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An extension of the optimal homotopy asymptotic method with applications to nonlinear coupled partial differential equations

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Abstract

In this paper, we applied an extension of optimal homotopy asymptotic method (EOHAM) for the approximate solution of coupled partial differential equations (PDEs). The obtained results are compared with other results for its efficiency. The order of convergence and residuals are plotted.

Keywords: EOHAM, HPM, exact, coupled PDEs.

2010 MSC: 35A25, 35C05.

1. Introduction

The differential equations have vital role in science and technology. Numerical and analytical methods are used for the solutions of nonlinear problems. Numerical methods required linearization, discritization and are slow convergence. To escape from these difficulties the Analytical methods such as like variational iterative method (VIM) [3], adomian decomposition method (ADM) [2], differential transform method (DTM) [17], and Perturbation method were introduced [4, 9]. These methods contain a small parameter which cannot be found easily. New analytic Homotopy analysis method (HAM) [8] and Homotopy perturbation method (HPM) [5] were introduced. These methods combined the homotopy with the perturbation techniques. Recently, Marinca et al. introduced OHAM [6, 8] for the solution of nonlinear problems which made the perturbation methods independent of the assumption of small parameters and huge computational work. The motivation of this paper is to implement the EOHAM for the solution of coupled PDEs. In [10–16] OHAM has been proved to be valuable for obtaining an approximate solution of single partial differential equation (PDE). In the succeeding section, the basic idea of EOHAM is formulated for the solution of NPDEs. The effectiveness and efficiency of EOHAM is shown in Section 3.

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2. Fundamental mathematical theory of EOHAM

Consider a system of three partial differential equations

$$A_1(\xi,\zeta)+s_1(x,t)=0,\ \ A_2(\xi,\zeta)+s_2(x,t)=0,\ x\in\Omega,\quad B_1(\xi,\frac{\partial\xi}{\partial x})=0,\ \ B_2(\zeta,\frac{\partial\zeta}{\partial x})=0,\ x\in\Gamma. \tag{2.1}$$

$A_1,\ A_2$ can be divided into two parts

$$A_1=L_1+N_1,\quad A_2=L_2+N_2. \tag{2.2}$$

L_1, L_2 contain the linear parts while N_1, N_2 contain the nonlinear parts of the system of partial differential equations. According to OHAM, construct a system of optimal homotopies

$$\alpha(x,t;p):\phi\times[0,1]\to\Re,\quad \beta(x,t;p):\psi\times[0,1]\to\Re$$

satisfying

$$\begin{aligned} K\Big(\alpha(x,t;p),p\Big)&=(1-p)\Big\{L_1(\alpha(x,t;p))+s_1(x,t)\Big\}-K_1(p)\Big\{A_1(\alpha(x,t;p))+s_1(x,t)\Big\},\\ K\Big(\beta(x,t;p),p\Big)&=(1-p)\Big\{L_2(\beta(x,t;p))+s_2(x,t)\Big\}-K_2(p)\Big\{A_2(\alpha(x,t;p))+s_2(x,t)\Big\}. \end{aligned} \tag{2.3}$$

We have

$$\begin{aligned} &p=0=L_1(\alpha(x,t;0))+s_1(x,t)=0,\\ &p=0=L_2(\alpha(x,t;0))+s_2(x,t)=0,\\ &p=1=K_1(1)\Big\{A_1(\alpha(x,t;1))+s_1(x,t)=0\Big\},\\ &p=1=K_2(1)\Big\{A_2(\beta(x,t;1))+s_2(x,t)=0\Big\}, \end{aligned}$$

Obviously, when $p=0$ and $p=1$ we obtain

$$\alpha(x,t;0)=\xi_0(x,t),\ \beta(x,t;0)=\zeta_0(x,t),\quad \alpha(x,t;1)=\xi_1(x,t),\ \beta(x,t;1)=\zeta_1(x,t).$$

For $r=0$

$$L_1(\xi_0(x,t))+s_1(x,t)=0,\ B_1(\xi_0,\frac{\partial\xi_0}{\partial x})=0,\quad L_2(\zeta_0(x,t))+s_2(x,t)=0,\ B_2(\zeta_0,\frac{\partial\zeta_0}{\partial x})=0.$$

We choose auxiliary functions $K_1(r), K_2(r)$ in the form

$$H_1(r)=rK_{11}+r^2K_{12}+r^3K_{13}+\cdots+r^mK_{1m},\quad H_2(r)=rK_{21}+r^2K_{22}+r^3K_{23}+\cdots+r^mK_{2m}. \tag{2.4}$$

