Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 20 (2020), 313–324
Research Article

Online: ISSN 2008-949X
Journal of Mathematics and Computer Science
Journal Homepage: www.isr-publications.com/jmcs

Exponential B-spline collocation method for solving the generalized Newell-Whitehead-Segel equation

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Abstract

In this work, we present a collocation method based on exponential basis spline functions for solving generalized Newell-Whitehead-Segel equation. The time derivative is discretized by finite difference scheme and the exponential basis spline functions are employed to interpolate spatial derivatives. The convergence and stability of the proposed algorithm are established. Numerical results demonstrate the accuracy of the proposed method.

Keywords: Non-linear generalized Newell-Whitehead-Segel equation, exponential B-spline collocation method, convergence, stability.

2010 MSC: 65M70, 65Z05, 65N12, 65D05, 65D07.

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1. Introduction

The generalized Newell-Whitehead-Segel (gNWS) equation is extensively used in fluid mechanics and its mathematical description can be stated as follows:

$$u_t = mu_{xx} + au + \psi(x,t,u,u_x,bu^q), \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant T, \tag{1.1}$$

where u is a function of x and t and m, a, b, are real constants with $m > 0$ and $q \in Z^+$. The initial and the boundary conditions are represented in equations (1.2), (1.3), and (1.4), respectively.

$$u(x,0) = f(x), \tag{1.2}$$
$$u(0,t) = g_1(t),\ u(1,t) = g_2(t), \tag{1.3}$$
$$u_x(0,t) = h_1(t),\ u_x(1,t) = h_2(t), \tag{1.4}$$

where f, g_i's, h_i's are known functions. By substituting $a = -1, b = 1 = m, q = 2, \psi = bu^q$ into gNWS equation (1.1), we obtain Fisher's equation [3] while the values $a = 1 = m, b = -1, q = 2, \psi = bu^q + uu_x$ convert gNWS equation into Burgers-Fisher equation. Huxley equation [3] can be achieved from gNWS

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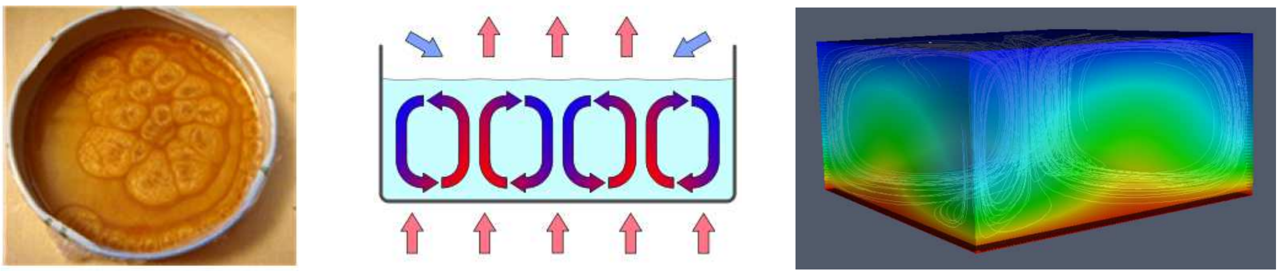
doi: 10.22436/jmcs.020.04.05

Received: 2019-10-16 Revised: 2019-12-30 Accepted: 2020-01-16

equation by setting $m = 1, a = -\beta, b = 1 + \beta, q = 2, \psi = bu^q - u^3$ and gNWS eqation takes the form of Burger Huxley equation when $m = 1, a = -\beta, b = 1 + \beta, q = 2, \psi = bu^q + uu_x - u^3$ are taken. The Newell-Whitehead-Segel (NWS) equation [14] is the special case of gNWS Eq (1.1) when $\psi = -bu^2$. The gNWS equation becomes Allen-Cahn [18] equation when $b = -a = 4, m = 1, q = 3, \psi = bu^q$ are substituted in it while gNWS equation takes the form of Nagumo reaction diffusion equation when $m = 1, a = \alpha, b = -1, q = 2, \psi = bu^q + u^3 - u^2$.

The NWS equation explains the dynamic attitude of dual blend fluid nearby bifurcation point of the Rayleigh-Benard convection of a binary fluid mixture. The layer of fluid heated from below advances a systematic design of convection cells named as Benard cells which takes place in a horizontal plane. A spontaneous change in severely heated fluid occurs which leads the hot fluid upwards while cold downwards. Because of analytical as well as experimental approachability, the phenomenon of this convection is extensively studied. The judiciously inspected examples of the self-organized nonlinear systems are the mentioned convection patterns [7]. These cells are produced due to gravity and Buoyancy forces. The primary movement is the upwelling of the warmer liquid from the heated bottom layer [16]. Two types of these shapes or patterns may be analyzed. Firstly, the liquid streamline from cylinders which might be bent or the produced spirals named as roll (stripe) pattern. Secondly, the fluid movement is separated into honey comb cells known as hexagonal shape. A part of liquid moves downwards, upwards in the middle of each cell and on the boundary between the cells, respectively while the remaining travels oppositely. For entirely different physical phenomena, the production of similar patterns can be observed, e.g., the systems with propagation of laser beams [8] in a nonlinear medium with reacting and diffusing species produce hexagonal pattern while visual cortex, human fingerprints and zebra's skin follow stripe patterns. The Rayleigh-Benard convection phenomenon and the convection cells in a gravity field are depicted in Figure 1.



(a) Benard cells. (b) Convection cells in gravity field. (c) Simulation of Rayleigh Benard convection in 3D.

Figure 1: Benard cells phenominon and convection cells in gravity field.

