



Determinants, inverses, norms, and spreads of skew circulant matrices involving the product of Fibonacci and Lucas numbers



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Abstract

In this paper, we investigate the invertibility of $n \times n$ skew circulant matrix involving the product of Fibonacci and Lucas numbers, whose determinant and inverse can be expressed by the $(n-1)^{\text{th}}$, n^{th} , $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$ product of Fibonacci and Lucas numbers. Some norms and bounds for the spread of these matrices are given, respectively. In addition, we generalize these results to skew left circulant matrix involving the product of Fibonacci and Lucas numbers. Finally, several numerical examples are illustrated to show the effectiveness of our theoretical results.

Keywords: Determinant, inverse, norm, spread, Fibonacci number, skew circulant matrix.

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1. Introduction

Circulant matrices play a vital role in nonlinear control systems, preconditioners, signal processing, image processing, Toeplitz matrix problems and various differential equations [3, 7, 8, 19, 23, 26–28]. Jin et al. [14] proposed the GMRES (Generalized Minimal Residual) method with the Strang-type block-circulant preconditioner for solving singular perturbation delay differential equations. Next, Pelletier and Cooperstock [22] put forward preconditioners based on circulant operators for speeding up the solution of edge-preserving image super resolution problems. Besides, circulant matrices have an important application in the optimization design of digital filters. Narasimha [21] decomposed linear convolution into a sum of skew circulant convolutions, which perform filtering procedure in approximately half of computational cost for real signals. Shortly afterwards, based on circulant, Toeplitz and shift circulant matrices, Daher et al. [1] developed a fast algorithm for optimal design of block digital filters, which reduced the computational complexity of the design process as well as decreased the memory requirements. Better

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yet, Fu et al. [4] proposed a skew circulant filter, which achieves the above advantages of [1] and even greater efficiency with only approximately half of the computational cost for the real signals.

On the other hand, Fibonacci sequences are shining stars in the vast array of integer sequences and Fibonacci numbers appear in many unexpected places, such as electrical networks, nature (earth, optics and botany), even music and so on. For centuries, they amaze engineers with their charm and applications such as electrical engineering, geometry, graph theory, physics, physiology, psychology, neurophysiology, biology, chemistry, stock market trading, water treatment, snow plowing, and trigonometry [6, 16, 17].

Recently, some authors have done some research on circulant matrices involving some famous numbers about their determinants, inverses, some norms, and spreads [2, 5, 10, 11, 13, 18, 20, 24, 25]. Recently, Zheng and Shon [30] studied the exact determinants and inverses of generalized Lucas skew circulant type matrices, while Jiang and Hong [12] gave that of Tribonacci r-circulant type matrices. What’s more, Ahmetİpek [9] investigated an improved estimation for spectral norms of circulant matrices with classical Fibonacci and Lucas number entries.

We present, therefore, the results on the combination of skew circulant matrix, skew left circulant matrix, Fibonacci numbers and Lucas numbers. More specifically, we study the invertibility, determinant, multiple norms, lower and upper bounds for the spread of these matrices, which are going to have potential to be useful for realistic application.

The main aims we study in the paper are as follows. Fibonacci numbers F_n and Lucas numbers L_n are often defined recursively [16] as

$$F_n = F_{n-1} + F_{n-2}, \text{ where } F_1 = 1, F_2 = 1 \text{ and } L_n = L_{n-1} + L_{n-2}, \text{ where } L_1 = 1, L_2 = 3.$$

The product \mathfrak{F}_n of Fibonacci and Lucas numbers satisfies the following recurrence relations [20]:

$$\mathfrak{F}_{n+2} = 3\mathfrak{F}_{n+1} - \mathfrak{F}_n, \text{ where } \mathfrak{F}_1 = 1, \mathfrak{F}_2 = 3. \tag{1.1}$$

The sequence $\{\mathfrak{F}_n\}$ is also given by the formula

$$\mathfrak{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.2}$$

where α and β are the roots of the characteristic equation $x^2 - 3x + 1 = 0$. The first few values of the sequence are given in the following table:

n	1	2	3	4	5	6	7	8	...
\mathfrak{F}_n	1	3	8	21	55	144	377	987	...

The product \mathfrak{F}_n of Fibonacci and Lucas numbers recurrence relations play a vital role in the following paper.

Consider a skew circulant matrix involving the product of Fibonacci and Lucas numbers

$$\text{SCirc}(\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_n) = \begin{pmatrix} \mathfrak{F}_1 & \mathfrak{F}_2 & \dots & \mathfrak{F}_n \\ -\mathfrak{F}_n & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathfrak{F}_2 \\ -\mathfrak{F}_2 & \dots & -\mathfrak{F}_n & \mathfrak{F}_1 \end{pmatrix}_{n \times n}. \tag{1.3}$$

Besides, define a skew left circulant matrix involving the product of Fibonacci and Lucas numbers

$$\text{SLCirc}(\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_n) = \begin{pmatrix} \mathfrak{F}_1 & \dots & \mathfrak{F}_{n-1} & \mathfrak{F}_n \\ \vdots & \ddots & \ddots & -\mathfrak{F}_1 \\ \mathfrak{F}_{n-1} & \ddots & \ddots & \vdots \\ \mathfrak{F}_n & -\mathfrak{F}_1 & \dots & -\mathfrak{F}_{n-1} \end{pmatrix}_{n \times n}. \tag{1.4}$$

2. Preliminaries

In this section, we list some lemmas that play a vital role in the proofs of main theorems.

Lemma 2.1. *Let $\{\tilde{\mathfrak{F}}_n\}$ be the product of Fibonacci and Lucas numbers. Then*

$$(i) \sum_{i=1}^n \tilde{\mathfrak{F}}_i = 2\tilde{\mathfrak{F}}_n - \tilde{\mathfrak{F}}_{n-1} - 1, \tag{2.1}$$

$$(ii) \sum_{i=1}^n i\tilde{\mathfrak{F}}_i = (n+2)\tilde{\mathfrak{F}}_{n+1} - (n+3)\tilde{\mathfrak{F}}_n - 2, \tag{2.2}$$

$$(iii) \sum_{i=1}^n \tilde{\mathfrak{F}}_i^2 = \frac{\tilde{\mathfrak{F}}_{n+1}^2 - \tilde{\mathfrak{F}}_n^2 - 8n - 7}{5}, \tag{2.3}$$

$$(iv) \sum_{i=1}^n \tilde{\mathfrak{F}}_i a^i = \frac{\tilde{\mathfrak{F}}_n a^{n+2} - \tilde{\mathfrak{F}}_{n+1} a^{n+1} + a}{a^2 - 3a + 1}, \quad (a \neq \frac{3 \pm \sqrt{5}}{2}). \tag{2.4}$$

Proof.

