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Numerical solutions for linear fractional differential equations of order $1 < \alpha < 2$ using finite difference method (FFDM)

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Abstract

The major goal of this paper is to find accurate solutions for linear fractional differential equations of order $1 < \alpha < 2$. Hence, it is necessary to carry out this goal by preparing a new method called Fractional Finite Difference Method (FFDM). However, this method depends on several important topics and definitions such as Caputo's definition as a definition of fractional derivative, Finite Difference Formulas in three types (Forward, Central and Backward) for approximating the second and third derivatives and Composite Trapezoidal Rule for approximating the integral term in the Caputo's definition. In this paper, the numerical solutions of linear fractional differential equations using FFDM will be discussed and illustrated. The purposed problem is to construct a method to find accurate approximate solutions for linear fractional differential equations. The efficiency of FFDM will be illustrated by solving some problems of linear fractional differential equations of order $1 < \alpha < 2$.

Keywords: Finite difference formulas, composite trapezoidal rule, numerical solutions, linear fractional differential equation.
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1. Introduction

Fractional calculus is an old mathematical concept dating back to 17th century and involves integration and differentiation of arbitrary order. In a letter on 30th September 1695, L'Hospital

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wrote to Leibnitz asking him about the differentiation of order 1/2. Leibnitz response was "apparent paradox from which one day useful consequences will be drawn". In the following centuries, fractional calculus developed significantly within pure mathematics [21]. However the applications of fractional calculus just emerged in last few decades in various areas of physics and engineering, namely in signal processing, control engineering [2, 9], electromagnetism [10], bioscience [15], fluid mechanics [16], electrochemistry [20], diffusion processes [28], dynamic of viscoelastic materials [14], continuum and statistical mechanics [16] and propagation of spherical flames [18]. Motivated by increasing number of applications of fractional differential equations, considerable attention has been given to provide efficient methods for exact and numerical solutions of fractional differential equations.

In general, most of the fractional differential equations do not have exact solutions. Particularly, there is no known method for solving fractional differential equations exactly. Therefore several methods for approximate solutions to classical differential equations are extended to solve differential equations of fractional order numerically. Some of these methods include Adomian Decomposition Method [7], Perturbation Method [1], Homotopy Analysis Method [11], Variational Iteration Method [26], Extrapolation Method [8], and Generalized Differential Transform Method [19].

Fractional Finite Difference Method is useful to solve the linear fractional differential equations of order $1 < \alpha < 2$ numerically. This method is a technique by which fractional derivatives of $y(t)$ are approximated by differences in the values of the $y(t)$ between a value of the independent variable t and small increment $(t + h)$. To solve fractional differential equations numerically, we can replace the derivatives in the linear fractional differential equation of order $1 < \alpha < 2$ with finite difference approximations.

Further, the FFDM can be used to find accurate approximate solutions to the general fractional differential equation of the form

$$\begin{gathered} D_*^{\alpha} y(t) + a_m y^{(m)}(t) + a_{m-1} y^{(m-1)}(t) + ... + a_0 y(t) + N(y(t), y^{'}(t)) = f(t)\,, \\ t \geq 0, \quad m-1 < \alpha \leq m, \end{gathered} \tag{1.1}$$

Subject to $y^{(i)}(0) = y_i\,, \quad i = 0, 1, ..., m-1.$

Where D_*^{α} is the derivative of y of order α in the sense of Caputo fractional differential operator, $y(t)$ is an unknown function of the independent variable t and N is a nonlinear differential operator.

FFDM's efficiency is demonstrated by comparing the obtained approximate solutions for some linear fractional differential equations of order $1 < \alpha < 2$ by that method with the exact solutions and with other approximate solutions obtained by other solvers and methods. For the linear fractional differential equations, the mentioned method is utilized to find approximate solutions for these equations. Moreover, Mathematica (Version. 9.0) software program is used to solve the system of linear equations that is obtained from some operations.