The NWS equation has been studied extensively in the last decades. Kheiri et al. [11] developed Homotopy analysis and Homotopy Pade methods for solving the modified Burgers-Korteweg-de Vries and the Newell-Whitehead equations. Ezzati and Shakibi [6] used the Adomian's Decomposition and multi-quadric qausi-interpolation techniques to obtain the solution of NWS equation. Macias-Diaz and Ramirez [12] computed numerical results for gNWS equation using finite difference algorithm. Aasaraai [1] used differential transformation method (DTM) with variable and constant coefficients to investigate the analytical solution of NWS equation. Nourazar et al. [13] developed Homotopy perturbation approach to investigate the exact solution of NWS equation. Recently, Zahra et al. [21] applied a collocation scheme based on cubic B-spline functions for the approximate solution of NWS equation using initial and boundary conditions.

Several numerical methods have been proposed for solving boundary value problems such as finite element, finite difference, spline interpolation, etc.. The B-spline method is one of the most efficient numerical methods due to its simplicity. Here we aim to use the exponential B-spline method (ExBSM) as it is capable to approximate the unknown function up to a certain smoothness. It has the potential to provide the approximation at non any point in the spatial domain with reasonable accuracy as compared to the typical finite difference approach. Moreover, it does not involve operators, procedure of linearization.

The main purpose of current study is to obtain the numerical solution of the proposed one dimensional nonlinear generalized Newell-Whitehead-Segel equation via exponential B-spline method. The time derivative is discretized by finite difference scheme while exponential B-spline functions interpolate spatial derivatives. The convergence and stability of the proposed method is established. The obtained numerical outcomes are show an excellent agreement with the true solutions. The computational results are compared with some other methods on the topic and it is observed that our proposed scheme preforms better in terms of accuracy and efficiency.

2. Exponential basis functions and temporal discretization

The time $[0, T]$ and spatial interval $[0, 1]$ are partitioned into equally divided intervals and taken as pairs (x_r, t_k). Here $x_r = a + rh, t_k = k\Delta t$ for $r = 0, 1, \dots, N, k = 0, 1, \dots, K$ where h and Δt represent spatial and time steps respectively. The exponential B-spline basis function (ExBSBF) can be stated as:

$$\eta_j(x) = \begin{cases} \beta_2\left((x_{r-2} - x) - \frac{1}{\sigma}(\sinh(\sigma(x_{r-2} - x)))\right), & x \in [x_{r-2}, x_{r-1}], \\ \alpha_1 + \beta_1(x_r - x) + \gamma e^{\sigma(x_r - x)} + \delta e^{-\sigma(x_r - x)}, & x \in [x_{r-1}, x_r], \\ \alpha_1 + \beta_1(x - x_r) + \gamma e^{\sigma(x - x_r)} + \delta e^{-\sigma(x - x_r)}, & x \in [x_r, x_{r+1}], \\ \beta_2\left((x - x_{r-2}) - \frac{1}{\sigma}(\sinh(\sigma(x - x_{r-2})))\right), & x \in [x_{r+1}, x_{r+2}], \\ 0, & \text{else}, \end{cases} \tag{2.1}$$

where, $\alpha_1 = \frac{\sigma hc}{\sigma hc - s}, \beta_1 = \frac{\sigma\left(c(c-1)+s^2\right)}{2(\sigma hc - s)(1-c)}, \beta_2 = \frac{\sigma}{2(\sigma hc - s)}, \gamma = \frac{e^{-\sigma h}(1-c)+s(e^{-\sigma h}-1)}{4(\sigma hc - s)(1-c)}, \quad \delta = \frac{(c-1)e^{\sigma h}+s(e^{\sigma h}-1)}{4(\sigma hc - s)(1-c)}$, with $c = \cosh(\sigma h), s = \sinh(\sigma h)$ and $\sigma > 0$ is a free parameter.

3. Description of exponential B-spline collocation approach

If $U(x, t)$ represents the true solution and $u(x, t)$ is the approximate solution, then using exponential B-spline functions we let [10, 19, 20]

$$u_j^k(x, t) = \sum_{j=-1}^{N+1} \delta_j^k(t)\, \eta_j(x), \tag{3.1}$$

where δ_j^k are to be determined. Using equations (2.1) and (3.1), we can evaluate the values of u and its first two derivatives at x_j as follows:

$$\begin{cases} u_j^k = l_1\delta_{j-1}^k + l_2\delta_j^k + l_1\delta_{j+1}^k, \\ (u_x)_j^k = l_3\delta_{j+1}^k + 0\,\delta_j^k + (-l_3)\delta_{j-1}^k, \\ (u_{xx})_j^k = l_4\delta_{j-1}^k + (-2l_4)\delta_j^k + l_4\delta_{j+1}^k, \end{cases} \tag{3.2}$$

where, $l_1 = \frac{s - \sigma h}{2(\sigma hc - s)}, l_2 = 1, l_3 = \frac{\sigma(1-c)}{2(\sigma hc - s)}, l_4 = \frac{s\sigma^2}{2(\sigma hc - s)}$. Using Eqs (1.3), (3.1), and (3.2), we can write

$$u(x_0, t_{k+1}) = l_1\delta_{-1}^{k+1}(t) + l_2\delta_0^{k+1}(t) + l_1\delta_1^{k+1}(t) = g_1(t_{k+1}), \tag{3.3}$$

$$u(x_N, t_{k+1}) = l_1\delta_{N-1}^{k+1}(t) + l_2\delta_N^{k+1}(t) + l_1\delta_{N+1}^{k+1}(t) = g_2(t_{k+1}). \tag{3.4}$$

Now implementing finite difference scheme to equation (1.1), the following expression can be obtained

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} = \frac{m(u_{xx})_j^{k+1} + (mu_{xx})_j^k}{2} + \frac{a(u)_j^{k+1} + (au)_j^k}{2} + \frac{(\psi)_j^{k+1} + (\psi)_j^k}{2}. \tag{3.5}$$