(i) According to (1.1)

$$\sum_{i=1}^n \tilde{\mathfrak{F}}_i = \tilde{\mathfrak{F}}_1 + \tilde{\mathfrak{F}}_2 + \dots + \tilde{\mathfrak{F}}_n = \tilde{\mathfrak{F}}_1 + (3\tilde{\mathfrak{F}}_1 - \tilde{\mathfrak{F}}_0) + \dots + (3\tilde{\mathfrak{F}}_{n-1} - \tilde{\mathfrak{F}}_{n-2}) = 2 \sum_{i=1}^n \tilde{\mathfrak{F}}_i - 2\tilde{\mathfrak{F}}_n + \tilde{\mathfrak{F}}_{n-1} + \tilde{\mathfrak{F}}_1 - \tilde{\mathfrak{F}}_0.$$

It is easy to check that

$$\sum_{i=1}^n \tilde{\mathfrak{F}}_i = 2\tilde{\mathfrak{F}}_n - \tilde{\mathfrak{F}}_{n-1} - 1. \tag{2.5}$$

(ii) From (1.1), we also get that

$$\begin{aligned} \sum_{i=1}^n i\tilde{\mathfrak{F}}_i &= \tilde{\mathfrak{F}}_1 + 2\tilde{\mathfrak{F}}_2 + \dots + n\tilde{\mathfrak{F}}_n = \tilde{\mathfrak{F}}_1 + 2(3\tilde{\mathfrak{F}}_1 - \tilde{\mathfrak{F}}_0) + \dots + n(3\tilde{\mathfrak{F}}_{n-1} - \tilde{\mathfrak{F}}_{n-2}) \\ &= 2 \sum_{i=1}^n i\tilde{\mathfrak{F}}_i + \sum_{i=1}^n \tilde{\mathfrak{F}}_i - (2n+1)\tilde{\mathfrak{F}}_n + (n+1)\tilde{\mathfrak{F}}_{n-1} + \tilde{\mathfrak{F}}_1 - 2\tilde{\mathfrak{F}}_0. \end{aligned} \tag{2.6}$$

By (2.5) and (2.6), we have

$$\sum_{i=1}^n i\tilde{\mathfrak{F}}_i = (n+2)\tilde{\mathfrak{F}}_{n+1} - (n+3)\tilde{\mathfrak{F}}_n - 2.$$

(iii) Let $X_i = \begin{pmatrix} \tilde{\mathfrak{F}}_{i-1} & \tilde{\mathfrak{F}}_i \\ \tilde{\mathfrak{F}}_i & \tilde{\mathfrak{F}}_{i+1} \end{pmatrix}$. By Lemma 1 in [29], for $i \geq 1$, we get

$$|X_i| = \tilde{\mathfrak{F}}_{i-1}\tilde{\mathfrak{F}}_{i+1} - \tilde{\mathfrak{F}}_i^2 = -1.$$

For $\{\tilde{\mathfrak{F}}_n\}$, the following equation is satisfied,

$$\tilde{\mathfrak{F}}_i = \frac{\tilde{\mathfrak{F}}_{i+1} + \tilde{\mathfrak{F}}_{i-1}}{3}. \tag{2.7}$$

Therefore, using (2.7), we obtain that

$$\sum_{i=1}^n \tilde{\mathfrak{F}}_i^2 = \sum_{i=1}^n \left(\frac{\tilde{\mathfrak{F}}_{i+1} + \tilde{\mathfrak{F}}_{i-1}}{3}\right)^2 = \frac{\sum_{i=1}^n \tilde{\mathfrak{F}}_{i+1}^2 + \sum_{i=1}^n \tilde{\mathfrak{F}}_{i-1}^2 + 2 \sum_{i=1}^n \tilde{\mathfrak{F}}_{i+1}\tilde{\mathfrak{F}}_{i-1}}{9}$$

$$\begin{aligned} &= \frac{\sum_{i=1}^n \mathfrak{F}_{i+1}^2 + \sum_{i=1}^n \mathfrak{F}_{i-1}^2 + 2 \sum_{i=1}^n \mathfrak{F}_i^2 - 2n}{9} \\ &= \frac{4 \sum_{i=1}^n \mathfrak{F}_i^2 + \mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - \mathfrak{F}_1^2 + \mathfrak{F}_0^2 - 2n}{9}. \end{aligned}$$

Hence

$$\sum_{i=1}^n \mathfrak{F}_i^2 = \frac{\mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - 8n - 7}{5}.$$

(iv) Let

$$S_n = \sum_{i=1}^n \mathfrak{F}_i a^i = \mathfrak{F}_1 a + \mathfrak{F}_2 a^2 + \dots + \mathfrak{F}_n a^n. \tag{2.8}$$

From (1.1) and (2.8), we get

$$(a^2 - 3a + 1)S_n = \mathfrak{F}_n a^{n+2} - \mathfrak{F}_{n+1} a^{n+1} + a.$$

Thus

$$S_n = \frac{\mathfrak{F}_n a^{n+2} - \mathfrak{F}_{n+1} a^{n+1} + a}{a^2 - 3a + 1}, \quad (a \neq \frac{3 \pm \sqrt{5}}{2}). \quad \square$$

Lemma 2.2. Let the form of the matrix $\mathcal{H} = [h_{ij}]_{i,j=1}^{n-2}$ be as follows:

$$h_{ij} = \begin{cases} 1 + \mathfrak{F}_{n+1}, & i = j, \\ -\mathfrak{F}_n, & i = j + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{H}^{-1} = [h'_{ij}]_{i,j=1}^{n-2}$ is given by

$$h'_{ij} = \begin{cases} \frac{\mathfrak{F}_n^{i-j}}{(1 + \mathfrak{F}_{n+1})^{i-j+1}}, & i \geq j, \\ 0, & i < j. \end{cases}$$

