Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 21 (2020), 150–157
Research Article

Online: ISSN 2008-949X
Journal of Mathematics and Computer Science
Journal Homepage: www.isr-publications.com/jmcs

Numerical solution of second order Painlevé differential equation

Hijaz Ahmad[a], Tufail A. Khan[a], Shao-Wen Yao[b,*]

[a]*Department of Basic Sciences, University of Engineering and Technology, Peshawar, Pakistan.*

[b]*School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China.*

Abstract

In this paper, the second order Painlevé differential equation is solved by variational iteration algorithm-I with an auxiliary parameter (VI-I with AP), how to optimally find the auxiliary parameter and Pade approximates for the numerical solution are explained. The effectiveness and suitability of the proposed method are shown by solving two types of second order Painlevé differential equation and the proposed method is compared with other methods to illustrate the accuracy and efficiency of the method.

Keywords: Painlevé equation, second order Painlevé differential equation, VIA-I with AP, RK4.

2020 MSC: 34M55, 65L99, 35A15.

©2020 All rights reserved.

1. Introduction

In present time, the Painlevé equations are extensively arise in several fields of pure and applied mathematics as well as theoretical physics. Painlevé considered wide class of second order equations and classified them to the nature of singularities. The Painlevé equations were introduced by Painlevé [10] and some of his partners during having under observation a non-linear second order differential equation. Painlevé and his colleagues discovered that there are 50 such canonical equations of the form

$$\left(\frac{d^2u}{dt^2}\right) = R\ (t, u, \frac{du}{dt}),$$

where R is a holomorphic function in t and rational in the other entries. Singularities of these equations have a unique property (called as Painlevé property) the only moveable singularities are poles. This vital property has been occupied by all linear ODE but rarely in non-linear equations. In a study about the classification of the singularity structure, Painlevé and Gambier found that up to certain transformations, every second-order differential equation with polynomial coefficients can be put into one of 50 canonical

*Corresponding author
Email address: yaoshaowen@hpu.edu.cn (Shao-Wen Yao)

doi: 10.22436/jmcs.021.02.06

Received: 2019-10-18 *Revised:* 2020-02-03 *Accepted:* 2020-03-03

forms. Which of these equations are irreducible and cannot be solved?, was the question arisen. Painlevé explained that 6 of these 50 equations are irreducible and required the introduction of new special functions for its solution. These six second-order nonlinear differential equations are nowadays known as the Painlevé equations. In this paper, two types of second order Painlevé differential equation are studied. Type I:

$$\frac{d^2u}{dt^2} = 2u^3 + tu + \alpha.$$

Type II:

$$\frac{d^2u}{dx^2} = 6u^2 + x.$$

As the Painlevé equations were created from strictly mathematical attentions and they have the remarkable importance of many physical problems including plasma physics, nonlinear waves, statistical mechanics, general relativity, quantum gravity, fibre optics and nonlinear optics [12]. This reality has become the reason for interest in the study of these equations in nearby years. In this paper, the solution to second order Painlevé equations of both the types have been analyzed by VI Algorithm-I, VI Algorithm-I with AP, RK4 and RKF45 method are incorporated and comparison with other methods are shown.

2. Variational iteration algorithm-I

Consider a nonlinear differential equation.

$$L\left[u(x)\right] + N\left[u(x)\right] = g(x), \tag{2.1}$$

where $L\left[u(x)\right]$ and $N\left[u(x)\right]$ denote, correspondingly, the linear and nonlinear term, while $g(x)$ is the source term. Constructing correction functional for (2.1), which reads

$$u_{k+1}(x) = u_k(x) + \int_0^x \lambda(\eta)\left[L\{u_k(\eta)\} + N\{\widetilde{u_k(\eta)}\} - g(\eta)\right] d\eta, \tag{2.2}$$

where λ is known as Lagrange multiplier [23], which can be found by variation theory [14–16, 20, 21], taking δ on both sides of the recurrence relation (2.2) w.r.t the variable $u_k(x)$,

$$\delta u_{k+1}(x) = \delta u_k(x) + \delta \int_0^x \lambda(\eta)\left[L\{u_k(\eta)\} + N\{\widetilde{u_k(\eta)}\} - g(\eta)\right] d\eta, \tag{2.3}$$