Separating terms of k^{th} and $(k+1)^{th}$ time levels, equation (3.5) takes the form

$$d_1 u_j^{k+1} - d_2 (u_{xx})_j^{k+1} - \Delta t(\psi)_j^{k+1} = r(x_j), \tag{3.6}$$

where, $d_1 = 2 - a\Delta t, d_2 = m\Delta t, d_3 = 2 + a\Delta t, r(x_j) = d_3 u_j^k + d_2 (u_{xx})_j^k + \Delta t(\psi)_j^k$. Using equation (3.2) in equation (3.6) and some simplification yields

$$p_1 \delta_{j-1}^{k+1} + p_2 \delta_j^{k+1} + p_1 \delta_{j+1}^{k+1} - \Delta t(\psi(u))_j^{k+1} = r(x_j), \quad j = 0, 1, \dots, N, \tag{3.7}$$

where, $p_1 = d_1 l_1 - d_2 l_4, p_2 = d_1 l_2 + 2d_2 l_4$. Eliminate the unknowns δ_{-1}^{k+1} and δ_{N+1}^{k+1} with the help of equations (3.3), (3.4), and (3.7), a system of order $(N+1) \times (N+1)$ can be generated as follows:

$$A\delta_j^{k+1} - \Delta t B = C, \tag{3.8}$$

where

$$A = \begin{pmatrix} p_1^* & 0 & 0 & & & & \\ p_1 & p_2 & p_1 & & & 0 & \\ & \ddots & \ddots & \ddots & & & \\ & 0 & & & p_1 & p_2 & p_1 \\ & & & & 0 & 0 & p_1^* \end{pmatrix}, \delta_j^{k+1} = \begin{bmatrix} \delta_0^{k+1} \\ \delta_1^{k+1} \\ \vdots \\ \delta_N^{k+1} \end{bmatrix}, B = \begin{bmatrix} \psi_0^{k+1} \\ \psi_1^{k+1} \\ \vdots \\ \psi_{N-1}^{k+1} \\ \psi_N^{k+1} \end{bmatrix}, C = \begin{bmatrix} r_0^{*k} \\ r_1^k \\ r_2^k \\ \vdots \\ r_{N-1}^k \\ r_N^{*k} \end{bmatrix},$$

where, $r_0^* = p_1 g_1(t) - l_1 r_0, r_N^* = p_1 g_2(t) - l_1 r_N, p_1^* = l_2 p_1 - l_1 p_2$. The above recurrence relation is solved after obtaining initial and first order approximation separately. The initial vector u^0 is computed from the initial condition (1.2). For the next approximation u^1, we use Taylor series expansion at $t = t_0 + \Delta t$ as follows:

$$u^1 = u^0 + \Delta t u_t^0 + \frac{(\Delta t)^2}{2!} u_{tt}^0 + O(\Delta t)^3, \tag{3.9}$$

where, $u_t^0 = (mu_{xx} + au + \psi)^0, u_{tt}^0 = (mu_{xxt} + au_t + (\psi)_t)^0$, and the values of u^0 and its derivatives are computed by initial condition. By putting these values in equation (3.9), we achieve first order approximation as below:

$$u^1 = u^0 + \Delta t[mu_{xx} + au + \psi]^0 + \frac{(\Delta t)^2}{2!}[mu_{xxt} + au_t + (\psi)_t]^0 + O(\Delta t)^3. \tag{3.10}$$

Theorem 3.1. *The rate of convergence of the presented scheme to discretize equation (1.1) is one in time direction.*

Proof. Suppose u^k be the spline approximation for exact solution U^k at time $t = t_k$ and local truncation error of equation (3.6) is $e_k = u^k - U^k$, we have [4]

$$e_{n+1} \leqslant \mu_k (\Delta t)^2, \quad k \geqslant 2.$$

By utilizing equation (3.10) for $k = 1$, we obtain

$$e_1 \leqslant \mu_1 (\Delta t)^3.$$

Choosing $\mu = \max\{\mu_1, \mu_2, \dots, \mu_n\}$ and taking global error $E_{n+1} = \sum_{k=1}^n e_k$ at $(n+1)^{th}$ time level we may obtain the following expression:

$$|E_{n+1}| = |\sum_{k=1}^n e_k| \leqslant \sum_{k=1}^n |e_k| \leqslant \mu_1 (\Delta t)^3 + \sum_{k=2}^n \mu_k (\Delta t)^2 \leqslant n\mu(\Delta t)^2 \leqslant n\mu(T/n)\Delta t = C\Delta t,$$

where $\Delta t \leqslant (T/n)$ and $C = \mu T$ which implies first order convergence in time direction. □

4. Convergence of the method

Suppose $u_j^k(x,t)=\sum_{j=-1}^{N+1}\delta_j(t)\eta_j(x)$ be the exponential B-spline approximation to the exact solution $U(x,t)$. Due to computational round off error let $S^*(x)=\sum_{j=-1}^{N+1}\delta_j^*(t)\eta_j(x)$ be the computed spline approximation to $u(x)$. Therefore, we must estimate the errors $\|u(x)-S^*(x)\|_\infty$ and $\|U(x)-S^*(x)\|_\infty$ separately to estimate the error $\|u(x)-U(x)\|_\infty$. Putting $S^*(x)$ into equation (3.8), we obtain

$$A\delta^*-\Delta t B^*=C^*. \tag{4.1}$$

Subtracting equation (3.10) and equation (4.1), we have

$$A(\delta^*-\delta)-\Delta t(B^*-B)=C^*-C. \tag{4.2}$$

Theorem 4.1. *Suppose* $f(x)\in C^4[a,b]$ *and* $|f^4(x)|\leqslant\kappa$, $\forall x$ *with* h *being step size of equally space partition of* $[a,b]$. *If* $S(x)$ *is the unique spline function that interpolates f(x) at the nodes then* $\exists$ *a constant* λ_j, *s.t.,*

$$\|f^j-S^j\|_\infty\leqslant\lambda_j\kappa h^{4-j}, j=0,1,2,3.$$

Proof. For proof see [5, 9]. □

Using triangular inequality and Theorem 4.1, equation (3.7) takes the following form $|r^*(x_j)-r(x_j)|$

