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Application of Shehu transform to Atangana-Baleanu derivatives

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Abstract

Recently, Shehu Maitama and Weidong Zhao proposed a new integral transform, namely, Shehu transform, which generalizes both the Sumudu and Laplace integral transforms. In this paper, we present new further properties of this transform. We apply this transformation to Atangana–Baleanu derivatives in Caputo and in Riemann–Liouville senses to solve some fractional differential equations.

Keywords: Shehu transform, Mittag-Leffler kernel, non-singular and non-local fractional operators.

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1. Introduction

The resolution of fractional differential equations has become a fertile mathematical field, Many researches that give innovative methods, meet the needs of other sciences. the derivatives of Caputo and Riemann–Liouville were the most appropriate way to model the various natural phenomena, which require non-integer derivatives, but its limitations in some respects led to the search for other derivatives. Both Caputo and Riemann–Liouville fractional derivatives have singular kernels. In addition, the Riemann–Liouville derivative of the constant does not equal zero. To overcome the disadvantage of the singular kernel, Caputo and Fabrizio [7] presented a fractional derivative operator without a singular kernel. Caputo-Fabrizio operator has proved to be effective through some applications [1, 12]. In [2], the authors proposed a genuine new fractional derivatives based on the Mittag-Leffler function. One is the ABR derivative (Riemann–Liouville sense), the other is the ABC derivative (Caputo sense). In addition to the fact that the kernel is non-singular and non-local, the ABC and ABR derivatives includes all the properties of the fractional derivatives except the semigroup property. However, recently these new fractional operators were recognized in one recently established classification of fractional operators [16].

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In [4] the authors have studied the relation between AB derivatives with Laplace, Sumudu, Fourier and Mellin transforms.

The current work is considered as a supplement to what was stated in, where we study the relationship between these two derivatives and a new integral transform called the Shehu transform, then we use the results obtained to solve some linear fractional equations.

The manuscript is organized as follows: In Section 2, we review the Sumudu and Shehu transforms and we prove some results, then, we examine the relationship between Atangana–Baleanu derivatives with Shehu transform, as well as some new results. In Section 3, we solve some fractional equations using the results of previous sections.

2. Definitions and preliminaries

Definition 2.1 ([13, 14])**.** The Sumudu transform is obtained over the set of functions

$$A = \left\{ \nu(t) : \exists N, \eta_1, \eta_2 > 0,\ |\nu(t)| < N \exp\left(\frac{|t|}{\eta_i}\right),\ \text{if } t \in (-1)^i \times [0, \infty) \right\},$$

by

$$\mathbb{S}[f(t)] = G(u) = \int_0^\infty f(ut) \exp(-t)\, dt, \quad u \in (-\eta_1, \eta_2).$$

Many authors have studied and used this transformation to solve problems in different disciplines of science [3, 5, 6, 8, 9] and [13–15]. The next theorem shows the relationship between Laplace and Sumudu transforms

Theorem 2.2 ([5])**.** *Let* G *and* F *be the Sumudu and the Laplace transforms of* $f(t) \in A$. *Then*

$$G(u) = \frac{F(1/u)}{u}. \tag{2.1}$$

The Shehu transform was introduced in [10] generalizes the Sumudu and the Laplace integral transforms, they have used it to solve ordinary and partial differential equations.

Definition 2.3. The Shehu transform is obtained over the set A by [10]:

$$\mathbb{H}[f(t)] = V(s, u) = \int_0^\infty \exp\left(-\frac{st}{u}\right) f(t)\, dt.$$

It is clear that the Shehu transform is linear as the Sumudu and Laplace transformations.