where $\widetilde{u_k(\eta)}$ is a restricted variable such that $\delta\widetilde{u_k(\eta)} = 0$. Utilizing optimality conditions, the estimation of Lagrange multiplier $\lambda(\eta)$ can be distinguished. An accurate arrangement can be acquired when k ways to deal with limitlessness,

$$u(x) = \lim_{k\to\infty} u_k(x).$$

The overhead strategy for finding the exact solution is known as variational iteration algorithm-I (VIA-I) proposed for the first time by a Chinese mathematicians Ji-Huan He [17–19]. The basic concept was taken from the general Lagrange multiplier method of Inokuti et al. [23]. Presently this strategy has been created to handle a vast range of problems started from different fields of sciences to obtain precise solutions of nonlinear problems and it has been modified further [1, 3–7, 22, 24–26]. In VIA-I, an unidentified helper parameter h will be presented which was used in HPM [27]. The ideal decision of h increases the accuracy and efficiency of the technique. After presenting h, equation (2.3) will yield the formula

$$\delta u_{k+1}(x) = \delta u_k(x) + h\delta \int_0^x \lambda(\eta)\left[L\{u_k(\eta)\} + N\{\widetilde{u_k(\eta)}\} - g(\eta)\right] d\eta.$$

Summarizing the iterative algorithm for equation (2.1) as,

$$\begin{cases} u_0(\eta) \text{ is an appropriate initial approximation,} \\ u_1(x,h) = u_0(x) + h\int\limits_0^x \lambda(\eta)[Nu_0(\eta) - g(\eta)]d\eta, \\ u_{k+1}(x,h) = u_k(x,h) + h\int\limits_0^x \lambda(\eta)[Nu_k(\eta,h) - g(\eta,h)]d\eta. \end{cases}$$

The estimated solution $u_k(x,h)$ has the helper parameter h, which guarantees the intermingling to the precise solution. This procedure is known as variational iteration algorithm-I with an auxiliary parameter (VIA-I with AP) [4, 11], which is extremely basic, less demanding to execute and is likewise able to approximate the solution with high exactness and accuracy in a vast solution domain. For the convergence of this method, see [9].

3. Type I of second order Painlevé differential equation

The second order Painlevé differential equation is formulated in the below form

$$\frac{d^2u}{dt^2} = 2u^3 + tu + \alpha, \tag{3.1}$$

with initial conditions: $u(0) = 1$, $u\prime(0) = 0$. First, we find the solution by the use of VIA-I. Constructing the correction functional for equation (3.1):

$$u_{k+1}(t) = u_k(t) + \int_0^t \lambda(\eta)\left\{\frac{d^2u_k(\eta)}{dt^2} - 2(\widetilde{u_k(\eta)})^3 - t(\widetilde{u_k(\eta)}) - (\widetilde{\alpha})\right\}d\eta. \tag{3.2}$$

Utilizing optimality conditions, the estimation of Lagrange multiplier $\lambda(\eta)$ is $\lambda(\eta) = \eta - t$. Utilizing this estimation of $\lambda(\eta)$ in equation (3.2) results in the below iterative scheme:

$$u_{k+1}(t) = u_k(t) + \int_0^t (\eta - t)\left\{\frac{d^2u_k(\eta)}{dt^2} - 2(u_k(\eta))^3 - t(u_k(\eta)) - (\alpha)\right\}d\eta. \tag{3.3}$$

Starting with the initial approximation, $u_0(t) = 1 + \alpha\frac{t^2}{2}$, one can get the below approximations by utilizing the iterative scheme (3.3),

$$\begin{aligned} u_0(t) &= 1 + \alpha\frac{t^2}{2}, \\ u_1(t) &= \frac{a^3(t^8)}{224} + \frac{a^2(t^6)}{20} + \frac{a(t^5)}{40} + \frac{a(t^4)}{4} + \frac{a(t^2)}{2} + \frac{(t^3)}{6} + t^2 + 1, \\ &\vdots \end{aligned}$$

We stop the solution procedure at $u_3(t)$ and combined the overhead series solution with the [4/4] Padé approximates to acquire our required approximate solution. For subtleties of Padé approximates, see [11]. Now we elucidate this second order Painlevé differential equation by VIA-I with AP. Using VIA-I with AP, recurrence relation for (3.1) is

$$u_{k+1}(t,\ h) = u_k(t,\ h) - h(\eta - t)\int_0^t \left\{\frac{\partial^2u_k(\eta,h)}{\partial\ t^2} - 2(u_k(\eta,h))^3 - t(u_k(\eta,h)) - (\alpha)\right\}d\eta. \tag{3.4}$$