2. Numerical Approximations and Fractional Calculus

The differentiation and integration is one of the most important topics in mathematics. This topic deals with the Integer Calculus (A calculus of the integer order derivatives and integrals). Integer Calculus is tightly related to the well-known fundamental theorem of classical calculus [23, 24]. Thus, the obtained results of integer order derivatives and integrals can often be carried over to the fractional case [23]. Fractional Calculus is a branch of mathematical analysis that studies the possibility of taking real or complex powers as orders.

The fractional derivative of order $\alpha > 0$ has several definitions. Riemann-Liouville and Caputo's definitions are the most commonly used for the derivative of this order. For the fractional derivative, the Caputo's definition is used, which is a modification of the Riemann-Liouville definition; because it has an advantage of dealing properly with the initial value problem since the initial condition is given in terms of the field variables and their integer order. This case is widely used in physical applications [6]. However, some basic concepts, definitions and properties of numerical approximations and fractional calculus are presented to be used in this section.

2.1. Numerical Differentiation and Integration

Usually, Taylor Series is used to obtain numerical differentiation methods. Methods using Taylor Series are: Backward Difference Method, Forward Difference Method and Central Difference Method to evaluate the derivative [3, 17]. Finite difference formulas are the most common formulas which are used to solve the ordinary and partial differential equations numerically [17]. The derivatives in these equations can be replaced with suitable finite difference approximations on a discretized domain. The accuracy of the solution depends on the number of mesh points such that if the number of mesh points is increased, then the solution will be more accurate [12]. However, Taylor Series can be used to derive some of finite difference approximations. For $f''(x_0)$, the well known formula is:

$$f''(x_0) = \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2} + O(h). \tag{2.1}$$

The central finite difference formula for the 3^{rd} derivative is known as

$$f'''(x_0) = \frac{-f(x_0-2h) + 2f(x_0-h) - 2f(x_0+h) + f(x_0+2h)}{2h^3} + O(h^2). \tag{2.2}$$

The forward finite difference formula for the 3^{rd} derivative similarly, which can be expressed as

$$f'''(x_0) = \frac{-f(x_0) + 3f(x_0+h) - 3f(x_0+2h) + f(x_0+3h)}{h^3} + O(h). \tag{2.3}$$

In addition, if we set $-h$ instead of h in Eq. (2.3), then the backward finite difference formula for the 3^{rd} derivative will be obtained as

$$f'''(x_0) = \frac{f(x_0) - 3f(x_0-h) + 3f(x_0-2h) - f(x_0-3h)}{h^3} + O(h). \tag{2.4}$$

On the other hand, there are several numerical methods for estimating definite integrals; one of these methods is called Composite Trapezoidal Rule [5],

Theorem 2.1. *Suppose that the interval $[a, b]$ is subdivided into n subintervals $[x_k, x_{k+1}]$ of width $h = \frac{b-a}{n}$ by using the equally spaced nodes $x_k = a + kh$, for $k = 0, 1, \ldots, n$. The Composite Trapezoidal Rule for n subintervals can be expressed as*

$$T(f,h) = \frac{h}{2}\sum_{k=1}^{n}\left[f(x_{k-1}) + f(x_k)\right]. \tag{2.5}$$

This is an approximation to the integral of $f(x)$ over $[a, b]$ since

$$\int_a^b f(x)\,.dx \approx T(f,h).$$

2.2. Fractional Derivatives

We start this subsection by illustrating some notations that will be used frequently. The notation $D_a^\alpha f(t)$ will be introduced as a notation which is denoted to the fractional derivative of a function $f(t)$ along t- axis of an arbitrary order $\alpha > 0$, where the subscript a is denoted to the lower limit of the integration. However, to get a simplified notation, we may drop the subscript a. Further, the fractional derivative can be defined using the definition of the fractional integral. The following definition illustrates that [25].