Proof. Let $e_{ij} = \sum_{k=1}^{n-2} h_{ik} h'_{kj}$. For $i < j$, $e_{ij} = 0$. For $i = j$, $e_{ii} = h_{ii} h'_{ii} = (1 + \mathfrak{F}_{n+1}) \cdot \frac{1}{1 + \mathfrak{F}_{n+1}} = 1$. For $i \geq j + 1$, we obtain

$$e_{ij} = \sum_{k=1}^{n-2} h_{ik} h'_{kj} = h_{i,i-1} h'_{i-1,j} + h_{ii} h'_{ij} = \frac{-\mathfrak{F}_n \cdot \mathfrak{F}_n^{i-j-1}}{(1 + \mathfrak{F}_{n+1})^{i-j}} + \frac{(1 + \mathfrak{F}_{n+1}) \cdot \mathfrak{F}_n^{i-j}}{(1 + \mathfrak{F}_{n+1})^{i-j+1}} = 0.$$

We get $\mathcal{H}\mathcal{H}^{-1} = I_{n-2}$, where I_{n-2} is $(n - 2) \times (n - 2)$ identity matrix. Similarly, we can verify $\mathcal{H}^{-1}\mathcal{H} = I_{n-2}$. Thus, the proof is completed. \square

Lemma 2.3 ([5]). Let the orthogonal skew left circulant matrix $\Delta = \text{SLCirc}(1, 0, 0, \dots, 0)$. Then

$$\text{SCirc}(a_1, a_2, \dots, a_n) = \Delta \text{SLCirc}(a_1, a_2, \dots, a_n).$$

Lemma 2.4 ([15]). Let $A = \text{SLCirc}(a_1, a_2, \dots, a_n)$ be a skew left circulant matrix and n be odd. Then

$$\lambda_j = \pm \left| \sum_{k=1}^n a_k \omega^{(j-\frac{1}{2})(k-1)} \right|, \quad (j = 1, 2, \dots, \frac{n-1}{2}), \quad \lambda_{\frac{n+1}{2}} = \sum_{k=1}^n |a_k (-1)^{k-1}|,$$

where $\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} = e^{\frac{2\pi i}{n}}$ and $\lambda_j (j = 1, 2, \dots, \frac{n-1}{2}, \frac{n+1}{2})$ are the eigenvalues of A .

3. Determinant, inverse, norms, and spread of skew circulant matrix involving the product of Fibonacci and Lucas numbers

In this section, let $A_n = \text{SCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be an $n \times n$ skew circulant matrix. We do some work on the determinant and inverse of A_n . Meanwhile, we also investigate the properties of skew circulant matrices involving the product of Fibonacci and Lucas numbers, such as their norms, lower and upper bounds for their spread.

3.1. Determinant and inverse of skew circulant matrix involving the product of Fibonacci and Lucas numbers

In this subsection, we first obtain an expression for the determinant of A_n in the following Theorem 3.1. Afterwards, we prove that A_n is invertible for every positive integer n in Theorem 3.2, and then we compute the inverse of A_n in Theorem 3.3.

Theorem 3.1. *Let $A_n = \text{SCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be a matrix as in (1.3) for a positive integer n . Then*

$$\det A_n = \frac{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}{1 - \mathfrak{F}_{n-1} + \mathfrak{F}_{n+1}}, \tag{3.1}$$

where \mathfrak{F}_n is the n^{th} product of Fibonacci and Lucas numbers.

Proof. Clearly, $\det A_1 = 1$, $\det A_2 = 10$, and $\det A_3 = 558$ meet (3.1). Under the circumstances ($n > 3$), construct two matrices

$$\Sigma = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 3 & \vdots & & & & \ddots & 1 \\ -1 & \vdots & & & \ddots & \ddots & -3 \\ 0 & \vdots & & \ddots & \ddots & \ddots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & -3 & 1 & 0 & \dots & 0 \end{pmatrix}_{n \times n}$$

and

$$\Omega_1 = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & (\frac{\mathfrak{F}_n}{1+\mathfrak{F}_{n+1}})^{n-2} & \vdots & & \ddots & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & (\frac{\mathfrak{F}_n}{1+\mathfrak{F}_{n+1}})^2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \frac{\mathfrak{F}_n}{1+\mathfrak{F}_{n+1}} & 1 & \ddots & & \vdots \\ 0 & 1 & 0 & \dots & \dots & 0 \end{pmatrix}_{n \times n}.$$

Multiplying A by Ω_1 and Σ from right and left, respectively,

$$\Sigma A_n \Omega_1 = \begin{pmatrix} \mathfrak{F}_1 & l'_n & \mathfrak{F}_{n-1} & \mathfrak{F}_{n-2} & \dots & \dots & \mathfrak{F}_2 \\ 0 & l_n & \mathfrak{F}_{n-2} & \mathfrak{F}_{n-3} & \dots & \dots & \mathfrak{F}_1 \\ \vdots & 0 & \mathfrak{F}_1 + \mathfrak{F}_{n+1} & 0 & \dots & \dots & 0 \\ \vdots & \vdots & -\mathfrak{F}_n & \ddots & \ddots & & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & -\mathfrak{F}_n & \mathfrak{F}_1 + \mathfrak{F}_{n+1} \end{pmatrix}_{n \times n}, \tag{3.2}$$

where

$$l'_n = \sum_{k=1}^{n-1} \mathfrak{F}_{k+1} \left(\frac{\mathfrak{F}_n}{1 + \mathfrak{F}_{n+1}}\right)^{n-1-k}, \quad l_n = 1 + 3\mathfrak{F}_n + \sum_{k=1}^{n-2} \mathfrak{F}_k \left(\frac{\mathfrak{F}_n}{1 + \mathfrak{F}_{n+1}}\right)^{n-1-k}.$$