Starting with the initial approximation, $u_0(t) = 1 + \alpha\frac{t^2}{2}$, one can get the below other approximations by utilizing the recurrence relation (3.4),

$$u_0(t) = 1 + \alpha\frac{t^2}{2},$$

$$u_1(t,\ h) = h\left(\frac{a^3(t^8)}{224} + \frac{a^2(t^6)}{20} + \frac{a(t^5)}{40} + \frac{a(t^4)}{4} + \frac{(t^3)}{6} + t^2\right) + \frac{a(t^2)}{2} + 1,$$
$$\vdots$$

We stop the solution process of the proposed technique at $u_3(t,h)$. Residual function for choosing optimal value of the helper parameter:

$$r_3(t,\ h) = \frac{\partial^2 u_3(\ t,h)}{\partial\ t^2} - 2\left(u_3(t,h)\right)^3 - t\left(u_3(t,h)\right) - (\alpha).$$

The square of residual function for 3rd-order approximation w.r.t. h for $(x,t)\,\epsilon\ [0,\ 1]\times[0,\ 1]$ is

$$\frac{1}{61}\sum_{i=0}^{10}\left(r_3\left(\frac{i}{30},h\right)\right)^2.$$

Square residual function gives minimum value at h=1.12966217481889, the traditional variational iteration method always leads to a good result as well but here this modification make it more convergent and efficient. Now solving the second order Painlevé differential equation using RK4 method. We change equation (3.1) into a system of first order IVP using $u(t) = u(t)$, $v(t) = u'(t)$. The above assumptions transform the second order Painlevé equation into the system

$$\frac{du}{dt} = v(t),$$
$$\frac{dv}{dt} = 2u^3 + tu + \alpha$$

with the initial conditions, $u(0) = 1, v(0) = 0$. RK4 method for a system of second order differential equations is

$$\begin{aligned}
f(t,u,v) = v(t),\ \ g(t,u,v) &= 2u^3 + tu + \alpha,\ \ t_0 = 0,\ \ u_0 = 1,\ v_0 = 0,\ h = 0.1,\\
k_{1,1} &= h *\ f(t_i,u_i,v_i),\\
k_{1,2} &= h *\ g(t_i,u_i,v_i),\\
k_{2,1} &= h *\ f(t_i + \frac{h}{2},\ u_i + \frac{1}{2}k_{1,1},\ v_i + \frac{1}{2}k_{1,2}),\\
k_{2,2} &= h *\ g(t_i + \frac{h}{2},\ u_i + \frac{1}{2}k_{1,1},\ v_i + \frac{1}{2}k_{1,2}),\\
k_{3,1} &= h *\ f(t_i + \frac{h}{2},\ u_i + \frac{1}{2}k_{2,1},\ v_i + \frac{1}{2}k_{2,2}),\\
k_{3,2} &= h *\ g(t_i + \frac{h}{2},\ u_i + \frac{1}{2}k_{21},\ v_i + \frac{1}{2}k_{2,2}),\\
k_{4,1} &= h *\ f(t_i + h,\ u_i + k_{3,1},\ v_i + k_{3,2}),\\
k_{4,2} &= h *\ g(t_i + h,\ u_i + k_{3,1},\ v_i + k_{3,2}),\\
u_{i+1} &= u_i + \frac{1}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right),\quad i = 0,1,2,3,\ldots,21.
\end{aligned}$$

At last, solving the second order Painlevé differential equation by using RK Fehlberg method, which is an approach for the numerical solution of ODEs suggested by Erwin Fehlberg. Each step requires the following coefficient equations

$$\begin{aligned}
k_1 &= h *\ f(t_i,u_i),\\
k_2 &= h *\ f(t_i + \frac{h}{4},\ u_i + \frac{1}{4}k_1),
\end{aligned}$$

$$k_3 = h * f(t_i + \frac{3h}{8}, u_i + \frac{3}{32}k_1 + \frac{9}{32}k_2),$$
$$k_4 = h * f(t_i + \frac{12h}{13}, u_i + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3),$$
$$k_5 = h * f(t_i + h, u_i + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4),$$
$$k_6 = h * f(t_i + \frac{h}{2}, u_i - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5).$$

Runge Kutta Method of order 4 is

$$u_{i+1} = u_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5.$$

Runge Kutta Method of order 5

$$v_{i+1} = u_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6.$$

Runge Kutta Fehlberg Method (RKF45) has a built-in routine existing in Maple. The results obtained using Maple and Matlab are displayed in Figure 1.

