Conformally transformed vector fields on special \((\alpha, \beta)\)-metric

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Abstract

In this paper, we characterize the PDE’s of conformal vector fields on Finsler space with special \((\alpha, \beta)\)-metrics. Further, we prove that conformally transformed vector field related by \(F\) and also corresponding conformal factors \(c\) and \(\tilde{c}\).

Keywords: Conformal vector field, special \((\alpha, \beta)\)-metrics.

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1. Introduction

The first to treat the conformal theory of Finsler metrics generally was M. S. Knebelman. He defined two metric functions \(F\) and \(\tilde{F}\) as conformal if the length of an arbitrary vector in the one is proportional to the length in the other, that is if \(g_{ij} = \phi g_{ij}\). The length of vector \(c\) means here the fact that \(\phi g_{ij}\), as well as \(g_{ij}\), must be Finsler metric tensor, he showed that \(\phi\) falls into a point function and also proved that if \((M, F)\) be an \(n\)-dimensional Finsler manifold and \(\phi\) a transformation on \(M\), then \(\phi\) is called the conformal transformation, if it preserves the angles. Let \(X\) be a vector field on \(M\) and \(\phi_t\) be the local one-parameter group of local transformations on \(M\) generated by \(X\). Then \(X\) is called a conformal vector field on \(M\) if each \(\phi_t\) is a local conformal transformation of \(M\).

Conformal vector fields play an important role in Finsler geometry. Some problems on \((\alpha, \beta)\)-metrics can be solved by constructing a conformal vector field on a Riemannian metric with certain curvature features. For two conformally related Finsler metrics on a manifold, their conformal vector field coincide [6].

In [4], Shen and Xia study the conformal vector fields on Randers spaces under certain curvature conditions. In [1], Huang and Mo shows that a conformal vector field of a Randers space of isotropic \(S\)-curvature must be homothetic. In [2], Kang characterizes the conformal vector fields of an \((\alpha, \beta)\)-metrics

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by some PDE’s in a special case \( \phi^1 \neq 0 \). Recently, Natesh et al. studied the conformal vector fields on Finsler space with special \((\alpha, \beta)\)-metrics [3].

As above reviews, this work characterize the PDE’s of conformal vector fields on Finsler space with special \((\alpha, \beta)\) metric. Then we proved the conformally transformed vector fields of Finsler metric \( \tilde{F} \) related by \( F \) and also with conformal factor related to each other (see Proposition 3.3, 4.2).

2. Preliminaries of conformal vector fields

Let \( F \) be a Finsler metric on a manifold \( M \) and \( V \) be a vector field on \( M \). Let \( \phi_t \) be the flow generated by \( V \). \( \phi_t : TM \to TM \) by \( \phi_t = (\phi_t(x), \phi_t(y)) \). \( V \) is said to be conformal if

\[
\phi_t^* \tilde{F} = e^{-2\sigma_t} F,
\]

where \( \sigma_t \) is a function \( M \) for every \( t \). Differentiating (2.1) by \( t \) at \( t = 0 \). We obtain

\[
X_v(F) = -2cF_t,
\]

where we define

\[
X_v = V^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial x^i} \frac{\partial V^j}{\partial y^j}, \quad c = \frac{d}{dt}|_{t=0} \sigma_t.
\]

In (2.2), the function \( c \) is called the conformal factor.

Remark 2.1. A vector field \( V \) is conformal satisfying (2.1) if and only if (2.2) holds for some scalar function \( c \). In this case, \( c \) and \( \sigma_t \) are related by

\[
\sigma_t = \int_0^t c(\phi_s) ds, \quad c = \frac{d}{dt}|_{t=0} \sigma_t.
\]

Remark 2.2. By (2.3) we easily see that \( \phi_t^* F = e^{-2\sigma_t} F \) for a scalar function \( c \) if and only if \( c \) is constant along every integral curve of \( V \).

Lemma 2.3 ([5]). Let \( \beta = b_l(x)y^i \) be a 1-form and \( V \) be a vector field on a Riemannian manifold \((M, \alpha)\) with \( \alpha = \sqrt{a_{ij}y^iy^j} \). Then we have

\[
X_v(\alpha^2) = 2V_{0i}, \quad X_v(\beta) = (V^l\frac{\partial b_l}{\partial x^i})y^i = (V^l b_{lj} + b^l V_{lj})y^i,
\]

where \( V_l = a_{ij}V^j \) and \( b^l = a^{lj}b_j \), and the covariant derivative is taken with respect to the Levi-Civita connection of \( \alpha \).