$$\begin{aligned}&=|\left(d_1S^*(x_j)-d_2S^*_{xx}(x_j)-\Delta t\psi(S^*(x_j))\right)-\left(d_1u_j^{k+1}-d_2(u_{xx})_j^{k+1}-\Delta t(\psi)_j^{k+1}\right)|\\&\leqslant|d_1\|S^*(x_j)-U(x_j)|+d_2|S^*_{xx}(x_j)-U_{xx}(x_j)|+\Delta t|\psi(x_j,S^*(x_j))-\psi(x_j,U(x_j))|\\&\leqslant d_1\kappa\lambda_0h^4+d_2\kappa\lambda_2h^2+\beta(|S^*(x_j)-U(x_j)|),\end{aligned}$$

where, $\|\psi'(z)\|\leqslant\beta, z\in R^3$ [15]. Finally, we are able to write

$$\|C^*-C\|\leqslant d_1\kappa\lambda_0h^4+d_2\kappa\lambda_2h^2+\beta(|S^*(x_j)-U(x_j)|).$$

Again using Theorem 4.1 yields

$$\|C^*-C\|\leqslant d_1\kappa\lambda_0h^4+d_2\kappa\lambda_2h^2+\beta\kappa\lambda_0h^4.$$

Also we can write

$$\|C^*-C\|\leqslant M_1h^2, \tag{4.3}$$

where $M_1=d_1\kappa\lambda_0h^2+d_2\kappa\lambda_2+\beta\kappa\lambda_0h^2$.

Now using Jacobian for nonlinear term on L.H.S. of equation (4.2), we obtain the following equation

$$\|B^*-B\|=\left(\frac{\partial\varphi(\xi_1)}{\partial u}J(\delta^*-\delta)\right), \tag{4.4}$$

where $\xi_1\in(0,1)$ and J is Jacobian given as

$$J=\begin{pmatrix}0&0&0&&&\\l_1&l_2&l_1&&0&\\&\ddots&\ddots&\ddots&&\\&&&l_1&l_2&l_1\\&0&&0&0&0\end{pmatrix}.$$

Substituting equation (4.4) into equation (4.2), the following expression is obtained

$$W(\delta^*-\delta)=(C^*-C), \tag{4.5}$$

where $W = A + \frac{\partial\varphi(\xi_1)}{\partial u}J$. Since matrix W is strictly diagonally dominant so non-singular, W^{-1} exists, hence equation (4.5) implies

$$(\delta^* - \delta) = W^{-1}(C^* - C).$$

Taking norm on both sides of the above equation and using equation (4.3), we obtain

$$\|\delta^* - \delta\|_\infty \leqslant \|W^{-1}\| M_1 h^2. \tag{4.6}$$

Suppose γ_j is the sum of j^{th} row of matrix $W = [\nu_{j,i}]$ for $i = 0, 1, \dots, N$, then we have

$$\begin{cases} \gamma_0 = l_2 p_1 - l_1 p_2, & \text{if} \quad j = 0, \\ \gamma_j = 2p_1 + p_2, & \text{if} \quad 1 \leqslant j \leqslant N-1, \\ \gamma_N = l_2 p_1 - l_1 p_2, & \text{if} \quad j = N. \end{cases}$$

From the properties of inverse of matrices, we can write

$$\sum_{j=0}^{N} \nu_{i,j}^{-1} \gamma_j = 1,$$

where $\nu_{i,j}^{-1}$ are the entries of W^{-1}.

$$\|W^{-1}\| = \sum_{j=0}^{N} \|\nu_{i,j}^{-1}\| \leqslant \frac{1}{\min(\gamma_j)} = \frac{1}{\upsilon_l} \leqslant \frac{1}{|\upsilon_l|}, \tag{4.7}$$

where l is some integer between 0 and N. Putting equation (4.7) in equation (4.6) implies the relation

$$\|\delta^* - \delta\|_\infty \leqslant \beta_2 h^2, \tag{4.8}$$

where $\beta_2 = \frac{M_1}{\upsilon_l}$.

Lemma 4.2. *The exponential B-Spline basis functions satisfy*

$$|\sum_{j=-1}^{N} \eta_j(x)| \leqslant 3, \quad 0 \leqslant x \leqslant 1. \tag{4.9}$$

Proof. We know that

$$|\sum_{j=-1}^{N+1} \eta_j(x)| \leqslant \sum_{j=-1}^{N+1} |\eta_j(x)|.$$

At any knot x_j we have

$$\sum_{j=-1}^{N+1} |\eta_j(x)| = |\eta_{j-1}(x)| + |\eta_j(x)| + |\eta_{j+1}(x)| = |l_1| + |l_2| + |l_1| \leqslant 2.$$

Also in each subinterval $x_{j-1} \leqslant x \leqslant x_j$

$$\eta_j(x_j) = l_2, \quad \eta_{j-1}(x_{j-1}) = l_2, \quad \eta_{j+1}(x_j) = l_1, \quad \eta_{j-2}(x_{j-1}) = l_1.$$

Hence for any $x_{j-1} \leqslant x \leqslant x_j$, it is verified that

$$\sum_{j=-1}^{N+1} |\eta_j(x)| = |\eta_{j-2}(x)| + |\eta_{j-1}(x)| + |\eta_j(x)| + |\eta_{j+1}(x)| \leqslant 3,$$

which completes the proof. □

Since

$$S^*(x) - u(x) = \sum_{j=-1}^{N+1} (\delta_j^* - \delta_j)\eta_j(x).$$

Applying norm on both sides of the above equation and using equations (4.8) and (4.9) enables us to write

$$||S^*(x) - u(x)|| = ||\sum_{j=-1}^{N+1} (\delta_j^* - \delta_j)\eta_j(x)|| = |\sum_{j=0}^{N} \eta_j(x)|||\delta_j^* - \delta_j|| \leqslant 3\beta_2 h^2.$$

Theorem 4.3. *If $u(x)$ is the exponential B-spline collocation approximation to exact solution $U(x)$ then the method has 2nd order convergence*

$$||u(x) - U(x)|| \leqslant \epsilon h^2,$$

where $\epsilon = \lambda_0 \kappa h^2 + 3\beta_2$ is a finite constant.