We further simplify the above results using (2.4) as follows:

$$l'_n = \frac{-1 + \mathfrak{F}_{n-1} + \mathfrak{F}_n(1 + \mathfrak{F}_{n+1})^{n-2}}{(1 - \mathfrak{F}_{n-1} + \mathfrak{F}_{n+1})(1 + \mathfrak{F}_{n+1})^{n-2}}, \tag{3.3}$$

$$l_n = \frac{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}{(1 - \mathfrak{F}_{n-1} + \mathfrak{F}_{n+1})(1 + \mathfrak{F}_{n+1})^{n-2}}. \tag{3.4}$$

So it holds that

$$\det(\Sigma A_n \Omega_1) = l_n(1 + \mathfrak{F}_{n+1})^{n-2} = \frac{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}{1 - \mathfrak{F}_{n-1} + \mathfrak{F}_{n+1}}.$$

Clearly, $\det \Sigma = \det \Omega_1 = (-1)^{\frac{(n-1)(n-2)}{2}}$, we have

$$\det A_n = \frac{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}{1 - \mathfrak{F}_{n-1} + \mathfrak{F}_{n+1}}. \quad \square$$

Theorem 3.2. Let $A_n = \text{SCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be a matrix as in (1.3) for every positive integer n . Then A_n is invertible.

Proof. In Theorem 3.1, $\det A_1 = 1 \neq 0$, hence A_1 is invertible. For $n > 1$, according to [13], we have the eigenvalues of A_n

$$f(\omega^k \eta) = \sum_{j=1}^n \mathfrak{F}_j (\omega^k \eta)^{j-1}, \quad (k = 1, 2, \dots, n-1),$$

where $\omega = \exp(\frac{2\pi i}{n}), \eta = \exp(\frac{\pi i}{n})$.

Note that $\mathfrak{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha + \beta = 3, \alpha\beta = 1$, we get

$$f(\omega^k \eta) = \frac{1}{\alpha - \beta} \sum_{j=1}^n (\alpha^j - \beta^j) (\omega^k \eta)^{j-1}, \quad (k = 1, 2, \dots, n-1).$$

Since $|\omega^k \eta| = 1$, and $|\alpha| \neq 1, |\beta| \neq 1$, we have $\omega^k \eta \neq \alpha$ and $\omega^k \eta \neq \beta$, and hence,

$$\begin{aligned} f(\omega^k \eta) &= \frac{\alpha}{\alpha - \beta} \cdot \frac{1 + \alpha^n}{1 - \alpha \omega^k \eta} - \frac{\beta}{\alpha - \beta} \cdot \frac{1 + \alpha^n}{1 - \alpha \omega^k \eta} \\ &= \frac{\alpha - \beta + \alpha^{n+1} - \beta^{n+1} - \alpha\beta \omega^k \eta (\alpha^n - \beta^n)}{(\alpha - \beta)(1 - \alpha \omega^k \eta)(1 - \beta \omega^k \eta)} = \frac{1 + \mathfrak{F}_{n+1} - \mathfrak{F}_n \omega^k \eta}{1 - 3\omega^k \eta + \omega^{2k} \eta^2}, \quad (k = 1, 2, \dots, n-1). \end{aligned}$$

Assume to the contrary that there exists $\omega^l \eta$ ($l = 1, 2, \dots, n-1$) such that $f(\omega^l \eta) = 0$. We obtain $1 + \mathfrak{F}_{n+1} - \mathfrak{F}_n \omega^l \eta = 0$, and hence it follows that $\omega^l \eta = \frac{1 + \mathfrak{F}_{n+1}}{\mathfrak{F}_n}$ is a real number. That is to say, the imaginary part of $\omega^l \eta$ is $\sin \frac{(2l+1)\pi}{n} = 0$. Therefore, $\omega^l \eta = \pm 1$, and $1 + \mathfrak{F}_{n+1} - \mathfrak{F}_n \omega^l \eta \neq 0$ ($n > 0$). Consequently, we obtain $f(\omega^k \eta) \neq 0$ for any $\omega^k \eta$ ($k = 1, 2, \dots, n-1$). \square

Theorem 3.3. Let $A_n = \text{SCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be a matrix as in (1.3) for a positive integer $n > 1$. Then

$$A_n^{-1} = \text{SCirc}(y_1, y_2, \dots, y_n), \tag{3.5}$$

where

$$y_1 = \frac{(\mathfrak{F}_{n-1} - 1)\mathfrak{F}_n^{n-2} + (1 + \mathfrak{F}_{n+1})^{n-1}}{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}, \tag{3.6}$$

$$y_2 = -\frac{\mathfrak{F}_n^{n-1} + (3 + \mathfrak{F}_{n+2})(1 + \mathfrak{F}_{n+1})^{n-2}}{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}, \tag{3.7}$$

$$y_k = \frac{(1 - \mathfrak{F}_{n-1} + \mathfrak{F}_{n+1})\mathfrak{F}_n^{k-3}(1 + \mathfrak{F}_{n+1})^{n-k}}{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}, \quad (k = 3, 4, \dots, n). \tag{3.8}$$

Proof. Let

$$\Omega_2 = \begin{pmatrix} 1 & -\frac{l'_n}{\mathfrak{F}_1} & \omega_3 & \omega_4 & \cdots & \omega_n \\ 0 & 1 & -\frac{\mathfrak{F}_{n-2}}{l_n} & -\frac{\mathfrak{F}_{n-3}}{l_n} & \cdots & -\frac{\mathfrak{F}_1}{l_n} \\ \vdots & \ddots & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n \times n},$$

where

$$\omega_j = \frac{l'_n}{l_n} \mathfrak{F}_{n+1-j} - \mathfrak{F}_{n+2-j}, \quad (j = 3, 4, \dots, n),$$

l'_n and l_n are respectively given by (3.3) and (3.4). Multiplying (3.2) by Ω_2 from right,

$$\Sigma A_n \Omega_1 \Omega_2 = \begin{pmatrix} \mathfrak{F}_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & l_n & \ddots & & & & \vdots \\ \vdots & 0 & \mathfrak{F}_1 + \mathfrak{F}_{n+1} & \ddots & & & \vdots \\ \vdots & \vdots & -\mathfrak{F}_n & \ddots & \ddots & & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\mathfrak{F}_n & \mathfrak{F}_1 + \mathfrak{F}_{n+1} \end{pmatrix}_{n \times n},$$

where Σ and Ω_1 are the same as Theorem 3.1.