The function Mittag-Leffler $E_\alpha(t)$ is a direct generalization of the exponential series. For $\alpha = 1$ we have $E_\alpha(t) = \exp(t)$. It is given by [11]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \quad \mathrm{Re}(\alpha) > 0.$$

Definition 2.4 ([2])**.** Let $f \in H^1(a, b)$, $b > a$, then for $\alpha \in (0, 1)$ the Atangana–Baleanu fractional derivative in Caputo sense is given as

$${}_{a}^{ABC}D_t^\alpha (f(t)) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(x) E_\alpha\left(-\alpha \frac{(t-x)^\alpha}{1-\alpha}\right) dx.$$

Definition 2.5 ([2])**.** Let $f \in H^1(a, b)$, $b > a$, then for $\alpha \in (0, 1)$ the Atangana–Baleanu fractional derivative in Riemann–Liouville sense is given as

$${}_{a}^{ABR}D_t^\alpha (f(t)) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t f(x) E_\alpha\left(-\alpha \frac{(t-x)^\alpha}{1-\alpha}\right) dx,$$

where $B(\alpha)$ is a normalization function under the conditions $B(0) = B(1) = 1$.

Theorem 2.6 ([2]). *The Laplace transform of Atangana–Baleanu fractional derivative in Caputo sense is given as*

$$\mathbb{L}\left\{{}_{a}^{ABC}D_t^{\alpha}\left(f\left(t\right)\right)\right\}\left(s\right)=\frac{B\left(\alpha\right)}{1-\alpha}\frac{s^{\alpha}F\left(s\right)-s^{\alpha-1}f\left(0\right)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}, \tag{2.2}$$

and the Laplace transform of the Atangana–Baleanu fractional derivative in Riemann–Liouville sense is given as

$$\mathbb{L}\left\{{}_{a}^{ABR}D_t^{\alpha}\left(f\left(t\right)\right)\right\}\left(s\right)=\frac{B\left(\alpha\right)}{1-\alpha}\frac{s^{\alpha}F\left(s\right)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}. \tag{2.3}$$

2.1. Main results

In what follows, we assume that $f\in H^1(a,b)$, $b>a$, $\alpha\in(0,1)$ and $f(t)\in A$.

Theorem 2.7. *The Sumudu transform of Atangana–Baleanu fractional derivative in Caputo sense is given as*

$$\mathbb{S}\left({}_{0}^{ABC}D_t^{\alpha}\left(f\left(t\right)\right)\right)=\frac{B\left(\alpha\right)}{1-\alpha+\alpha u^{\alpha}}\left(G\left(u\right)-f\left(0\right)\right). \tag{2.4}$$

Proof. Using (2.2) and (2.1), we obtain

$$\mathbb{S}\left({}_{0}^{ABC}D_t^{\alpha}\left(f\left(t\right)\right)\right)=\frac{1}{u}\left(\frac{B\left(\alpha\right)}{1-\alpha}\frac{\left(\frac{1}{u}\right)^{\alpha}F\left(1/u\right)-\left(\frac{1}{u}\right)^{\alpha-1}f\left(0\right)}{\left(\frac{1}{u}\right)^{\alpha}+\frac{\alpha}{1-\alpha}}\right)=\left(\frac{1}{u}\right)^{\alpha}\frac{B\left(\alpha\right)}{1-\alpha}\frac{G\left(u\right)-f\left(0\right)}{\left(\frac{1}{u}\right)^{\alpha}+\frac{\alpha}{1-\alpha}}.$$

Then we obtain the desired result

$$\mathbb{S}\left({}_{0}^{ABC}D_t^{\alpha}\left(f\left(t\right)\right)\right)=\frac{B\left(\alpha\right)}{1-\alpha+\alpha u^{\alpha}}\left(G\left(u\right)-f\left(0\right)\right).$$ □

Theorem 2.8. *The Sumudu transform of Atangana–Baleanu fractional derivative in Riemann–Liouville sense is given as*

$$\mathbb{S}\left({}_{0}^{ABR}D_t^{\alpha}\left(f\left(t\right)\right)\right)=\frac{B\left(\alpha\right)}{1-\alpha+\alpha u^{\alpha}}G\left(u\right). \tag{2.5}$$