Definition 2.2. Let $\alpha \in \mathbb{R}^+$. For a positive integer m such that $m-1 < \alpha \leq m$, then the Riemann-Liouville Fractional Derivative of a function $f(t)$ of order α is defined by

$$D_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dt}\right)^m \int_a^t (t-x)^{m-\alpha-1} f(x).dx.$$

In addition, one observation that should be appointed, that's the most used version of D_a^α is when $a = 0$, so

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dt}\right)^m \int_0^t (t-x)^{m-\alpha-1} f(x).dx.$$

In 1967, M. Caputo presented a new definition of a fractional derivative called Caputo's Fractional Derivative which is a modification of the Riemann-Liouville Fractional Derivative [6]. However, the definition of the Caputo's Fractional Derivative uses almost the same notations of the definition of the Riemann-Liouville Fractional Derivative.

Definition 2.3. Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $n-1 < \alpha < n$, then the Caputo's Fractional Derivative of order α is defined by

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}.dx. \tag{2.6}$$

In addition, if $a = 0$ in Eq. (2.6), then the most used version of the Caputo's Fractional Derivative is obtained, i.e.

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}.dx. \tag{2.7}$$

3. Approximating the Fractional Derivative of order $1 < \alpha < 2$ using FFDM

For the fractional derivative, the Caputo's definition is chosen. Now, according to Eq. (2.7), if $1 < \alpha < 2$, then the Caputo's Fractional Derivative is defined by

$$D^\alpha y(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{y''(x)}{(t-x)^{\alpha-1}}.dx \tag{3.1}$$

for $t \geq 0$ and $\alpha \in \mathbb{R}^+$.
By applying integral by parts on the right hand side of Eq. (3.1), we get

$$D^\alpha y(t) = \frac{1}{(2-\alpha)\Gamma(2-\alpha)}\left(y''(0)\, t^{2-\alpha} + \int_0^t (t-x)^{2-\alpha} y'''(x)\,.dx\right). \tag{3.2}$$

Last integral in Eq. (3.2) can be approximated using the Composite Trapezoidal Rule, Eq. (2.5), as follows

$$\int_0^t (t-x)^{2-\alpha} y^{'''}(x)\,.dx \approx \frac{h}{2}\left[t^{2-\alpha} y^{'''}(0) + 2\sum_{j=1}^{n-1} (t-x_j)^{2-\alpha} y^{'''}(x_j) + (t-b)^{2-\alpha} y^{'''}(b)\right], \tag{3.3}$$

with $h = (b-a)/n$ and $x_j = a + jh$ for each $j = 0, 1, \dots, n-1$.
Now, by substituting Eq. (3.3) in Eq. (3.2), we obtain,

$$\begin{aligned} D^{\alpha} y(t) \approx & \frac{1}{(2-\alpha)\,\Gamma(2-\alpha\ )} \Bigg(y^{''}(0)\, t^{2-\alpha} \\ & + \frac{h}{2}\left[t^{2-\alpha} y^{'''}(0) + 2\sum_{j=1}^{n-1} (t-x_j)^{2-\alpha} y^{'''}(x_j) + (t-b)^{2-\alpha} y^{'''}(b)\right] \Bigg). \end{aligned} \tag{3.4}$$

Here, $y^{''}$ and $y^{'''}$ can be approximated at specific points. Forward finite difference formula for the 2^{nd} order derivative, Eq. (2.1), is used to approximate $y^{''}(0)$ such that

$$y^{''}(0) \approx \frac{y(0) - 2y(h) + y(2h)}{h^2} \tag{3.5}$$

for small values of h.
To approximate $y^{'''}(0)$, also forward finite difference formula for the 3^{rd} derivative, Eq. (2.3), is used as follows

$$y^{'''}(0) \approx \frac{-y(0) + 3y(h) - 3y(2h) + y(3h)}{h^3} \tag{3.6}$$

for small values of h.
While the central finite difference formula for the 3^{rd} derivative, Eq. (2.2), is used to approximate $y^{'''}(x_j)$ in the sum term of Eq. (3.4) as

$$y^{'''}(x_j) \approx \frac{-y(x_j - 2h) + 2y(x_j - h) - 2y(x_j + h) + y(x_j + 2h)}{2h^3} \tag{3.7}$$