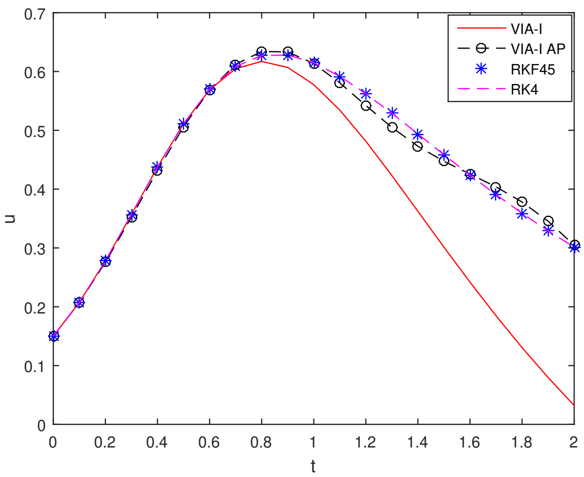

Figure 1: Comparison of the approximate solutions by VIA-I with AP, VIA-I, RKF45, and RK4.

Table 1: Comparison of the results for Painlevé differential equation (3.1).

t	RKF45	RK4	VIA-I	VIA-I with AP
0.1	1.015	1.015	1.015	1.015
0.2	1.063	1.063	1.063	1.063
0.3	1.146	1.146	1.146	1.146
0.4	1.274	1.274	1.273	1.275
0.5	1.459	1.459	1.457	1.468
0.6	1.725	1.725	1.716	1.743
0.7	2.118	2.118	2.087	2.156
0.8	2.737	2.736	2.635	2.806
0.9	3.834	3.830	3.496	3.934
1.0	6.311	6.274	5.005	6.259

The graphical result displays that at t=0.7, the VIA-I starts diverging while VIA-I with AP, RKF45 and RK4 show almost similar results. Here, four methods have been analyzed to solve Painlevé II differential equation. As there is no exact solution of the said equation, in this way, for examination VIA-I, VIA-I with AP, RKF45 and RK4 have been utilized. The graphical and numerical outcomes demonstrate that VIA-I with AP is a progressively precise and dependable technique for solving second order Painlevé differential equation than VIA-I.

4. Type II of second order Painlevé differential equation

The second order Painlevé differential equation [9, 13, 22] is formulated in the below form

$$\frac{d^2u}{dx^2} = 6u^2 + x, \tag{4.1}$$

with below initial conditions: $u(0) = 0$, $u\prime(0) = 1$. First, we find the solution by the use of VIA-I. Constructing the correction functional for equation (4.1):

$$u_{k+1}(x) = u_k(x) + \int_0^t \lambda(\eta) \left\{ \frac{d^2 u_k(\eta)}{dx^2} - 6(\widetilde{u_k(\eta)})^2 - \eta \right\} d\eta. \tag{4.2}$$

Utilizing optimality conditions [8], the estimation of Lagrange multiplier $\lambda(\eta)$ is $\lambda(\eta) = \eta - t$. Utilizing this estimation of $\lambda(\eta)$ in equation (4.2) results in the below iterative scheme:

$$u_{k+1}(x) = u_k(x) + \int_0^t (\eta - x) \left\{ \frac{d^2 u_k(\eta)}{dx^2} - 6(u_k(\eta))^2 - \eta \right\} d\eta. \tag{4.3}$$

Starting with the initial approximation $u_0(t) = x + \frac{1}{6}x^3$, one can get the below approximations by utilizing the iterative scheme (4.3),

$$\begin{aligned} u_0(t) &= 1 + \alpha\frac{t^2}{2}, \\ u_1(x) &= x + \frac{(x^4(5x^4 + 112x^2 + 840))}{1680} + \frac{x^3}{6}, \\ &\vdots \end{aligned}$$

We stop the solution procedure at $u_3(t)$ and combined the overhead series solution with the [4/4] Padé approximates to acquire our required approximate solution. For subtleties of Padé approximates, see [11]. Now we elucidate this second order Painlevé differential equation by VIA-I with AP. Using VIA-I with AP, recurrence relation for (4.1) is

$$u_{k+1}(x, h) = u_k(x, h) - h(\eta - x) \int_0^x \left\{ \frac{\partial^2 u_k(\eta, h)}{\partial x^2} - 6(u_k(\eta, h))^2 - \eta \right\} d\eta. \tag{4.4}$$