Proof. Using (2.3) and (2.1), we obtain

$$\mathbb{S}\left\{{}_{a}^{ABR}D_t^{\alpha}\left(f\left(t\right)\right)\right\}\left(s\right)=\frac{1}{u}\frac{B\left(\alpha\right)}{1-\alpha}\frac{\left(\frac{1}{u}\right)^{\alpha}F\left(\frac{1}{u}\right)}{\left(\frac{1}{u}\right)^{\alpha}+\frac{\alpha}{1-\alpha}}=\frac{1}{u}\frac{B\left(\alpha\right)}{1-\alpha}\frac{\left(\frac{1}{u}\right)^{\alpha}uG\left(u\right)}{\left(\frac{1}{u}\right)^{\alpha}+\frac{\alpha}{1-\alpha}}=\frac{B\left(\alpha\right)}{1-\alpha}\frac{G\left(u\right)}{1+\frac{\alpha}{1-\alpha}u^{\alpha}}.$$

Then we obtain the desired result:

$$\mathbb{S}\left({}_{0}^{ABR}D_t^{\alpha}\left(f\left(t\right)\right)\right)=\frac{B\left(\alpha\right)}{1-\alpha+\alpha u^{\alpha}}G\left(u\right).$$ □

The link between the Shehu and Sumudu transforms is illustrated by the following theorem.

Theorem 2.9. *Let* $G\left(u\right)$ *and* $V\left(s,u\right)$ *be the Sumudu and the Shehu transforms of* $f(t)\in A$. *Then*

$$V\left(s,u\right)=\frac{u}{s}G\left(\frac{u}{s}\right). \tag{2.6}$$

Proof. If $f(t)\in A$, then

$$V\left(s,u\right)=\int_0^{\infty}\exp\left(-\frac{st}{u}\right)f\left(t\right)dt.$$

If we set $\tau=st/u(t=u\tau/s)$, then the right-hand side can be written as

$$V\left(s,u\right)=\int_0^{\infty}\exp\left(-\tau\right)f\left(\frac{u}{s}\tau\right)\frac{u}{s}d\tau=\frac{u}{s}\int_0^{\infty}\exp\left(-\tau\right)f\left(\frac{u}{s}\tau\right)d\tau.$$

The integral on the right-hand side is clearly $G\left(\frac{u}{s}\right)$, thus yielding (2.6). It is noticeable that

$$V\left(s,1\right)=\frac{1}{s}G\left(\frac{1}{s}\right)=F\left(s\right),$$

where $F\left(s\right)$ is the Laplace transform of $f\left(t\right)$. □

The following important properties are obtain using the relationship between Shehu and Sumudu transforms (2.6).

Theorem 2.10. *The Shehu transform of* t^{x-1} *is*

$$V(s,u) = \Gamma(x)\left(\frac{u}{s}\right)^{x}, \quad x > 0.$$

Proof. When $x > 0$, the Sumudu of t^{x-1} is given by [5]

$$G(u) = \Gamma(x)\, u^{x-1},$$

where $\Gamma(x)$ is the Gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Then, by using (2.6), we get the desired result. □

Theorem 2.11. *Let* $\alpha, \omega \in \mathbb{C}$, *with* $\mathrm{Re}(\alpha) > 0$. *Shehu transform of* $E_\alpha(\omega t^\alpha)$ *is given by*

$$\mathbb{H}\left(E_\alpha\left(\omega t^\alpha\right)\right) = \frac{u}{s}\left(1 - \omega\left(\frac{u}{s}\right)^{\alpha}\right)^{-1}. \tag{2.7}$$

Proof. Based on the reference [8], we have

$$\mathbb{S}\left(E_\alpha\left(\omega t^\alpha\right)\right) = \left(1 - \omega u^\alpha\right)^{-1},$$

then, by using (2.6), we have

$$\mathbb{H}\left(E_\alpha\left(\omega t^\alpha\right)\right) = \left(\frac{u}{s}\right)\left(1 - \omega\left(\frac{u}{s}\right)^{\alpha}\right)^{-1}. \qquad \square$$