Proof. Using triangular inequality and Theorem 4.1, the following relation may be achieved

$$\begin{aligned} ||u(x) - U(x)|| &= ||u(x) - S^*(x) + S^*(x) - U(x)|| \\ &\leqslant ||S^*(x) - U(x)|| + ||u(x) - S^*(x)|| \leqslant \lambda_0 \kappa h^4 + 3\beta_2 h^2 = \epsilon h^2, \end{aligned}$$

where $\epsilon = \lambda_0 \kappa h^2 + 3\beta_2$. □

Now if $u(x, t)$ approximates the exact solution $U(x, t)$, then

$$||u(x, t) - U(x, t)|| \leqslant \omega(\Delta t + h^2),$$

where ω is constant. Hence our method is of first order convergence in time while second order in spatial direction.

5. Stability

For Von Neumann stability analysis, consider $\psi = bu^q$ in equation (1.1), the equation (3.7) takes the following form

$$q_1\delta_{j-1}^{k+1} + q_2\delta_j^{k+1} + q_1\delta_{j+1}^{k+1} = q_3\delta_{j-1}^{k} + q_4\delta_j^{k} + q_3\delta_{j+1}^{k}, \tag{5.1}$$

where, $d_4 = b\Delta t, q_1 = d_1 l_1 - d_2 l_4 - d4(l_1)^q$, $q_2 = d_1 l_2 + 2d_2 l_4 - d4(l_2)^q$, $q_3 = d_3 l_1 + d_2 l_4 + d_4(l_1)^q$, $q_4 = d_3 l_2 - 2d_2 l_4 + d_4(l_2)^q$. Put $\delta_j^k = \rho^k e^{i\eta jh}$ [17] in equation (5.1), we obtain

$$q_1\rho^{k+1}e^{i\eta(j-1)h} + q_2\rho^{k+1}e^{i\eta(j)h} + q_1\rho^{k+1}e^{i\eta(j+1)h} = q_3\rho^k e^{i\eta(j-1)h} + q_4\rho^k e^{i\eta(j)h} + q_3\rho^k e^{i\eta(j+1)h}.$$

Take $\phi = \eta h$ and dividing both sides by $\rho^k e^{i\eta(j)h}$ yields

$$\rho\left(q_1 e^{-i\phi} + q_2 + q_1 e^{i\phi}\right) = q_3 e^{-i\phi} + q_4 + q_3 e^{i\phi}.$$

Since $e^{i\phi} = \cos(\phi) + i\sin(\phi)$, so above equation yields

$$\rho = \frac{2q_3\cos(\phi) + p_4}{2q_1\cos(\phi) + q_2}. \tag{5.2}$$

Substituting equations (3.2), (3.6), and (5.1) into equation (5.2) and after simplification the following relation can be obtained as

$$\rho = -1 + \frac{2^{2+q}\left(\mu\cos(\phi) - \nu\right)}{2b\nu\Delta t(\frac{\mu}{\nu})^q\cos(\phi) + 2^q\left(\nu(-2 + (a+b)\Delta t) - s\nu_1 + (\nu_2 + s(-2 + a\Delta t + \nu_1))\cos(\phi)\right)}, \tag{5.3}$$

where $\nu = \sigma hc - s, \nu_1 = m\Delta t\sigma^2, \mu = s - h\sigma, \nu_2 = h(2 - a\Delta t)\sigma, c = \cosh(h\sigma), s = \sin(h\sigma)$. It is obvious from Eq. (5.3) that $|\rho| < 1$ which demonstrates the proposed method unconditionally stable.

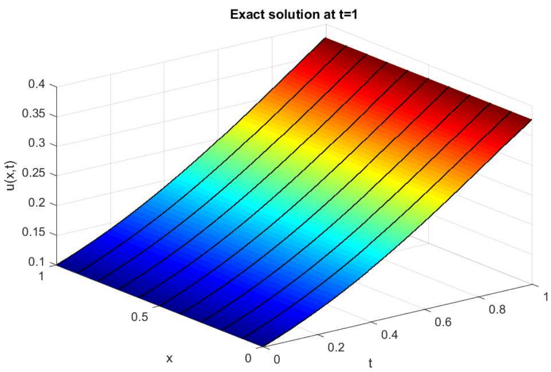
6. Numerical experiments

The numerical method presented in this paper is tested for getting solution of the generalized Newell-Whitehead-Segel equation for the numerical accuracy. Throughout the paper, we set free parameter $\sigma = 1.175$. All the computations are performed in MATLAB R2015b. In the first three examples, the initial and Dirichlet boundary conditions are obtained from the exact solution while in the last example Neumann boundary conditions are used. The following formulas are used to compute the numerical results.