Starting with the initial approximation $u_0(t) = 1 + \alpha\frac{t^2}{2}$, one can get the below other approximations by utilizing the recurrence relation (4.4),

$$\begin{aligned} u_0(t) &= 1 + \alpha\frac{t^2}{2}, \\ u_1(x) &= x + \frac{(hx^4(5x^4 + 112x^2 + 840))}{1680} + \frac{x^3}{6}, \\ &\vdots \end{aligned}$$

We stop the solution process of the proposed technique at $u_3\,(t,h)$. Residual function for choosing optimal value of the helper parameter is

$$r_3\,(x,\ h) = \frac{\partial^2 u_3\,(\,x,h)}{\partial\, x^2} - 6\,(u_3(x,h))^2 - x.$$

The square of residual function for 3rd-order approximation w.r.t. h for $(x,t) \in\ [0,\ 1] \times [0,\ 1]$ is

$$\frac{1}{11}\sum_{i=0}^{10}\left(r_3\left(\frac{i}{10},h\right)\right)^2.$$

Square residual function gives minimum value at h=1.07490094963061, the traditional variational iteration method always leads to a good result as well but here this modification make it more convergent and efficient.

Table 2: Comparison of the approximate solution by VIA-I with AP with VIA-I, RKF45 and RK4.

x	CP and CM [13]	ADM [9]	MADM [9]	VIM [22]	Present	Maple Results [22]
0.1	0.1097190000	0.1002158320	0.1002601271	0.1002167477	0.1002167477	0.1002167980
0.2	0.2416430000	0.2021177613	0.2021288956	0.2021394527	0.2021394528	0.2021395810
0.3	0.3725930000	0.3086306243	0.3086306987	0.3086307490	0.3086307481	0.3086309826
0.4	0.4814830000	0.4239851458	0.4239860367	0.4239862788	0.4239862923	0.4239865878
0.5	0.5554420000	0.5543356243	0.5543370146	0.5543399110	0.5543401816	0.5543405192
0.7	0.6641910000	0.8992174326	0.8992199875	0.8992296944	0.8992421070	0.8992504369
0.9	1.2976000000	1.4812013434	1.4814889672	1.481778951	1.4819866745	1.4825252525
1.0	2.3111300000	1.9371446837	1.9416721356	1.959421042	1.9601244148	1.9631292204

5. Conclusion

In this article, the Variational iteration algorithm-I with an auxiliary parameter has been employed for solving second order Painlevé differential equation. Furthermore, this article shows that variational iteration algorithm-I, RK4 and RK Fehlberg method are appropriate to present Painlevé II differential equation. Graphical and numerical outcomes demonstrate that VIA-I with AP has great precision and is more exact and dependable approach than VIA-I for the solution of Painlevé II differential equation.

Acknowledgment

The authors thank the Editor-in-Chief and unknown referees for the fruitful comments and significant remarks that helped them in improving the quality and readability of the paper, which led to a significant improvement of the paper. This work was supported by National Natural Science Foundation of China (No. 71601072) and Key Scientific Research Project of Higher Education Institutions in Henan Province of China (No. 20B110006).

References

[1] H. Ahmad, *Variational iteration algorithm–II with an auxiliary parameter and its optimal determination*, Nonlinear Sci. Lett. A, **9** (2018), 62–72. 2

[2] H. Ahmad, *Variational iteration method with an auxiliary parameter for solving differential equations of the fifth order*, Nonlinear Sci. Lett. A, **9** (2018), 27–35.

[3] H. Ahmad, *Variational Iteration Algorithm–I with an Auxiliary Parameter for Solving Fokker–Planck Equation*, Earthline J. Math. Sci., **2** (2019), 29–37.. 2

[4] H. Ahmad, T. A. Khan, *Variational iteration algorithm-I with an auxiliary parameter for wave-like vibration equations*, J. Low Frequency Noise Vibr. Active Control, **38** (2019), 1113–1124. 2

[5] H. Ahmad, T. A. Khan, *Variational iteration algorithm I with an auxiliary parameter for the solution of differential equations of motion for simple and damped mass–spring systems*, Noise & Vibration Worldwide, **51** (2020), 12–20.