Let $\Sigma A_n \Omega_1 \Omega_2 = \mathcal{D} \oplus \mathcal{H}$, where $\mathcal{D} = \text{diag}(1, l_n)$ is a diagonal matrix, \mathcal{H} is as defined in Lemma 2.2, and $\mathcal{D} \oplus \mathcal{H}$ is the direct sum of \mathcal{D} and \mathcal{H} . Denote $\Omega = \Omega_1 \Omega_2$, then we obtain $A_n^{-1} = \Omega(\mathcal{D}^{-1} \oplus \mathcal{H}^{-1})\Sigma$.

Since the last row elements of the matrix Ω are $(0, 1, -\frac{\mathfrak{F}_{n-2}}{l_n}, -\frac{\mathfrak{F}_{n-3}}{l_n}, \dots, -\frac{\mathfrak{F}_2}{l_n}, -\frac{\mathfrak{F}_1}{l_n})$, the last row elements of the matrix $\Omega(\mathcal{D}^{-1} \oplus \mathcal{H}^{-1})$ are $(0, \frac{1}{l_n}, T_3, T_4, \dots, T_n)$, where

$$T_k = \sum_{i=1}^{n+1-k} -\frac{\mathfrak{F}_{n+2-k-i}\mathfrak{F}_n^{i-1}}{l_n(1 + \mathfrak{F}_{n+1})^i}, \quad (k = 3, 4, \dots, n).$$

Using (2.4), we simplify the above equation as

$$T_k = -\frac{\mathfrak{F}_n^{n+2-k}(1 + \mathfrak{F}_{n+1})^{k-3} + [\mathfrak{F}_{n+1-k}(1 - \mathfrak{F}_{n-1}) + \mathfrak{F}_{n-k}\mathfrak{F}_n](1 + \mathfrak{F}_{n+1})^{n-2}}{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}, \quad (k = 3, 4, \dots, n). \tag{3.9}$$

The last row of $A_n^{-1} = \text{SCirc}(y_1, y_2, \dots, y_n)$ is $(-y_2, -y_3, \dots, -y_n, y_1)$, which is given by the following equations:

$$-y_2 = \frac{3}{l_n} - T_3,$$

$$\begin{aligned} -y_3 &= T_n, \\ -y_4 &= T_{n-1} - 3T_n, \\ -y_k &= T_{n-k+3} - 3T_{n-k+4} + T_{n-k+5}, \quad k = 5, 6, \dots, n, \\ y_1 &= \frac{1}{l_n} - 3T_3 + T_4. \end{aligned}$$

Applying (3.9) into above equations, we compute

$$\begin{aligned} y_1 &= \frac{(\mathfrak{F}_{n-1} - 1)\mathfrak{F}_n^{n-2} + (1 + \mathfrak{F}_{n+1})^{n-1}}{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}, \\ y_2 &= -\frac{\mathfrak{F}_n^{n-1} + (3 + \mathfrak{F}_{n+2})(1 + \mathfrak{F}_{n+1})^{n-2}}{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}, \\ y_k &= \frac{(1 - \mathfrak{F}_{n-1} + \mathfrak{F}_{n+1})\mathfrak{F}_n^{k-3}(1 + \mathfrak{F}_{n+1})^{n-k}}{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}, \quad (k = 3, 4, \dots, n). \end{aligned} \quad \square$$

3.2. Norms and spread of skew circulant matrix involving the product of Fibonacci and Lucas numbers

In this subsection, we further investigate the properties of skew circulant matrices involving the product of Fibonacci and Lucas numbers, such as their norms, lower and upper bounds for their spread.

Theorem 3.4. Let $A_n = \text{SCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be a matrix as in (1.3). Then the maximum column sum matrix norm, the maximum row sum matrix norm and the Frobenius norm of A_n are given by

$$\|A_n\|_1 = \|A_n\|_\infty = 2\mathfrak{F}_n - \mathfrak{F}_{n-1} - 1, \tag{3.10}$$

$$\|A_n\|_F = \sqrt{\frac{n(\mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - 8n - 7)}{5}}, \tag{3.11}$$

where \mathfrak{F}_n is the n th product of Fibonacci and Lucas numbers.

Proof. According to the Definition 4 in [18], (2.1), and (2.3), we have

$$\begin{aligned} \|A_n\|_1 &= \sum_{i=1}^n |\mathfrak{F}_i| = 2\mathfrak{F}_n - \mathfrak{F}_{n-1} - 1, \\ \|A_n\|_\infty &= \sum_{i=1}^n |\mathfrak{F}_i| = 2\mathfrak{F}_n - \mathfrak{F}_{n-1} - 1, \\ \|A_n\|_F &= \sqrt{n \sum_{i=1}^n |\mathfrak{F}_i|^2} = \sqrt{\frac{n(\mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - 8n - 7)}{5}}. \end{aligned} \quad \square$$

Theorem 3.5. Let $A'_n = \text{SCirc}(\mathfrak{F}_1, -\mathfrak{F}_2, \dots, -\mathfrak{F}_{n-1}, \mathfrak{F}_n)$ be an odd-order alternative skew circulant matrix. Then

$$\|A'_n\|_2 = 2\mathfrak{F}_n - \mathfrak{F}_{n-1} - 1,$$

where \mathfrak{F}_n is the n th product of Fibonacci and Lucas numbers.