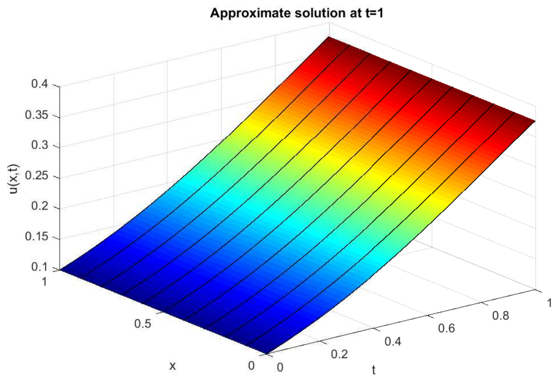
$$\text{Absolute error} = |U_j - u_j^k|, \quad L_\infty = \max_j |U_j - u_j|.$$

Example 6.1. Consider gNWS equation (1.1) with $a = 2, m = 1, q = 2, b = -3, \psi = bu^q$ for which the exact solution given in [1, 21] is

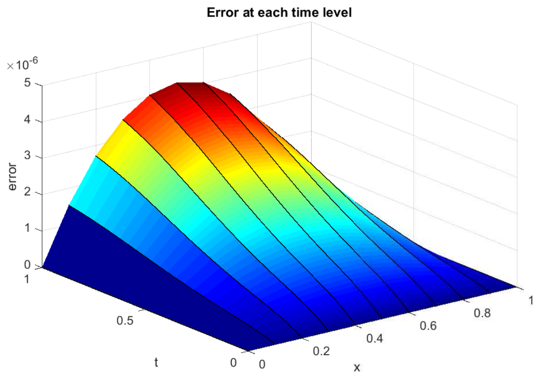
$$u(x,t) = \frac{-2\lambda e^{2t}}{-2 + 3\lambda\left(1 - e^{2t}\right)}.$$



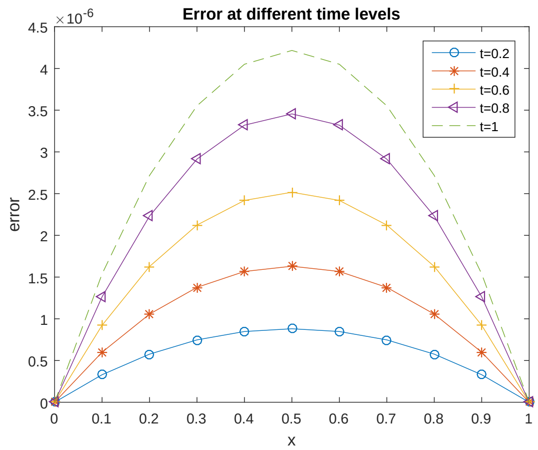
(a) Exact



(b) Approximate



(c) 3D error



(d) 2D error

Figure 2: Solutions and Error graphs for $t \in [0, 1]$ of Example 6.1.

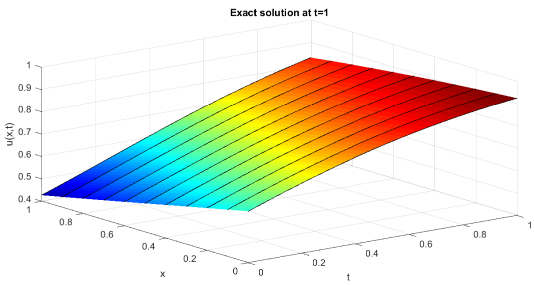
The absolute maximum errors of Example 6.1 are provided in Table 1 at different time levels and knots. The numerical results of ExBSM are compared with the existing methods named as Uniform cubic B-spline (UCBS) [21], Trigonometric cubic B-spline (TCBS) [21], Extended cubic B-spline (ECBS) [21]. Figure 2 exhibits the solutions and errors graphically. It is obvious that the proposed method is more reliable and efficient as compared to others.

Table 1: Absolute maximum errors at $\lambda = 0.1$ of Example 6.1.

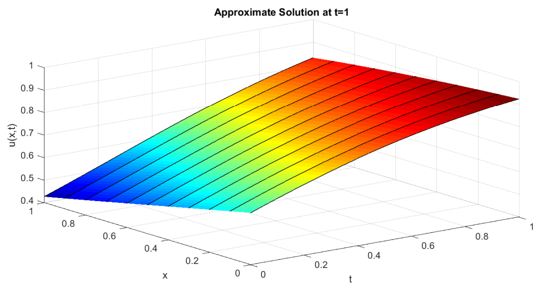
x	t	ExBSM	UCBS[21]	TCBS[21]	ECBS[21]
0.2	0.2	5.760E-07	8.323E-04	8.295E-04	6.068E-04
0.6		8.471E-07	1.226E-03	1.222E-03	9.013E-04
0.8		5.760E-07	8.323E-04	8.295E-04	6.068E-04
0.2	1.0	2.714E-06	1.991E-05	3.239E-05	6.339E-05
0.6		4.046E-06	4.932E-06	2.342E-05	8.366E-05
0.8		2.714E-06	1.991E-05	3.239E-05	6.339E-05

Example 6.2. Consider gNWS equation (1.1) with $a = 1 = m, q = 4, b = -1, \psi = bu^q$ and the exact solution given in [13, 21] is

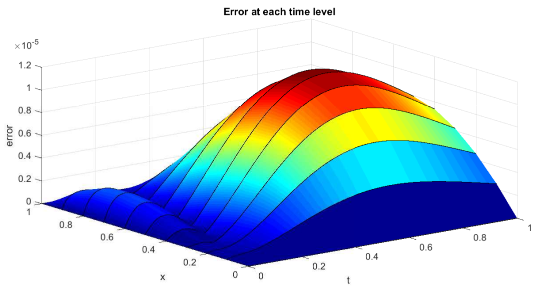
$$u(x,t) = (\frac{1}{2}\tanh\left(\frac{-2}{2\sqrt{10}}(x - \frac{7t}{\sqrt{10}})\right) + \frac{1}{2})^{\frac{-2}{3}}.$$



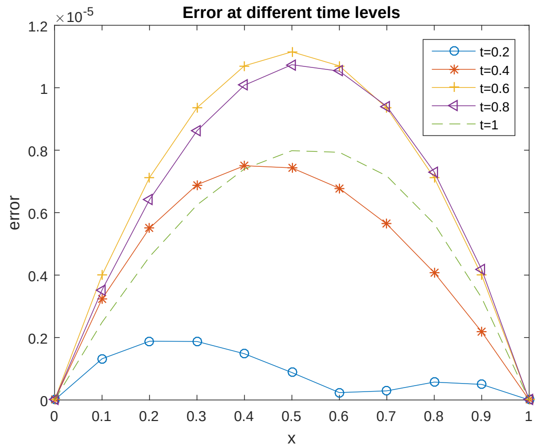
(a) Exact



(b) Approximate



(c) 3D error



(d) 2D error

Figure 3: Solutions and Error graphs for $t \in [0, 1]$ of Example 6.2.