for small values of h and for $j = 0, 1, \dots, n-1$.
Finally, $y^{'''}(b)$ can be approximated using backward finite difference formula for the 3^{rd} derivative, Eq. (2.4), as follows

$$y^{'''}(b) \approx \frac{y(b) - 3y(b-h) + 3y(b-2h) - y(b-3h)}{h^3} \tag{3.8}$$

for small values of h.
By substituting Eq.s (3.5), (3.6), (3.7) and (3.8) in Eq. (3.4), we get

$$D^{\alpha} y(t) \approx \frac{1}{(2-\alpha)\,\Gamma(2-\alpha\ )} (A + \frac{h}{2}(B + 2C + D)), \tag{3.9}$$

where

$$A = \frac{y(0) - 2y(h) + y(2h)}{h^2} t^{2-\alpha},$$

$$B = \frac{-y(0) + 3y(h) - 3y(2h) + y(3h)}{h^3} t^{2-\alpha},$$

$$C = \sum_{j=1}^{n-1} \left[\frac{-y\left(x_j - 2h\right) + 2y(x_j - h) - 2y\left(x_j + h\right) + y(x_j + 2h)}{2h^3} \left(t - x_j\right)^{2-\alpha} \right],$$

and

$$D = \frac{y\left(b\right) - 3y\left(b - h\right) + 3y\left(b - 2h\right) - y(b - 3h)}{h^3} \left(t - b\right)^{2-\alpha}.$$

We observe that, according to Eq. (3.9), the approximation of $D^\alpha(y(t))$ depends on the value of h such that if the value of h is decreased, then the approximate result of $D^\alpha y\left(t\right)$ will be more accurate. Moreover, Eq. (3.9) is the purposed equation that needs an algorithm to approximate the fractional derivative of order $1 < \alpha < 2$ of a given function $y(t)$ which is clear. Also, the value of h according to the Composite Trapezoidal Rule, Eq. (2.5), is $(b - a)/n$ So;

$$n = \frac{b - a}{h}.$$

Now, we can find $D^\alpha y\left(t\right)$ approximately by taking increasingly n's, that's we may take n={10, 100, 1000, 10000}. Also, we should note that we use $a = 0$ and $b = 1$.

4. Numerical Solutions for Linear Fractional Differential Equations of order $1 < \alpha < 2$ using FFDM

To show the efficiency of the FFDM, we will approximate the solutions for some linear fractional differential equations of order $1 < \alpha < 2$ using a prepared program in Mathematica (Ver. 9.0) and compare them with the exact solutions and with other approximate solutions obtained by other solvers and methods. As we mentioned before, Eq. (3.9) is considered as an approximation for $D^\alpha y\left(t\right)$; so whenever we find $D^\alpha y\left(t\right)$ in any linear fractional differential equation of order $1 < \alpha < 2$ of the form of Eq. (1.1), we may replace it by that approximation.

Example 4.1 ([22])**.** Consider the following linear fractional differential equation:

$$D^{1.5}y\left(t\right) = t^{1.5}.y\left(t\right) + 4\sqrt{\frac{t}{\pi}} - t^{3.5},$$

with the following initial conditions

$$y\left(0\right) = 0.$$

Note that, the exact solution for this problem is $y\left(t\right) = t^2$.

By taking $h = 0.1$, Table 1 illustrates the concluded results using FFDM by computing the approximate solutions for this problem at specific points of t.

Example 4.2 ([4, 27])**.** Consider the following linear fractional differential equation:

$$D^2y\left(t\right) + D^{1.5}y\left(t\right) + y\left(t\right) = t^2 + 2 + 4\sqrt{\frac{t}{\pi}},$$

with the following initial condition

$$y\left(0\right) = 0.$$

Note that, the exact solution for this problem is $y\left(t\right) = t^2$.

Here, we take $h = 0.1$, Table 2 illustrates the concluded results using FFDM by computing the approximate solutions for this problem at specific points of t.