Proof. In [13], we have the eigenvalues of A'_n

$$\lambda_j(A'_n) = \sum_{i=1}^n (-1)^{i-1} \mathfrak{F}_i (\omega^j)^{i-1}, \quad (j = 0, 1, \dots, n-1).$$

Using the absolute value inequality,

$$|\lambda_j(A'_n)| \leq \sum_{i=1}^n |(-1)^{i-1} \mathfrak{F}_i| \cdot |(\omega^j \eta)^{i-1}| = \sum_{i=1}^n \mathfrak{F}_i, \quad (j = 0, 1, \dots, n-1).$$

Since n is odd, $\sum_{i=1}^n \mathfrak{F}_i$ is an eigenvalue of A'_n , which is

$$\begin{pmatrix} \mathfrak{F}_1 & -\mathfrak{F}_2 & \mathfrak{F}_3 & \cdots & \mathfrak{F}_n \\ -\mathfrak{F}_n & \ddots & \ddots & \ddots & \vdots \\ \mathfrak{F}_{n-1} & \ddots & \ddots & \ddots & \mathfrak{F}_3 \\ \vdots & \ddots & \ddots & \ddots & -\mathfrak{F}_2 \\ \mathfrak{F}_2 & \cdots & \mathfrak{F}_{n-1} & -\mathfrak{F}_n & \mathfrak{F}_1 \end{pmatrix}_{n \times n} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} = \sum_{i=1}^n \mathfrak{F}_i \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix}.$$

To sum up, we obtain

$$\max_{0 \leq j \leq n-1} |\lambda_j(A'_n)| = \sum_{i=1}^n \mathfrak{F}_i. \tag{3.12}$$

Since skew circulant matrices A'_n satisfies $(A'_n)^H A'_n = A'_n (A'_n)^H$ ($(A'_n)^H$ is the conjugate transpose of A'_n), skew circulant matrices are normal. By Lemma 7 in [18], (2.1), and (3.12), we have

$$\|A'_n\|_2 = \sum_{i=1}^n \mathfrak{F}_i = 2\mathfrak{F}_n - \mathfrak{F}_{n-1} - 1. \quad \square$$

Theorem 3.6. Let $A_n = \text{SCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be a matrix as in (1.3). Then a lower bound and an upper bound for the spread $s(A_n)$ of A_n are given as follows:

$$\left| \frac{2 - 2n + (n + 4)\mathfrak{F}_n - (n + 2)\mathfrak{F}_{n+1}}{n - 1} \right| \leq s(A_n) \leq \sqrt{\frac{2n(\mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - 8n - 7)}{5}} - 2n, \tag{3.13}$$

where \mathfrak{F}_n is the n^{th} product of Fibonacci and Lucas numbers.

Proof. Note that the trace of A_n is $\text{tr}A_n = n\mathfrak{F}_1 = n$. The sum of the off-diagonal of A_n is

$$\ell(A_n) = \sum_{k=2}^n (n - k + 1)\mathfrak{F}_k - \sum_{k=2}^n (k - 1)\mathfrak{F}_k = (n + 2) \sum_{k=2}^n \mathfrak{F}_k - 2 \sum_{k=2}^n k\mathfrak{F}_k.$$

By (2.1) and (2.2), we have

$$\ell(A_n) = 2 - 2n + (n + 4)\mathfrak{F}_n - (n + 2)\mathfrak{F}_{n+1}.$$

Since skew circulant matrix A_n satisfies $A_n^H A_n = A_n A_n^H$ (A_n^H is the conjugate transpose of A_n), skew circulant matrix is normal. Since A_n is a normal and real matrix, by Definition 5 and Lemma 6 in [18] and (3.11), we get

$$\left| \frac{2 - 2n + (n + 4)\mathfrak{F}_n - (n + 2)\mathfrak{F}_{n+1}}{n - 1} \right| \leq s(A_n) \leq \sqrt{\frac{2n(\mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - 8n - 7)}{5}} - 2n. \quad \square$$

4. Determinant, inverse, norms and spread of skew left circulant matrix involving the product of Fibonacci and Lucas numbers

In this section, let $A_n'' = \text{SLCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be a skew left circulant matrix. According to Lemma 2.3, we know that skew circulant matrix and skew left circulant matrix can be transformed into each other. Therefore, using the obtained conclusions in Section 3.1, we get a determinant explicit formula for A_n'' . Besides, we prove that A_n'' is invertible for every positive integer n and compute the inverse of A_n'' . We also study the properties of skew left circulant matrices involving the product of Fibonacci and Lucas numbers.

4.1. Determinant and inverse of skew left circulant matrix involving the product of Fibonacci and Lucas numbers

In this subsection, we get a determinant explicit formula for A_n'' in Theorem 4.1. Besides, we prove that A_n'' is invertible for every positive integer n and compute the inverse of A_n'' in Theorems 4.2 and 4.3, respectively.

Theorem 4.1. Let $A_n'' = \text{SLCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be a matrix as in (1.4) for a positive integer n . Then

$$\det A_n'' = (-1)^{\frac{n(n-1)}{2}} \frac{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}{1 - \mathfrak{F}_{n-1} + \mathfrak{F}_{n+1}}, \tag{4.1}$$

where \mathfrak{F}_n is the n^{th} product of Fibonacci and Lucas numbers.

Theorem 4.2. Let $A_n'' = \text{SLCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be a matrix as in (1.4) for every positive interger n . Then A_n'' is invertible.

Theorem 4.3. Let $A_n'' = \text{SLCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be a matrix as in (1.4) for a positive integer $n > 1$. Then

$$(A_n'')^{-1} = \text{SLCirc}(y_1'', y_2'', \dots, y_n''), \tag{4.2}$$

where

$$y_1'' = \frac{(\mathfrak{F}_{n-1} - 1)\mathfrak{F}_n^{n-2} + (1 + \mathfrak{F}_{n+1})^{n-1}}{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}, \tag{4.3}$$

$$y_k'' = -\frac{\mathfrak{F}_n^{n-1-k}(1 + \mathfrak{F}_{n+1})^{k-2}[\mathfrak{F}_n^2 + (1 - \mathfrak{F}_{n-1})(1 + \mathfrak{F}_{n+1})]}{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}, \quad (k = 2, 3, \dots, n - 1), \tag{4.4}$$

$$y_n'' = \frac{\mathfrak{F}_n^{n-1} + (3 + \mathfrak{F}_{n+2})(1 + \mathfrak{F}_{n+1})^{n-2}}{\mathfrak{F}_n^n + (1 + \mathfrak{F}_{n+1})^n}. \tag{4.5}$$

4.2. Norms and spread of skew left circulant matrix involving the product of Fibonacci and Lucas numbers

Like Section 3.2, we study some properties of skew left circulant matrices involving the product of Fibonacci and Lucas numbers.