To get the approximate solutions, we expand $\alpha(x,t;r,K_{1i}),\beta(x,t;r,K_{2i})$ by Taylors series about p in the following manner,

$$\alpha(x,t;r,K_{1i})=f_0(x,t)+\sum_{k\geqslant 1}f_k(x,t;k_{1i})r^k,\quad \beta(x,t;r,K_{2i})=g_0(x,t)+\sum_{l\geqslant 1}g_l(x,t;k_{2i})r^k, \tag{2.5}$$

where $k=l=n=i=1,2,3,\ldots$. Using Eq. (2.4)-(2.5) into Eq. (2.3) and equating the coefficient of like powers of r, we have

$$\begin{aligned} &L_1(\xi_1(x,t))-L_1(\xi_0(x,t))=K_{11}\Big(L_1(\xi_0(x,t))\Big)+N_1\Big(\xi_0(x,t)\Big),\ B_1(\xi_1,\frac{\partial\xi_1}{\partial x})=0,\\ &L_2(\zeta_1(x,t))-L_2(\zeta_0(x,t))=K_{21}\Big(L_2(\zeta_0(x,t))\Big)+N_2\Big(\zeta_0(x,t)\Big),\ B_2(\zeta_1,\frac{\partial\zeta_1}{\partial x})=0, \end{aligned}$$

$$
\begin{aligned}
L_1(\xi_k(x,t)) - L_1(\xi_{k-1}(x,t)) &= \sum_{i=1}^{k} C_{1i}\Big[L_1\Big(\xi_{k-i}(x,t),\zeta_{k-i}(x,t)\Big) \\
&\qquad + N_1\Big(\xi_{k-i}(x,t),\zeta_{k-i}(x,t)\Big)\Big], B_1(\xi_k,\frac{\partial \xi_k}{\partial x}) = 0, \\
L_2(\zeta_k(x,t)) - L_2(\zeta_{k-1}(x,t)) &= \sum_{i=1}^{k} C_{2i}\Big[L_2\Big(\zeta_{k-i}(x,t),\zeta_{k-i}(x,t)\Big) + N_2\Big(\xi_{k-i}(x,t),\zeta_{k-i}(x,t)\Big)\Big], \\
& B_2(\zeta_k,\frac{\partial \zeta_k}{\partial x}) = 0,\ k = 2,3,\ldots.
\end{aligned}
$$

It has been observed that the convergence of the series (2.5) depends upon the auxiliary constants. If it is convergent at $r = 1$, one has

$$
\alpha^r(x,t;K_{1i}) = \xi_0(x,t) + \sum_{k\geqslant 1}\xi_k(x,t;K_{1i}),\ \beta^r(x,t;K_{2i}) = \zeta_0(x,t) + \sum_{l\geqslant 1}\zeta_l(x,t;K_{2i}), i = 1,2,\ldots,m. \tag{2.6}
$$

Substituting Eq. (2.6) into Eq. (2.1), it results the following expression for residuals

$$
\begin{aligned}
R_1(x,t;K_{1i}) &= L_1(\alpha^r(x,t;K_{1i})) + s_1(x,t) + N_1(\alpha^r(x,t;K_{1i})), \\
R_2(x,t;K_{2i}) &= L_2(\beta^r(x,t;K_{2i})) + s_2(x,t) + N_2(\beta^r(x,t;K_{2i})).
\end{aligned}
$$

If $R_1(x,t;C_{1i}) = 0, R_2(x,t;C_{2i}) = 0$, then $\alpha^r(x,t;C_{1i}), \beta^r(x,t;C_{2i})$ will be the exact solutions of the problem. Generally it does not happen, especially in nonlinear problems.

For the computation of auxiliary constants, one can apply the method of least squares as under

$$
J_1(K_{1i}) = \int_0^t\int_\Omega R_1^2(x,t;K_{1i})dxdt, \quad J_2(K_{2i}) = \int_0^t\int_\psi R_2^2(x,t;K_{2i})dxdt,
$$

and

$$
\frac{\partial J_1}{\partial K_{ij}} = \frac{\partial J_2}{\partial K_{ij}} = 0,\ i = j = 1,2,\ldots,m.
$$

The m^{th} order approximate solution can be obtained by these constants.

3. Application of EOHAM to nonlinear coupled Burgers equations

Example 3.1. Consider the nonlinear coupled Burgers equations with initial conditions [1]

$$
\frac{\partial \xi}{\partial t} - \frac{\partial^2 \xi}{\partial x^2} - 2\xi\frac{\partial \xi}{\partial x} + \xi\frac{\partial \zeta}{\partial x} + \zeta\frac{\partial \xi}{\partial x}, \quad \frac{\partial \zeta}{\partial t} - \frac{\partial^2 \zeta}{\partial x^2} - 2\zeta\frac{\partial \zeta}{\partial x} - \xi\frac{\partial \zeta}{\partial x} - \zeta\frac{\partial \xi}{\partial x}, \tag{3.1}
$$

$$
\zeta(x,0) = \sin x, \quad \xi(x,0) = \sin x. \tag{3.2}
$$

The exact solutions of Eq. (3.2) are

$$
\xi(x,t) = e^{-t}\sin x, \quad \zeta(x,t) = e^{-t}\sin x.
$$

Applying the technique discussed in Section 2 we have

$$
\begin{aligned}
(1-l)\frac{\partial \xi(x,t)}{\partial t} - U_1(l)\Big[\frac{\partial \xi}{\partial t} - \frac{\partial^2 \xi}{\partial x^2} - 2\xi\frac{\partial \xi}{\partial x} + \xi\frac{\partial \zeta}{\partial x} + \zeta\frac{\partial \xi}{\partial x}\Big] &= 0, \\
(1-l)\frac{\partial \zeta(x,t)}{\partial t} - U_2(l)\Big[\frac{\partial \zeta}{\partial t} - \frac{\partial^2 \zeta}{\partial x^2} - 2\zeta\frac{\partial \zeta}{\partial x} - \zeta\frac{\partial \zeta}{\partial x} - \zeta\frac{\partial \xi}{\partial x}\Big] &= 0.
\end{aligned}
$$