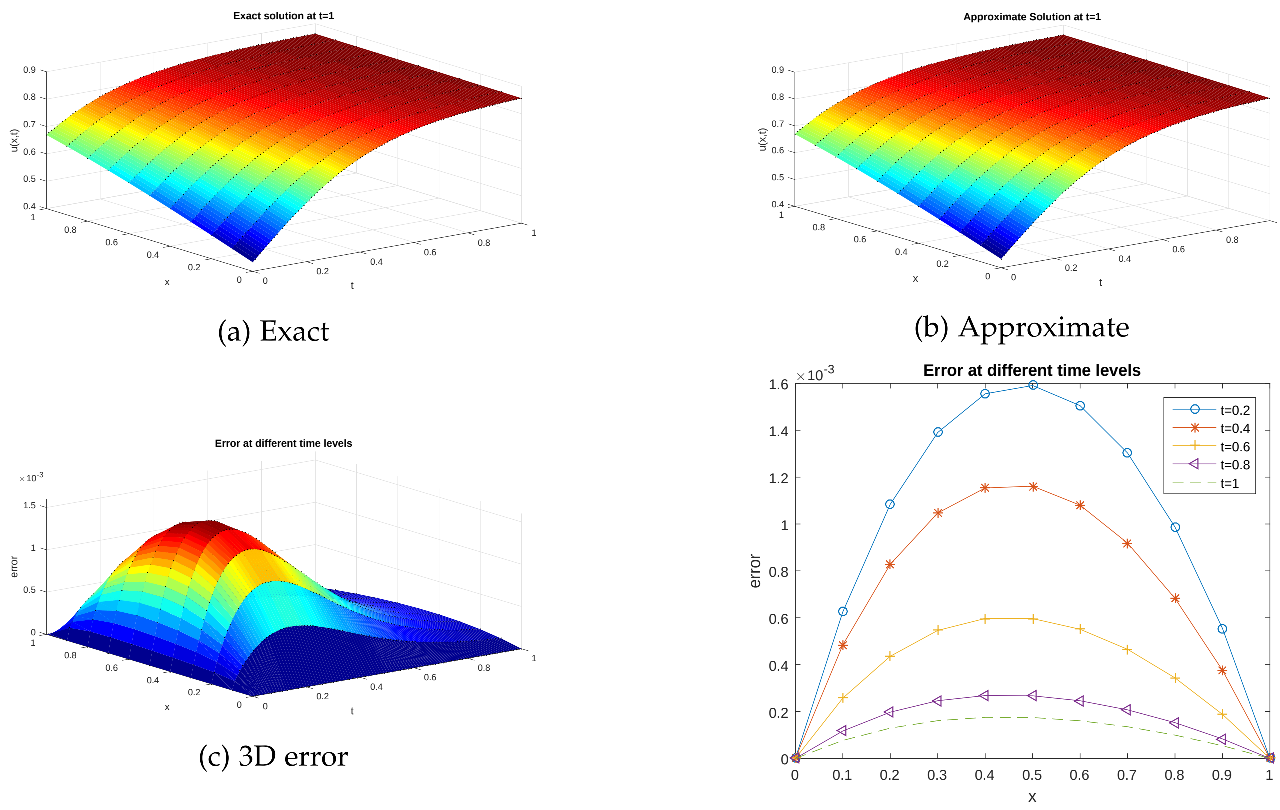
The error terms of Example 6.2 at different time levels and knots are provided in Table 2. The numerical outputs of ExBSM are compared with the methods (UCBS, TCBS, ECBS). The solutions and error terms are displayed in Figure 3. It can be seen that proposed scheme is more reliable and efficient.

Example 6.3. Consider gNWS equation (1.1) with $a = 3, b = -4, m = 1, q = 3, \psi = bu^q$ with the true solution given in [13, 21] as

$$u(x,t) = \frac{e^{x\sqrt{6}}\sqrt{(\frac{3}{4})}}{e^{x\sqrt{6}} + e^{\left(\frac{x\sqrt{6}}{2} - \frac{9t}{2}\right)}}.$$

Table 2: Absolute maximum errors for $x, t \in [0, 1]$ of Example 6.2.

x	t	ExBSM	UCBS[21]	TCBS[21]	ECBS[21]
0.2	0.2	1.881E-06	3.800E-04	3.951E-04	9.673E-04
0.6		1.985E-06	2.890E-04	3.111E-04	8.863E-04
0.8		2.122E-06	1.120E-04	1.270E-04	4.224E-04
0.2	1.0	7.143E-06	1.080E-03	1.103E-03	2.251E-03
0.6		1.107E-05	1.703E-03	1.734E-03	3.558E-03
0.8		7.534E-06	1.172E-03	1.192E-03	2.441E-03



(a) Exact

(b) Approximate

(c) 3D error

(d) 2D error

Figure 4: Solutions and Error graphs for $t \in [0, 1]$ of Example 6.3.