3. Conformal vector fields on special \((\alpha, \beta)\)-metric, \( F = \alpha + \frac{\beta^2}{\alpha} \)

In this, section, we study the conformally transformed vector field on Finsler space with special \((\alpha, \beta)\)-metric, \( F = \alpha + \frac{\beta^2}{\alpha} \). For this, first we prove the following lemma.

Lemma 3.1. A vector field \( V \) on a Finsler space with special \((\alpha, \beta)\) metric \( F = \alpha + \frac{\beta^2}{\alpha} \) is conformal with conformal factor \( c \) if and only if it satisfies

\[
X_v(F^2) = -4 \left( cF^2 + F^l \frac{\partial V^l}{\partial y^k} y^k \right),
\]

or

\[
V_{0i} = -2 \left( cF^2 + F^l \frac{\partial V^l}{\partial y^k} y^k \right),
\]

where

\[
F^1 = 2\alpha k_0 + m_0 r_{00} b^l.
\]
Proof. By Remark 2.1, it is sufficient to show that \( X_v(t^2) = 2V_{00} + t^1 \frac{\partial V}{\partial y^i} y^i \). Let us consider the spray coefficients \( G^i \) as

\[
G^i = G^i_\alpha + Py^i + Q^i,
\]

where

\[
P = \alpha^{-1} \theta (-2\alpha Q\lambda_0 + r_{00}),
\]

\[
Q^i = \alpha Q\lambda_0^i + \psi (-2\alpha Q\lambda_0 + r_{00}) b^i,
\]

\[
\theta = \frac{\phi' - s(\phi'' + \phi' \phi')}{2\phi ((\phi - s\phi') + (b^2 - s^2)\phi'')},
\]

\[
Q = \frac{\phi'}{\phi - s\phi'},
\]

\[
\psi = \frac{1}{2 \phi - s\phi'} + (b^2 - s^2)\phi''.
\]

By calculation we get,

\[
\theta = \frac{4s^3}{(2 + 2s^2)(1 - 3s^2 + 2b^2)},
\]

\[
Q = \frac{1 + s^2}{1 - s^2},
\]

\[
\psi = \frac{1}{1 - s^2 + 2b^2},
\]

\[
P = \frac{4s^3}{\alpha(1 - s^2)(2 + 2s^2)} \left\{ \frac{-2(\alpha + s^2)s - 0 + r_{00}(1 - s^2)}{(1 - 3s^2 + 2b^2)} \right\},
\]

\[
Q' = \alpha \left( \frac{1 + s^2}{1 - s^2} \right) s_0^i - \frac{2\alpha s_0 - r_{00}}{1 - s^2 + 2b^2} b^i.
\]

Let

\[
G^i = g^{im} G_m = \frac{1}{4} \left\{ [F^2]_{x^i y^i} Y^k - [F^2]_{x^i} \right\},
\]

\[
G^k = \frac{\partial G^k}{\partial y^i},
\]

\[
G^k_{ij} = \frac{\partial^2 G^k}{\partial y^i \partial y^j}.
\]

Then the spray coefficients of class of Finsler metric \( F = \alpha + \frac{\beta^2}{\alpha} \) is

\[
G^k = G^k - (Ls_0 + Mr_{00}) y^i + \alpha ks_0^i - Rs_0 r_{00} b^i,
\]

where

\[
L = \frac{s^3(2\alpha - 2s^2)}{(2\alpha + 2\alpha s^4)(1 - 3s^2 + 2b^2)},
\]

\[
M = \frac{4s^2(1 - s^2)}{2\alpha(1 + s^4)(1 - 3s^2 + 2b^2)},
\]

\[
K = \frac{1 + s^2}{1 - s^2}.
\]
\[ R = \frac{2\alpha}{1 - s^2 + 2b}. \]

From [6] by computation shows that

\[ X_\nu(F^2) = V^i(F^2)_{x^i} + 2\frac{\partial V^i}{\partial x^j} y^j y^i = V^i(F^2)_{x^i} + 2\frac{\partial V^i}{\partial x^j} y^j y^i - \frac{\partial y^i}{\partial x^j} V^j y^i, \]