Table 1: Numerical solutions for Example 4.1 using FFDM, when $h = 0.1$

T	App.	Exact	Absolute Error
0.1	0.00999999999999998	0.01	2.08167E-17
0.2	0.03999999999999981	0.04	2.01228E-16
0.3	0.08999999999999946	0.09	5.96745E-16
0.4	0.15999999999999906	0.16	9.99201E-16
0.5	0.24999999999999858	0.25	1.9984E-15
0.6	0.35999999999999790	0.36	2.9976E-15
0.7	0.48999999999999705	0.49	2.9976E-15
0.8	0.63999999999999600	0.64	3.9968E-15
0.9	0.80999999999999470	0.81	6.10623E-15
1.0	0.99999999999999340	1.00	6.99441E-15

Table 2: Numerical solutions for Example 4.2 using FFDM, when $h = 0.1$

T	App.	Exact	Absolute Error
0.1	0.01000000000000325	0.01	3.20056E-15
0.2	0.04000000000000937	0.04	9.29812E-15
0.3	0.09000000000001489	0.09	1.48076E-14
0.4	0.16000000000001902	0.16	1.89848E-14
0.5	0.25000000000002354	0.25	2.29816E-14
0.6	0.36000000000002824	0.36	2.80331E-14
0.7	0.49000000000003320	0.49	3.30291E-14
0.8	0.64000000000003820	0.64	3.79696E-14
0.9	0.81000000000004320	0.81	4.29656E-14
1.0	1.00000000000004860	1.00	3.9968E-14

Example 4.3 ([13]). Consider the following linear fractional differential equation:

$$D^2 y\left(t\right) + D^{1.5} y\left(t\right) + y\left(t\right) = 1 + t,$$

with the following initial condition

$$y\left(0\right) = 1.$$

Regarding to [13], the author's develop a new method called Generalized Taylor Collocation Method (GTCM) to give approximate solutions for linear fractional differential equations with variable coefficients. They found that this method is easy to write in computer codes and can be used as an effective method for obtaining analytic and approximate solutions for fractional differential equations. Moreover, the previous problem was solved within their paper by their method. They introduce the fourth order approximation solution for that problem which is as

$$y\left(t\right) = 1 + t.$$

However, our FFDM is also applied to solve this problem and compared with GTCM according to their approximate solutions. by taking $h = 0.1$, Figure 1 shows that the concluded results using FFDM and the results using GTCM coincide.

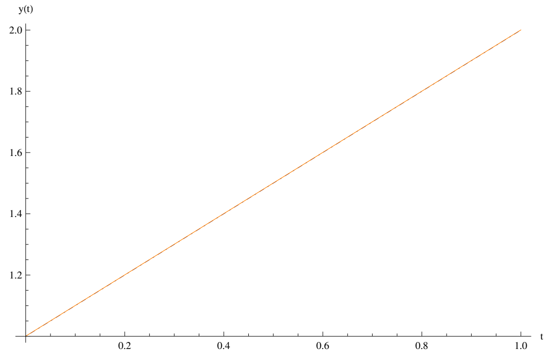

Figure 1: The comparison between the FFDM solutions (where $h = 0.1$) and the approximate solutions which are obtained by GTCM, according to Example 4.3

5. Conclusions

The major goal of this paper is to find accurate approximate solutions for linear fractional differential equations of order $1 < \alpha < 2$. Hence, we carry out this goal by preparing a new method called Fractional Finite Difference Method (FFDM). In this paper, we discussed and illustrated the numerical solutions of linear fractional differential equations using FFDM. Also, the efficiency of FFDM was illustrated by solving some examples of linear fractional differential equations of order $1 < \alpha < 2$.

Successfully, FFDM was applied to approximate the linear fractional differential equations of order $1 < \alpha < 2$. All ideas were illustrated to be efficient in applying the proposed technique to several linear fractional differential equations of order $1 < \alpha < 2$. We found that our method is powerful and efficient in finding numerical solutions for those equations. Moreover, the FFDM solutions demonstrate excellent approximations in comparison with the exact solutions and with other methods and solvers through the applicable domain.

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