Theorem 2.12. *Let* $G(u)$ *and* $V(s,u)$ *be the Sumudu and the Shehu transforms of* $f(t) \in A$. *Then, the Shehu transform of Atangana–Baleanu fractional derivative in Caputo sense is given as*

$$\mathbb{H}\left({}_0^{ABC}D_t^{\alpha}\left(f(t)\right)\right) = \frac{B(\alpha)}{1 - \alpha + \alpha\left(\frac{u}{s}\right)^{\alpha}}\left(V(s,u) - \frac{u}{s} f(0)\right).$$

Proof. Using (2.4) and the relationship between Sumudu and Shehu transforms (2.6), we obtain

$$\begin{aligned}\mathbb{H}\left({}_0^{ABC}D_t^{\alpha}\left(f(t)\right)\right) &= \frac{u}{s}\frac{B(\alpha)}{1 - \alpha + \alpha\left(\frac{u}{s}\right)^{\alpha}}\left(G\left(\frac{u}{s}\right) - f(0)\right)\\ &= \frac{B(\alpha)}{1 - \alpha + \alpha\left(\frac{u}{s}\right)^{\alpha}}\left(V(s,u) - \frac{u}{s} f(0)\right). \qquad \square\end{aligned}$$

Theorem 2.13. *Let* $G(u)$ *and* $V(s,u)$ *be the Sumudu and the Shehu transforms of* $f(t) \in A$. *Then, the Shehu transform of Atangana–Baleanu fractional derivative in Riemann–Liouville sense is given as*

$$\mathbb{H}\left({}_0^{ABR}D_t^{\alpha}\left(f(t)\right)\right) = \frac{B(\alpha)}{1 - \alpha + \alpha\left(\frac{u}{s}\right)^{\alpha}}\left(V(s,u) - \frac{u}{s} f(0)\right).$$

Proof. Using (2.5) and the relationship between Sumudu and Shehu transforms (2.6), we obtain

$$\mathbb{H}\left({}_0^{ABR}D_t^{\alpha}\left(f(t)\right)\right) = \frac{u}{s}\frac{B(\alpha)}{1 - \alpha + \alpha\left(\frac{u}{s}\right)^{\alpha}} G\left(\frac{u}{s}\right) = \frac{B(\alpha)}{1 - \alpha + \alpha\left(\frac{u}{s}\right)^{\alpha}} V(s,u). \qquad \square$$

3. Applications

Consider the following fractional initial value problem

$$\begin{cases} {}_{0}^{ABC}D_{t}^{\alpha}\left(y\left(t\right)\right)=g\left(t,y\left(t\right)\right), & t>0, \\ y\left(0\right)=c, & c\in\mathbb{R}. \end{cases} \tag{3.1}$$

Let $y, g \in A$, having Shehu transforms $V(s,u)$ and $W(s,u)$, respectively. Then, by applying Shehu transform to of both sides of (3.1), we obtain

$$\frac{B\left(\alpha\right)}{1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}}\left(V\left(s,u\right)-\frac{u}{s}f\left(0\right)\right)=W(s,u),$$

thus

$$V\left(s,u\right)=\frac{1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}}{B\left(\alpha\right)}W(s,u)+\frac{u}{s}f\left(0\right). \tag{3.2}$$

Now, by taking the inverse Shehu transform of (3.2), we get the exact solution.

In the same way, we can study cases that contain a derivatives in Riemann–Liouville sense.