(3.1)

\[ \frac{\partial V^i}{\partial x^j} y^j y^i = V_{0i0} + 2(\alpha k s_0^i - M s_0 r_{00} b^i) \frac{\partial V^i}{\partial y^k} y^i + 2V_i G^k, \]

(3.2)

\[ 4V_k G^k = 2V_i \frac{\partial y^i}{\partial x^j} y^k - V^i \left\{ \left( \frac{1}{2\alpha} + \beta^2 \right) \frac{\partial a_{ij}}{\partial x^i} y^i y^j + 4\beta(1 - 2\alpha^3 \beta^2) \frac{\partial b_i}{\partial y^j} y^i \right\}. \]

(3.3)

Substitute (3.2) and (3.3) in (3.1) then we have

\[ X_F(F^2) = 2V_{0i0} + 2(\alpha k s_0^i - M s_0 r_{00} b^i) \frac{\partial V^i}{\partial y^k} y^i. \]

(3.4)

Therefore, it desired the claim. \( \square \)

**Remark 3.2.** From (3.4), suppose to be \( \frac{\partial V^i}{\partial y^k} y^i = 0 \). Then Lemma 3.2 becomes as, if \( V \) be vector field and \( F \) be special \( (\alpha, \beta) \)-metric, then it is conformal with conformal factor \( c \) if and only if it satisfies \( X_\nu(F^2) = -4cF^2 \) or \( V_{0i0} = -2cF^2 \).

**Proposition 3.3.** Let \( (M, \tilde{F} = (\alpha + \frac{\beta^2}{\alpha}) \) be conformally transformation of special \( (\alpha, \beta) \) metric \( F(= \alpha + \frac{\beta^2}{\alpha}) \) related with \( \tilde{F} = e^\alpha F \) for a scalar function \( \alpha \). Then \( V \) is a conformal vector field of \( (M, F) \) if and only if \( V \) is a conformal vector field of \( (M, \tilde{F}) \). Further, their conformal factor \( c \) and \( \tilde{c} \) are related by, \( \tilde{c} = c - \frac{1}{4} V(\sigma) \).

**Proof.** Assume that \( V \) is conformal vector field of \( F \) with the conformal factor \( c \). Then by Remark 3.2 we have

\[ X_\nu(F^2) = -4cF^2. \]

Then by conformally related with \( \tilde{F} = e^\alpha F \), we have

\[ X_\nu(F^2) = X_\nu(e^\alpha F^2) + e^\alpha X_\nu(F^2), \]

\[ X_\nu(F^2) = (V(\sigma) - 4c) \tilde{F}^2. \]

(3.5)

Again by Remark 3.2 implies that \( V \) is also a conformal vector field of \( \tilde{F} \) and the conformal factor \( \tilde{c} \). Therefore, from (3.5) we get

\[ \tilde{c} = c - \frac{1}{4} V(\sigma). \]

(3.6)

\( \square \)

**Theorem 3.4.** Let \( F = \alpha + \frac{\beta^2}{\alpha} \) be Finsler metric and \( V = V_i(x) \frac{\partial}{\partial x^i} \) be a vector field. Then \( V \) is a conformal vector field of \( F \) with the conformal factor \( c \) if and only if it satisfies \( V_{i,j} + V_{j,i} = -4c a_{ij} \) and \( V b_{L,j} + b^l V_{j,i} = -2c b_i \), where \( c \) is a scalar function.

**Proof.** Since we says that \( V \) is a conformal vector field of \( F \) if and only if \( X_\nu(F^2) = -4cF^2 \). From [3] by computation shows that

\[ X_\nu(F^2) = (1 - s^4) X_\nu(\alpha^2) + 4\alpha s(1 + s^2) X_\nu(\beta). \]
Now, plugging (3.1), we see that

\[ X_v(F^2) = -4cF^2, \]

is written as

\[ V_{0,0} + \alpha \left( \frac{2s}{1 - s^2} \right) (V^i b_{j,i} + b^i V_{i,j}) y^j = \frac{4sc}{1 - s^2} \alpha^2, \tag{3.6} \]

where \(\cdot\) represents the covariant derivative with respect to \(\alpha\).

