0^{\tau_k} |x(\int_0^s g(\theta,x(\theta))d\theta) - x(\int_0^s g^*(\theta,x^*(\theta)d\theta))|ds\\
&+ bB^{-1}\sum_{k=1}^m a_k \int_0^{\tau_k} |x(\int_0^s g^*(\theta,x^*(\theta)d\theta)) - x^*(\int_0^s g^*(\theta,x^*(\theta)d\theta))|ds\\
&+ b\int_0^t |x(\int_0^s g(\theta,x(\theta))d\theta) - x(\int_0^s g^*(\theta,x^*(\theta)d\theta))|ds\\
&+ b\int_0^t |x(\int_0^s g^*(\theta,x^*(\theta)d\theta)) - x^*(\int_0^s g^*(\theta,x^*(\theta)d\theta))|ds\\
&\leqslant bLB^{-1}\sum_{k=1}^m a_k \int_0^{\tau_k}\int_0^s |g(\theta,x(\theta)) - g^*(\theta,x^*(\theta))|d\theta ds + bT\|x-x^*\|ds\\
&+ bL\int_0^t\int_0^s |g(\theta,x(\theta)) - g^*(\theta,x^*(\theta))|d\theta ds + bT\|x-x^*\|ds\\
&\leqslant bLB^{-1}\sum_{k=1}^m a_k \int_0^{\tau_k}\int_0^s [|g(\theta,x(\theta)) - g(\theta,x^*(\theta))|\\
&+ |g(\theta,x^*(\theta)) - g^*(\theta,x^*(\theta))|]d\theta ds + bT\|x-x^*\|ds\\
&+ bL\int_0^t\int_0^s [|g(\theta,x(\theta)) - g(\theta,x^*(\theta))| + |g(\theta,x^*(\theta)) - g^*(\theta,x^*(\theta))|]d\theta ds + bT\|x-x^*\|ds\\
&\leqslant 2bLb_1T^2\|x-x^*\| + 2bT\|x-x^*\| + 2bLT^2\delta.
\end{aligned}
$$

Hence

$$\|x-x^*\| \leqslant \frac{2bLT^2\delta}{[1-(2bLb_1T^2+2bT)]} = \epsilon.$$

This mean that the solution of the nonlocal problem (1.1)-(1.2) depends continuously on the functional g. □

8. Examples

In this section we offer some examples to illustrate our results.

Example 8.1. Consider the following nonlinear differential equation

$$\frac{dx}{dt} = \frac{1}{3}t^3 + \frac{\ln(1+|x(\int_0^t \frac{\cos^2(x)}{s+e^{|x(s)|}})|)}{5+t^2}, \quad t\in(0,1], \tag{8.1}$$

with nonlocal condition

$$\sum_{k=1}^{\infty} \frac{1}{k^4} x(\frac{k-1}{k}) = 1. \tag{8.2}$$

Set

$$f\left(t, x(\int_0^t g(s,x(s))ds)\right) = \frac{1}{3}t^3 + \frac{\ln(1+|x(\int_0^t \frac{\cos^2(x)}{s+e^{|x(s)|}})|)}{5+t^2}.$$

Then

$$\left|f\left(t, x(\int_0^t g(s,x(s))ds)\right)\right| \leqslant \frac{1}{3}t^3 + \frac{1}{5}(|x|),$$

and also

$$|g(s,x(s))| \leqslant 1.$$

It is clear that the assumptions 1-3 of Theorem 3.1 are satisfied with $|c(t)| = |\frac{1}{3}t^3| \leqslant \frac{1}{3}$ is measurable bounded, $b = \frac{1}{5}$, $2bT = \frac{2}{5} < 1$, $L = \frac{(\frac{1}{5})(\frac{\pi^4}{90})+\frac{1}{3}}{1-\frac{2}{5}} \simeq 0.92$, and the series: $\sum_{k=1}^{\infty} \frac{1}{k^4}$ is convergent. Therefore, by applying Theorem 3.1, the given nonlocal problem (8.1)-(8.2) has a solution.

Example 8.2. Consider the following nonlinear differential equation

$$\frac{dx}{dt} = \frac{1}{4}(t+1) + \frac{x(\int_0^t \frac{s}{\sin^2(s)+e^{|x(s)|}})|)}{\sqrt{t+9}}, \quad t \in (0,1], \tag{8.3}$$

with nonlocal condition

$$\sum_{k=1}^{\infty} \frac{1}{k^6} x(\frac{k^2+k-1}{k^2+k}) = 0. \tag{8.4}$$

Set

$$f\left(t, x(\int_0^t g(s,x(s))ds)\right) = \frac{t}{4} + \frac{x(\int_0^t \frac{\sin^2(s)}{s+e^{|x(s)|}})|)}{\sqrt{t+9}}.$$

Then

$$\left|f\left(t, x(\int_0^t g(s,x(s))ds)\right)\right| \leqslant \frac{t}{4} + \frac{1}{3}|x|,$$

and also

$$|g(s,x(s))| \leqslant 1.$$

It is clear that the assumptions 1-3 of Theorem 3.1 are satisfied with $|c(t)| = |\frac{t}{4}| \leqslant \frac{1}{4}$ is measurable bounded, $b = \frac{1}{3}$, $2bT = \frac{2}{3} < 1$, $L = \frac{\frac{1}{4}}{1-\frac{2}{3}} = \frac{3}{4}$, and the series: $\sum_{k=1}^{\infty} \frac{1}{k^6}$ is convergent. Therefore, by applying Theorem 3.1, the given nonlocal problem (8.3)-(8.4) has a solution.

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