Theorem 4.4. Let $A_n'' = \text{SLCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be a matrix as in (1.4). Then three kinds norms of A_n'' are given by

$$\|A_n''\|_1 = \|A_n''\|_\infty = 2\mathfrak{F}_n - \mathfrak{F}_{n-1} - 1, \tag{4.6}$$

$$\|A_n''\|_F = \sqrt{\frac{n(\mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - 8n - 7)}{5}}, \tag{4.7}$$

where \mathfrak{F}_n is the n^{th} product of Fibonacci and Lucas numbers.

Proof. Using the similar method in Theorem 3.4, the conclusion is obtained. □

Theorem 4.5. Let $A_n''' = \text{SLCirc}(\mathfrak{F}_1, -\mathfrak{F}_2, \dots, -\mathfrak{F}_{n-1}, \mathfrak{F}_n)$ be an odd-order alternative skew left circulant matrix. Then

$$\|A_n'''\|_2 = 2\mathfrak{F}_n - \mathfrak{F}_{n-1} - 1,$$

where \mathfrak{F}_n is the n^{th} product of Fibonacci and Lucas numbers.

Proof. By Lemma 2.4, we have

$$\lambda_j(A_n''') = \pm \left| \sum_{i=1}^n (-1)^{i-1} \mathfrak{F}_i \omega^{(j-\frac{1}{2})(k-1)} \right|, \quad (j = 1, 2, \dots, \frac{n-1}{2}), \quad \lambda_{\frac{n+1}{2}}(A_n''') = \sum_{i=1}^n \mathfrak{F}_i. \quad (4.8)$$

Therefore,

$$|\lambda_j(A_n''')| \leq \sum_{i=1}^n |(-1)^{i-1} \mathfrak{F}_i (-1)^{i-1}| = \sum_{i=1}^n \mathfrak{F}_i, \quad (j = 1, 2, \dots, \frac{n-1}{2}). \quad (4.9)$$

By (4.8) and (4.9), we gain

$$\max_{0 \leq i \leq n+1} |\lambda_i(A_n''')| = \sum_{i=1}^n \mathfrak{F}_i. \quad (4.10)$$

Since skew left circulant matrix A_n''' satisfies $(A_n''')^H A_n''' = A_n''' (A_n''')^H$ ($(A_n''')^H$ is the conjugate transpose of A_n'''), skew circulant matrix is normal. By Lemma 7 in [18], (2.1), and (4.10), we obtain

$$\|A_n'''\|_2 = 2\mathfrak{F}_n - \mathfrak{F}_{n-1} - 1. \quad \square$$

Theorem 4.6. Suppose that $A_n'' = \text{SLCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$ be a matrix as in (1.4). The lower and upper bounds for the spread of A_n'' are

$$2\mathfrak{F}_n \leq s(A_n'') \leq \begin{cases} \sqrt{\frac{2n(\mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - 8n - 7)}{5} - \frac{2(\mathfrak{F}_{n+1} + \mathfrak{F}_n + 1)^2}{25n}}, & \text{if } n \text{ is odd,} \\ \sqrt{\frac{2n(\mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - 8n - 7)}{5}}, & \text{if } n \text{ is even,} \end{cases}$$

where \mathfrak{F}_n is the n^{th} product of Fibonacci and Lucas numbers.

Proof. Since A_n'' is a symmetric matrix, by Definition 5 and Lemma 6 in [18],

$$2 \max |A_i''| \leq s(A_n'') \leq \sqrt{2\|A_n''\|_F^2 - \frac{2}{n} |\text{tr} A_n''|^2}.$$

For A_n'' ,

$$2 \max |A_i''| = 2\mathfrak{F}_n.$$

On the other hand, according to Theorem 4.4,

$$\|A_n''\|_F = \sqrt{\frac{n(\mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - 8n - 7)}{5}}.$$

If n is odd, we have

$$\text{tr}(A_n'') = \mathfrak{F}_1 - \mathfrak{F}_2 + \mathfrak{F}_3 - \dots + \mathfrak{F}_n = \sum_{i=0}^{\frac{n-1}{2}} \mathfrak{F}_{2i+1} - \sum_{i=1}^{\frac{n-1}{2}} \mathfrak{F}_{2i}.$$

Note that

$$\mathfrak{F}_{n+1} = 3\mathfrak{F}_n - \mathfrak{F}_{n-1}, \quad \text{where } \mathfrak{F}_1 = 1, \mathfrak{F}_2 = 3,$$

we conclude that

$$\begin{aligned} \sum_{i=0}^{\frac{n-1}{2}} \mathfrak{F}_{2i+1} - \sum_{i=1}^{\frac{n-1}{2}} \mathfrak{F}_{2i} &= 2 \sum_{i=1}^{\frac{n-1}{2}} \mathfrak{F}_{2i} - \sum_{i=0}^{\frac{n-1}{2}} \mathfrak{F}_{2i+1} + \mathfrak{F}_1 + \mathfrak{F}_n, \\ \sum_{i=1}^{\frac{n-1}{2}} \mathfrak{F}_{2i} &= 3 \sum_{i=0}^{\frac{n-1}{2}} \mathfrak{F}_{2i+1} - 3\mathfrak{F}_1 - 3\mathfrak{F}_n - \sum_{i=1}^{\frac{n-1}{2}} \mathfrak{F}_{2i} + \mathfrak{F}_2 + \mathfrak{F}_{n-1}. \end{aligned}$$

Hence

$$\text{tr}(A_n'') = \frac{1}{5}(\mathfrak{F}_{n+1} + \mathfrak{F}_n + 1).$$

If n is even, then

$$\text{tr}(A_n'') = \mathfrak{F}_1 - \mathfrak{F}_1 + \mathfrak{F}_3 - \mathfrak{F}_3 + \dots - \mathfrak{F}_{n-1} = 0.$$

So the result is as follows:

$$2\mathfrak{F}_n \leq s(A_n'') \leq \begin{cases} \sqrt{\frac{2n(\mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - 8n - 7)}{5}} - \frac{2(\mathfrak{F}_{n+1} + \mathfrak{F}_n + 1)^2}{25n}, & \text{if } n \text{ is odd,} \\ \sqrt{\frac{2n(\mathfrak{F}_{n+1}^2 - \mathfrak{F}_n^2 - 8n - 7)}{5}}, & \text{if } n \text{ is even.} \end{cases}$$

□

5. Algorithms and numerical examples

In this section, we give two algorithms for computing inverses of skew circulant and skew left circulant matrices involving the product of Fibonacci and Lucas numbers. Then, several examples are given to show the correctness of these algorithms. Besides, we compute determinants, three kinds of norms and lower & upper bounds for spread of these matrices.