In order to simplify (3.6). We choose a special coordinate system \((s, y^a)\) at a fixed point on a manifold as usually used. Fix an arbitrary point \(x \in M\) and take an orthogonal basis \(e_i\) at \(x\) such that

\[ \alpha = \sqrt{\sum_{i=1}^{n} (y^i)^2}, \quad \beta = by^i. \]

It follows that \(\beta = s\alpha\) such that

\[ y^i = \frac{s}{\sqrt{b^2 - s^2} \bar{\alpha}} \left( \bar{\alpha} = \sqrt{\sum_{a=2}^{n} (y^a)^2} \right). \]

Then if we change coordinate \((y^i)\) to \((s, y^a)\) we get

\[ \alpha = \frac{b}{\sqrt{b^2 - s^2} \bar{\alpha}}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2} \bar{\alpha}}. \]

Let,

\[ \nabla_{0,0} = V_{a,b} y^a y^b, \quad \nabla_{1,0} = V_{1,a} y^a, \]

\[ \nabla_{0,1} = V_{a,1} y^a, \quad \nabla_{0,1} = b_{a,i} y^a. \]

Note, that under the coordinate \((s, y^a)\), we have \(b_1 = b\), \(\nabla_0 = 0\), but generally \(\nabla_{0,1} \neq 0\).

Under these coordinate, Equation (3.6) is equivalent to

\[ 0 = b \left( b \nabla_{1,0} + V^i \nabla_{0,1} \right) \frac{2s}{1 - s^2} + (\nabla_{1,0} + \nabla_{0,1}) s, \tag{3.7} \]

\[ 0 = b \left[ b(2bc + V^i b_{j,i} + V_{j,1} V_{1,i}) \frac{2s^2}{1 - s^2} + 2b^2c + V_{j,1} s^2 \right] \alpha^2 + (b^2 - s^2) \nabla_{0,0}. \tag{3.8} \]

For (3.7), we will prove

\[ b \nabla_{1,0} + V^i \nabla_{0,1} = 0, \quad \nabla_{1,0} + \nabla_{0,1} = 0. \tag{3.9} \]

If \(b \nabla_{1,0} + V^i \nabla_{0,1} \neq 0\) at a point, then we prove \(Q = \frac{2s}{1 - s^2} = ks\), for some constant \(k\). Solving \(Q = ks\) with \(\phi(0) = 1\), we get \(\phi(s) = \sqrt{1 + ks}\) and hence \(F\) is Riemannian. So we must have the first equation in (3.9).

Again by (3.8), we have the second equation in (3.9). For (3.8) we will prove

\[ \nabla_{0,0} = -2c \bar{\alpha}^2, \quad V^i b_{j,i} + b V_{j,1} = -2bc, \quad V_{j,1} = -2cF. \tag{3.10} \]

Putting \(s = 0\) in (3.8) we have

\[ \nabla_0 = 0, = -2c \bar{\alpha}^2. \]

By this, (3.8) is reduced to

\[ b \left( 2bc + V^i b_{j,i} + V_{j,1} V_{1,i} \right) \frac{2s}{1 - s^2} + (V_{j,1} + 2c) s = 0. \]

Similarly, repeating this procedure of (3.7) we obtain second and third values of (3.10). Therefore, it follows from (3.9) and (3.10) then we get the claim. \(\square\)
4. Conformal vector fields on special \((\alpha, \beta)\)-metric, \(F = \alpha + \beta + \frac{\beta^2}{\alpha}\)

In this section, we study the conformally transformed vector field on Finsler space with special \((\alpha, \beta)\)-metric. Now we prove the following lemma.

**Lemma 4.1.** A vector field \(V\) on a Finsler space with special \((\alpha, \beta)\) metric is conformal with conformal factor \(c\), if and only if it satisfies \(X_{\nu}(F^2) = -4cF^2\) or \(V_{00} = -2cF^2\).