[6] H. Ahmad, T. A. Khan, C. Cesarano, *Numerical Solutions of Coupled Burgers' Equations*, Axioms, **8** (2019), 16 pages.

[7] H. Ahmad, A. R. Seadawy, T. A. Khan, *Numerical solution of Korteweg-de Vries-Burgers equation by the modified variational Iteration algorithm–II arising in shallow water waves*, Phys. Scr., **95** (2020), 12 pages. 2

[8] N. Anjum, J.-H. He, *Laplace transform: Making the variational iteration method easier*, Appl. Math. Lett., **92** (2019), 134–138. 4

[9] S. S. Behzadi, *Convergence of Iterative Methods for Solving Painlevé Equation*, Appl. Math. Sci. (Ruse), **4** (2010), 1489–1507. 2, 4, 2

[10] A. V. Borisov, N. A. Kudryashov, *Paul Painlevé and his contribution to science*, Regul. Chaotic Dyn., **19** (2014), 1–19. 1

[11] J. P. Boyd, *Padé approximant algorithm for solving nonlinear ordinary differential equation boundary value problems on an unbounded domain*, Comput. Phys., **11** (1997), 299–303. 2, 3, 4

[12] P. A. Clarkson, *Painlevé equations–nonlinear special functions*, J. Comput. Appl. Math., **153** (2003), 127–140. 1

[13] E. Hashemizadeh, F. Mahmoudi, *Numerical solution of Painlevé equation by Chebyshev polynomials*, J. Interpolat. Approx. Sci. Comput., **2016** (2016), 26–31. 4, 2

[14] J.-H. He, *A fractal variational theory for one-dimensional compressible flow in a microgravity space*, Fractals, **27** (2019), 11 pages. 2

[15] J.-H. He, *A modified Li–He's variational principle for plasma*, Int. J. Numer. Methods Heat Fluid Flow, **2019** (2019), 11 pages.

[16] J.-H. He, *Lagrange Crisis and Generalized Variational Principle for 3D unsteady flow*, Int. J. Numer. Methods Heat Fluid Flow, **2019** (2019), 15 pages. 2

[17] J.-H. He, *The simpler, the better: Analytical methods for nonlinear oscillators and fractional oscillators*, J. Low Frequency Noise Vibr. Active Control, **2019** (2019), 12 pages. 2

[18] J.-H. He, *Generalized Variational Principles for Buckling Analysis of Circular Cylinders*, Acta Mech., **231** (2020), 899–906.

[19] J.-H. He, H. Latifizadeh, *A general numerical algorithm for nonlinear differential equations by the variational iteration method*, Int. J. Numer. Methods Heat Fluid Flow, **2020** (2020), 11 pages. 2

[20] C.-H. He, Y. Shen, F.-Y. Ji, J.-H. He, *Taylor series solution for fractal Bratu-type equation arising in electrospinning process*, Fractals, **28** (2020), 13 pages. 2

[21] J.-H. He, C. Sun,*A variational principle for a thin film equation*, J. Math. Chem., **57** (2019), 2075–2081. 2

[22] E. Hesameddini, A. Peyrovi, *The Use of Variational Iteration Method and Homotopy Perturbation Method for Painlevé Equation I*, Appl. Math. Sci., **3** (2009), 1861–1871. 2, 4, 2

[23] M. Inokuti, H. Sekine, T. Mura, *General use of the Lagrange multiplier in nonlinear mathematical physics*, Pergamon Press, Oxford, (1978). 2, 2

[24] M. Nadeem, H. Ahmad, *Variational Iteration Method for Analytical Solution of the Lane-Emden Type Equation with Singular Initial and Boundary Conditions*, Earthline J. Math. Sci., **2** (2019), 127–142. 2

[25] M. Nadeem, F. Li, H. Ahmad, *He's variational iteration method for solving non–homogeneous Cauchy Euler differential equations*, Nonlinear Sci. Lett. A Math. Phys. Mech., **9** (2018), 231–237.

[26] M. Nadeem, F. Li, H. Ahmad, *Modified Laplace variational iteration method for solving fourth–order parabolic partial differential equation with variable coefficients*, Comput. Math. Appl., **2019** (2019), 10 pages. 2

[27] D.-N. Yu, J.-H. He, A. G. Garcıa, *Homotopy perturbation method with an auxiliary parameter for nonlinear oscillators*, J. Low Frequency Noise Vibr. Active Control, **2018** (2018), 11 pages. 2