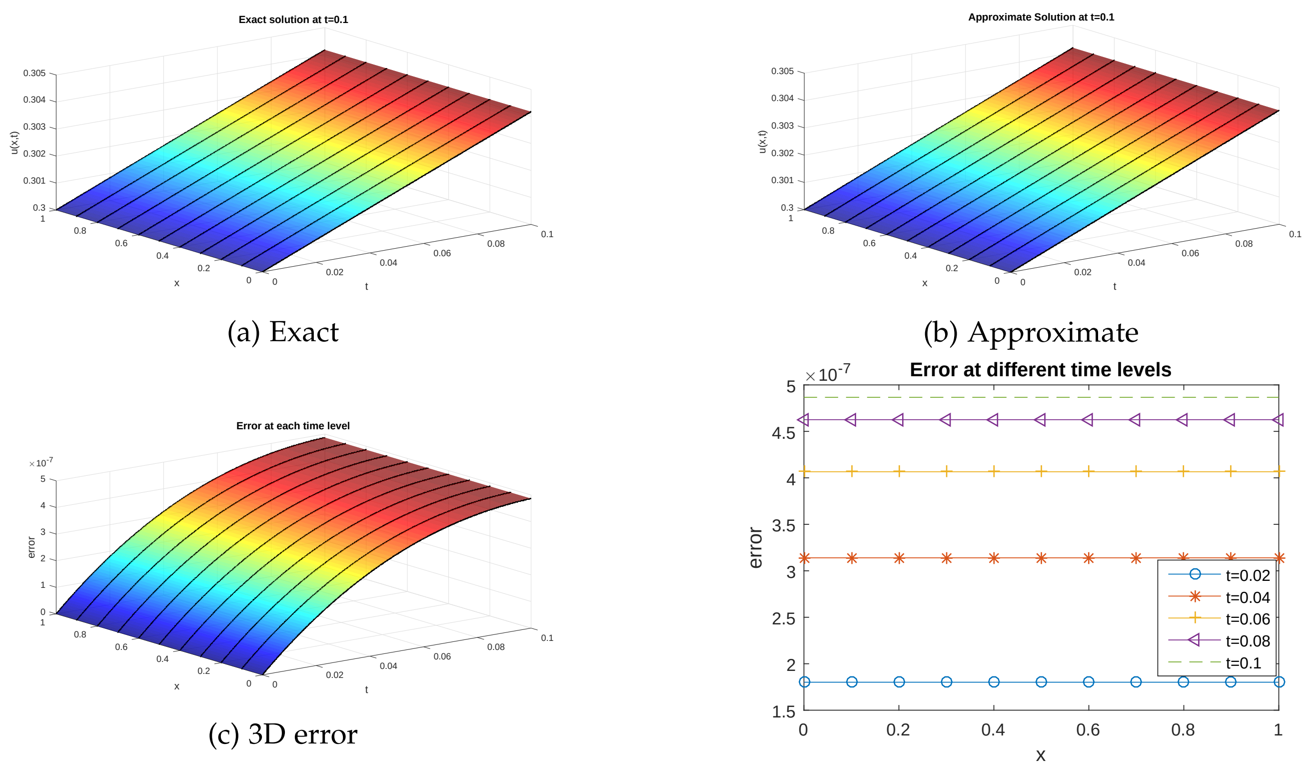
Table 3: Absolute maximum errors of Example 6.3.

x	t	ExBSM	UCBS[21]	TCBS[21]	ECBS[21]
0.2	0.2	1.100E-03	6.129E-02	6.134E-02	5.166E-02
0.6		1.500E-03	8.030E-02	8.036E-02	6.461E-02
0.8		9.883E-04	4.857E-02	4.862E-02	3.721E-02
0.2	1.0	1.100E-03	1.560E-02	1.563E-02	1.518E-03
0.6		1.500E-03	9.878E-03	9.969E-03	9.679E-03
0.8		9.883E-04	2.862E-03	2.915E-03	9.118E-03

For Example 6.3., the absolute maximum errors are tabulated in Table 3. A comparison of the ExBSM with the existing methods (UCBS, TCBS, ECBS) can be analyzed. Figure 4 exhibits the solutions and errors graphs. It may be observed that our method is more efficient, accurate and well organized.

Example 6.4. Consider gNWS equation (1.1) with $a = 0 = b, m = 1, q = 3, \psi = bu^q + u(\alpha - u)(1 - u), \alpha = 0.5, l = 1, \eta = 0.3$ with the exact solution given in [17, 21] as

$$u(x,t) = \begin{cases} 0.3 + 0.04201052049t - 0.002794271318t^2 - 0.0001237074665t^3 \\ \quad - \left(6.498086711 \times (10)^{-10}t + 2.063100590 \times (10)^{-9}t^2 - 1.502107790 \times (10)^{-9}t^3\right) x^2(l-x)^2 \\ \quad - \left(5.530644662 \times (10)^{-9}t - 1.727330167 \times (10)^{-8}t^2 + 1.237824484 \times (10)^{-8}t^3\right) x^3(l-x)^3. \end{cases}$$



(a) Exact (b) Approximate

(c) 3D error (d) 2D error

Figure 5: Solutions and Error graphs for $t \in [0, 0.1]$ of Example 6.4.

Table 4: Exact and approximate solution of Example 6.4.

x	t	ExBSM	UCBS[21]	AVF[17]	t	proposed	UCBS[21]	AVF[17]
0.1	0.02	0.3008	0.2991	0.3026	0.08	0.3033	0.2966	0.3107
0.5		0.3008	0.2991	0.3026		0.3033	0.2966	0.3107
0.9		0.3008	0.2991	0.3026		0.3033	0.2966	0.3107

In Table 4, the exact and approximate solutions of Example 6.4 are listed. The numerical results obtained by ExBSM are in close agreement with the true solution. It can be seen that our method is more accurate as compared to the methods named (UCBS) and Variational formulation (VF) [17]. Figure 5 depict the solutions and errors patterns.

7. Conclusion:

The current study presents direct implementation of exponential B-spline method to generalized-Whitehead-Segel equation to obtain its numerical solution. For this purpose, three different types of boundary conditions are considered. The convergence of the proposed method is established both in space and time directions. The proposed scheme has been shown to be stable without any condition. The obtained numerical results are in good agreement with the exact analytical solutions. A comparison shows that the proposed method furnishes more accurate results as compared to the methods named as Uniform cubic B-spline (UCBS) [21], Trigonometric cubic B-spline (TCBS) [21], Extended cubic B-spline (ECBS) [21], and Variational formulation (VF) [17].

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