**Proof.** By Remark 2.1, it is sufficient to show \(X_{\nu}(F^2) = 2V_{00}\). Let

\[
G^i = g^{im}G_m = \frac{1}{4}\{[F^2]_{x^i}y^kV^k - [F^2]_{x^i}\}, \quad G^k = \frac{\partial G^k}{\partial y^l}G^l + \frac{\partial^2 G^k}{\partial y^l\partial y^r}.
\]

In the view of section third the spray coefficients \(G^i\) is on the Finsler metric \(F = \alpha + \beta + \frac{\beta^2}{\alpha}\), we get,

\[
G^k = G^k + A_1A_2y^i + A_3s_0^i - (A_4s_0 + r_{00})b^i,
\]

where

\[
A_1 = \frac{r_{00}(1 - s^2) - 2\alpha(1 + 2s}s_0}{2 - 2s - 4b^2 - 4s^2b^2 - 4s^2 + 6s^4},
\]

\[
A_2 = \frac{1 - 3s^2 - 4s^3}{\alpha(1 - s^2)},
\]

\[
A_3 = \frac{\alpha(1 + 2s)}{1 - s^2},
\]

\[
A_4 = \frac{2\alpha(1 + 2s)}{(1 - s^2)(1 - 3s^2 - 2b^2)s_0}.
\]

We know that

\[
X_{\nu}(F^2) = V^i(F^2)x^i + 2\frac{\partial V^i}{\partial x^j}y^jy^l = V^i(F^2)x^i + 2\frac{\partial V^i}{\partial x^j}y^jy^l - \frac{\partial y^i}{\partial x^j}V^k y^j,
\]

\[
\frac{\partial V^i}{\partial x^j}y^jy^l = V_{00} + 2(A_3s_0^i - A_4s_0r_{00}b^i)\frac{\partial V^i}{\partial y^k}y^j + 2V_k G^k,
\]

\[
4V_k G^k = 2V^i \frac{\partial y^i}{\partial x^k}y^k - \frac{\partial a_{ij}}{\partial x^l}y^l \left( \frac{1}{2\alpha} + \frac{\beta^2}{2\alpha^3}\right) + \frac{\partial b^i}{\partial x^l}y^l \left( 6\beta - 4\beta^3 + 2\alpha + \frac{6\beta^2}{\alpha}\right) + \frac{\partial c^i}{\partial x^l}y^l \left( 2\beta - \frac{2\beta^3}{\alpha^2}\right).
\]

Substitute (4.2) and (4.3) in (4.1) then we have

\[
X_{\nu}(F^2) = 2V_{00} + 2(A_3s_0^i - A_4s_0r_{00}b^i)\frac{\partial V^i}{\partial y^k}y^j.
\]

Therefore, it is desired. \(\Box\)

**Proposition 4.2.** Let \((M, \tilde{F} = (\tilde{\alpha} + \tilde{\beta} + \frac{\tilde{\beta}^2}{\tilde{\alpha}}))\) be conformally transformation of special \((\alpha, \beta)\) metric \(F = \alpha + \beta + \frac{\beta^2}{\alpha}\) related with \(\tilde{F} = e^{\sigma} F\) for a scalar function. Then \(V\) is a conformal vector field of \((M, \tilde{F})\) if and only if \(V\) is a conformal vector field of \((M, F)\). Further, their conformal factor \(c\) and \(\tilde{c}\) are related by, \(\tilde{c} = c - \frac{1}{4}V(\sigma)\).
Proof. Assume that $V$ is conformal vector field of $F$ with the conformal factor $c$. Then by Remark 3.2 we have

$$X_v(F^2) = -4cF^2.$$

Then by conformally related with $\tilde{F} = e^{\sigma}F$, we have

$$X_v(F^2) = X_v(e^\sigma) F^2 + e^\sigma X_v(F^2),$$

$$X_v(F^2) = (V(\sigma) - 4c) F^2. \quad (4.4)$$

Again, by Remark 3.2 implies that $V$ is also a conformal vector field of $\tilde{F}$ and the conformal factor $\tilde{c}$. Therefore, from (4.4) we get

$$\tilde{c} = c - \frac{1}{4} V(\sigma).$$

Theorem 4.3. Let $F = \alpha + \beta + \frac{\beta^2}{\alpha}$ be Finsler metric and $V = V^i(x) \frac{\partial}{\partial x^i}$ be a vector field. Then $V$ is a conformal vector field of $F$ with the conformal factor $c$, if and only if it satisfies

$$V_{ij} + V_{j;i} = -4ca_{ij},$$

and

$$V^l b_{ij} + b^l V_{j;i} = -2cb_i,$$

where $c$ is a scalar function.