Based on Theorem 3.1, given the skew circulant matrix involving the product of Fibonacci and Lucas numbers $A_n = \text{SCirc}(\mathfrak{F}_1, \dots, \mathfrak{F}_n)$, the algorithm for finding its inverse is as follows.

Algorithm 5.1.

Step 1: Input n and generate the product of Fibonacci and Lucas numbers by (1.2).

Step 2: Compute y_i ($i = 1, 2, \dots, n$) via (3.6)-(3.8).

Step 3: Output $A_n^{-1} = \text{SCirc}(y_1, y_2, \dots, y_n)$ by (3.5).

Example 5.2. Consider a 5×5 skew circulant matrix involving the product of Fibonacci and Lucas numbers A_5

$$A_5 = \begin{bmatrix} 1 & 3 & 8 & 21 & 55 \\ -55 & 1 & 3 & 8 & 21 \\ -21 & -55 & 1 & 3 & 8 \\ 8 & -21 & -55 & 1 & 3 \\ 3 & 8 & -21 & -55 & 1 \end{bmatrix}.$$

Above all, according to (3.1), $\det A_5 = 516805000$. In particular, by (3.10) and (3.11), the three kinds norms of A_5 are given by

$$\|A_5\|_1 = \|A_5\|_\infty = 88, \quad \|A_5\|_F = \sqrt{17664}.$$

By (3.13), the lower and upper bounds for spread of A_5 are

$$\frac{521}{4} \leq s(A_5) \leq \sqrt{35318}.$$

As for A_5^{-1} , according to Algorithm 5.1, we have

$$y_1 = \frac{142521}{20672200}, \quad y_2 = -\frac{373641}{20672200}, \quad y_3 = \frac{841}{20672200}, \quad y_4 = \frac{319}{20672200}, \quad y_5 = \frac{121}{20672200}.$$

Finally,

$$(A_5)^{-1} = \begin{bmatrix} \frac{142521}{20672200} & -\frac{373641}{20672200} & \frac{841}{20672200} & \frac{319}{20672200} & \frac{121}{20672200} \\ -\frac{121}{20672200} & \frac{142521}{20672200} & -\frac{373641}{20672200} & \frac{841}{20672200} & \frac{319}{20672200} \\ -\frac{319}{20672200} & -\frac{121}{20672200} & \frac{142521}{20672200} & -\frac{373641}{20672200} & \frac{841}{20672200} \\ -\frac{841}{20672200} & -\frac{319}{20672200} & -\frac{121}{20672200} & \frac{142521}{20672200} & -\frac{373641}{20672200} \\ \frac{373641}{20672200} & -\frac{841}{20672200} & -\frac{319}{20672200} & -\frac{121}{20672200} & \frac{142521}{20672200} \end{bmatrix}.$$

Meanwhile, based on Theorem 4.3, given a skew left circulant matrix involving the product of Fibonacci and Lucas numbers $A_n'' = \text{SLCirc}(\tilde{f}_1, \dots, \tilde{f}_n)$, the algorithm for finding its inverse is as follows.

Algorithm 5.3.

- Step 1: Input n and generate the product of Fibonacci and Lucas numbers by (1.2).
- Step 2: Compute y_i'' ($i = 1, 2, \dots, n$) via (4.3)-(4.5).
- Step 3: Output $A_n''^{-1} = \text{SLCirc}(y_1'', y_2'', \dots, y_n'')$ by (4.2).

Example 5.4. Consider a 5×5 skew left circulant matrix involving the product of Fibonacci and Lucas numbers A_5'' :

$$A_5'' = \begin{bmatrix} 1 & 3 & 8 & 21 & 55 \\ 3 & 8 & 21 & 55 & -1 \\ 8 & 21 & 55 & -1 & -3 \\ 21 & 55 & -1 & -3 & -8 \\ 55 & -1 & -3 & -8 & -21 \end{bmatrix}.$$

Firstly, we get $\det A_5'' = 516805000$ from (4.1) and the three kinds norms of A_5'' by (4.6) and (4.7)

$$\|A_5''\|_1 = \|A_5''\|_\infty = 88, \quad \|A_5''\|_F = \sqrt{17664}.$$

In Theorem 4.6, the lower and upper bounds for spread of A_5'' are

$$110 \leq s(A_5'') \leq \sqrt{34688}.$$

As for $A_5''^{-1}$, according to Algorithm 5.3, compute

$$y_1'' = \frac{142521}{20672200}, \quad y_2'' = -\frac{121}{20672200}, \quad y_3'' = -\frac{319}{20672200}, \quad y_4'' = -\frac{841}{20672200}, \quad y_5'' = \frac{373641}{20672200}.$$

Finally,

$$(A_5'')^{-1} = \begin{bmatrix} \frac{142521}{20672200} & -\frac{121}{20672200} & -\frac{319}{20672200} & -\frac{841}{20672200} & \frac{373641}{20672200} \\ -\frac{121}{20672200} & -\frac{319}{20672200} & -\frac{841}{20672200} & \frac{373641}{20672200} & -\frac{142521}{20672200} \\ -\frac{319}{20672200} & -\frac{841}{20672200} & \frac{373641}{20672200} & -\frac{142521}{20672200} & \frac{121}{20672200} \\ -\frac{841}{20672200} & \frac{373641}{20672200} & -\frac{142521}{20672200} & \frac{121}{20672200} & \frac{319}{20672200} \\ \frac{373641}{20672200} & -\frac{142521}{20672200} & \frac{121}{20672200} & \frac{319}{20672200} & \frac{841}{20672200} \end{bmatrix}.$$

6. Conclusions

We discuss the invertibility of the skew circulant and skew left circulant matrices involving the product of Fibonacci and Lucas numbers and present the determinant and inverse matrices by constructing the transformation matrices. The four kinds of norms and bounds for the spread of these matrices are given, respectively. To test the validity of our method, we design two algorithms (Algorithms 5.1 and 5.3), and then conduct two examples.

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