Example 3.1. Consider the following fractional initial value problem

$$\begin{cases} {}_{0}^{ABC}D_{t}^{\alpha}\left(y\left(t\right)\right)=\lambda t, \quad \lambda\in\mathbb{R},\ t>0, \\ y\left(0\right)=0. \end{cases} \tag{3.3}$$

We first apply the Shehu transform of both sides of (3.3) to get

$$\mathbb{H}\left({}_{0}^{ABC}D_{t}^{\alpha}\left(y\left(t\right)\right)\right)=\mathbb{H}\left(\lambda t\right),$$

which yields

$$\frac{B\left(\alpha\right)}{1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}}\left(V\left(s,u\right)-\frac{u}{s}f\left(0\right)\right)=\lambda\left(\frac{u}{s}\right)^{2},$$

$$\begin{aligned} V\left(s,u\right)&=\frac{\lambda}{B\left(\alpha\right)}\left(\frac{u}{s}\right)^{2}\left(1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}\right)+\frac{u}{s}f\left(0\right) \\ &=\frac{\lambda}{B\left(\alpha\right)}\left(\left(1-\alpha\right)\left(\frac{u}{s}\right)^{2}+\alpha\left(\frac{u}{s}\right)^{\alpha+2}\right). \end{aligned} \tag{3.4}$$

When we apply inverse Shehu transform of (3.4) by using Theorem 2.10, we obtain exact solution of (3.3) as follows:

$$y\left(t\right)=\frac{\lambda}{B\left(\alpha\right)}\left(\left(1-\alpha\right)t+\frac{\alpha}{\Gamma\left(\alpha+2\right)}t^{\alpha+1}\right). \tag{3.5}$$

Notice that as $\alpha\to 0$, (3.5) goes to $y\left(t\right)=\lambda t$, and if $\alpha\to 1$, we obtain $y\left(t\right)=\lambda\frac{t^{2}}{2}$. For $\alpha=\frac{1}{2}$, we have $y\left(t\right)=\frac{\lambda}{B(1/2)}\left(\frac{1}{2}t+\frac{2}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)$.

Example 3.2. Consider the following fractional initial value problem

$$\begin{cases} {}_{0}^{ABC}D_{t}^{\alpha}\left(y\left(t\right)\right)=y\left(t\right), & t>0, \\ y\left(0\right)=1. \end{cases} \tag{3.6}$$

We first apply the Shehu transform of both sides of (3.6) to get

$$\mathbb{H}\left({}_{0}^{ABC}D_{t}^{\alpha}\left(y\left(t\right)\right)\right)=\mathbb{H}\left(y\left(t\right)\right),$$

which yields

$$\frac{B(\alpha)}{1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}}\left(V(s,u)-\frac{u}{s}f(0)\right)=V(s,u),$$

$$\Longrightarrow\left(B(\alpha)-\left(1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}\right)\right)V(s,u)=B(\alpha)\frac{u}{s}f(0).$$

$$\begin{aligned}V(s,u)&=\frac{B(\alpha)}{\left(B(\alpha)-1+\alpha-\alpha\left(\frac{u}{s}\right)^{\alpha}\right)}\frac{u}{s}\\&=\frac{B(\alpha)}{\left(B(\alpha)-1+\alpha\right)\left(1-\frac{\alpha}{B(\alpha)-1+\alpha}\left(\frac{u}{s}\right)^{\alpha}\right)}\frac{u}{s},\end{aligned}$$

$$V(s,u)=\frac{B(\alpha)}{B(\alpha)-1+\alpha}\frac{u}{s}\left(1-\frac{\alpha}{B(\alpha)-1+\alpha}\left(\frac{u}{s}\right)^{\alpha}\right)^{-1}, \tag{3.7}$$

applying the inverse of Shehu transform of (3.7), by using (2.7), we have the approximate solution of equation (3.6),

$$y(t)=\frac{B(\alpha)}{B(\alpha)-1+\alpha}E_{\alpha}\left(\frac{\alpha}{B(\alpha)-1+\alpha}t^{\alpha}\right).$$

When $\alpha\to 1$, we obtain $y(t)=E_1(t)=e^t$.

4. Conclusions

In this work, the Shehu transform is discussed and new related proprieties are established, the relationship between Shehu and Sumudu transforms is proved. Also, this transformation is applied to fractional Atangana–Baleanu derivatives. Finally, some differential equations were solved using the previous reported results.

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