Proof. Since we say that $V$ is a conformal vector field of $F$ if and only if $X_v(F^2) = -4cF^2$.

From [3] by computation shows that

$$X_v(F^2) = (1 - s^4)X_v(\alpha^2) + 4\alpha s(1 + s^2)X_v(\beta).$$

Now, plugging Equation (2.4), we see that

$$X_v(F^2) = -4cF^2,$$

is written as

$$V_{0,0} + \alpha \left( \frac{2s}{1 - s^2} \right) (V^i b_{j;i} + b^i V_{j;i}) y^l = \frac{4sc}{1 - s^2} \alpha^2, \quad (4.5)$$

where “;” represents the covariant derivative with respect to $\alpha$.

In order to simplify (4.5), we choose a special coordinate system $(s, y^a)$ at a fixed point on a manifold as usually used. Fix an arbitrary point $x \in M$ and take an orthogonal basis $e_i$ at $x$ such that

$$\alpha = \sqrt{\sum_{i=1}^{n} (y_i)^2}, \quad \beta = by^i.$$

It follows that $\beta = s\alpha$ such that

$$y^l = \frac{s}{\sqrt{b^2 - s^2}} \tilde{\alpha} \left( \tilde{\alpha} = \sqrt{\sum_{a=2}^{n} (y^a)^2} \right).$$
Then if we change coordinate \((y^1)\) to \((s, y^a)\) we get
\[
\alpha = \frac{b}{\sqrt{b^2 - s^2}} \alpha, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \beta.
\]

Let,
\[
\nabla_{0;0} = V_{a;b} y^a y^b, \quad \nabla_{1;0} = V_{1;a} y^a,
\]
\[
\nabla_{0;1} = V_{a;1} y^a, \quad b_{0;1} = b_{a;1} y^a.
\]

Note that under the coordinate \((s, y^a)\), we have \(b_1 = b, \quad b_0 = 0\), but generally \(b_{0;1} \neq 0\). Under these coordinate, (4.5) is equivalent to
\[
0 = b (b \nabla_{1;0} + V^i b_{0;1}) \frac{2s}{1 - s^2} + (\nabla_{1;0} + \nabla_{0;1}) s,
\]
\[
0 = b \left( b(2bc + V^i b_{1|1} + b V_{1|1}) \frac{2s^2}{1 - s^2} + 2b^2 c + V_{1|1} s^2 \right) \alpha^2 + (b^2 - s^2) \nabla_{0;0}.
\]

For (4.6), we will prove
\[
b \nabla_{1;0} + V^i b_{0;1} = 0, \quad \nabla_{1;0} + \nabla_{0;1} = 0.
\]

If \(b \nabla_{1;0} + V^i b_{0;1} \neq 0\) at a point, then we prove \(Q = \frac{2s}{1 - s^2} = ks\) for some constant \(k\). Solving \(Q = ks\) with \(\phi(0) = 1\). We get \(\phi(s) = \sqrt{T + ks}\) and hence \(F\) is Riemannian. So we must have the first equation in (4.8).

Again by (4.7), we have the second equation in (4.8).

For (4.7) we will prove
\[
\nabla_{0;0} = -2c \tilde{\alpha}^2, \quad V^i b_{1|1} + b V_{1|1} = -2bc, \quad V_{1|1} = -2c \tilde{F}.
\]

Putting \(s = 0\) in (4.7) we have
\[
\nabla_{0;0} = -2c \tilde{\alpha}^2.
\]

By this, (4.7) is reduced to
\[
b \left( 2bc + V^i b_{1|1} + b V_{1|1} \right) \frac{2s}{1 - s^2} + (V_{1|1} + 2c) s = 0.
\]

Similarly, repeating this procedure of (4.7) we obtain the second and third values of (4.9). Therefore, it follows from (4.8) and (4.9) then we get the claim.

\[\square\]

\textbf{Remark 4.4.} Let \(F = \alpha \phi(\frac{\beta}{\tilde{\alpha}})\) be an \((\alpha, \beta)\)-metric with \(\phi(0) = 1\). For this, holds for all class of \((\alpha, \beta)\)-metrics.

\textbf{References}