We consider

$$
\xi = \xi_0 + l\xi_1 + l^2\xi_2, \quad \zeta = \zeta_0 + l\zeta_1 + l^2\zeta_2, \qquad U_1(l) = lC_{11} + l^2C_{12}, \quad U_2(l) = lC_{21} + l^2C_{22}.
$$

3.1. Zeroth order system

$$\frac{\partial \xi_0}{\partial t} = 0, \ \frac{\partial \xi_0}{\partial t} = 0.$$

with initial conditions

$$\xi_0(x,0) = \sin x, \ \zeta_0(x,0) = \sin x.$$

Its solution is

$$\xi_0(x,t) = \sin x, \ \zeta_0(x,t) = \sin x. \tag{3.3}$$

3.2. First order system

$$\begin{aligned}
\frac{\partial \xi_1(x,t)}{\partial t} &= (1+C_{11})\frac{\partial \xi_0}{\partial t} - 2C_{11}\xi_0\frac{\partial \xi_0}{\partial x} + C_{11}\xi_0\frac{\partial \xi_0}{\partial x} + C_{11}\xi_0\frac{\partial \zeta_0}{\partial x} - C_{11}\frac{\partial^2 \xi_0}{\partial x^2},\\
\frac{\partial \zeta_1(x,t)}{\partial t} &= (1+C_{21})\frac{\partial \zeta_0}{\partial t} - C_{21}\zeta_0\frac{\partial \xi_0}{\partial x} - C_{21}\xi_0\frac{\partial \zeta_0}{\partial x} - 2C_{21}\zeta_0\frac{\partial \zeta_0}{\partial x} - C_{21}\frac{\partial^2 \zeta_0}{\partial x^2},
\end{aligned}$$

with initial conditions

$$\xi_1(x,0) = 0, \ \zeta_1(x,0) = 0.$$

Its solution is

$$\xi_1(x,t;C_{11}) = C_{11}\sin x, \ \zeta_1(x,t;C_{21}) = C_{21}t(\sin x - 4\cos x \sin x). \tag{3.4}$$

3.3. Second order system

$$\begin{aligned}
\frac{\partial \xi_2(x,t;C_{11},C_{12})}{\partial t} &= \Bigg[C_{12}\frac{\partial \xi_0}{\partial t} + (1+C_{11})\frac{\partial \xi_1}{\partial t} - 2C_{12}\xi_0\frac{\partial \xi_0}{\partial x} - 2C_{11}\xi_1\frac{\partial \xi_0}{\partial x} + C_{12}\zeta_0\frac{\partial \xi_0}{\partial x} + C_{11}\zeta_1\frac{\partial \xi_0}{\partial x}\\
&\quad - 2C_{11}\xi_0\frac{\partial \xi_1}{\partial x} + C_{11}\zeta_0\frac{\partial \xi_1}{\partial x} + C_{12}\xi_0\frac{\partial \zeta_1}{\partial x} + C_{11}\xi_1\frac{\partial \zeta_0}{\partial x} + C_{11}\xi_0\frac{\partial \zeta_1}{\partial x} - C_{12}\frac{\partial^2 \xi_0}{\partial x^2} - C11\frac{\partial^2_1}{\partial x^2}\Bigg],\\
\frac{\partial \zeta_2(x,t;C_{21},C_{22})}{\partial t} &= \Bigg[C_{22}\frac{\partial \zeta_0}{\partial t} + (1+C_{21})\frac{\partial \zeta_1}{\partial t} - C_{22}\zeta_0\frac{\partial \xi_0}{\partial x} - C_{21}\zeta_1\frac{\partial \xi_0}{\partial x} - C_{21}\zeta_0\frac{\partial_1}{\partial x} - C_{22}\xi_0\frac{\partial \zeta_0}{\partial x} - C_{21}\xi_1\frac{\partial \zeta_0}{\partial x}\\
&\quad - 2C_{22}\zeta_0\frac{\partial \zeta_1}{\partial x} - 2C_{21}\zeta_1\frac{\partial \zeta_0}{\partial x} - C_{21}\xi_0\frac{\partial \zeta_1}{\partial x} - 2C_{21}\zeta_0\frac{\partial \zeta_1}{\partial x} - C_{22}\frac{\partial^2 \zeta_0}{\partial x^2} - C_{21}\frac{\partial^2 \zeta_1}{\partial x^2}\Bigg],
\end{aligned}$$

with initial conditions

$$\xi_2(x,0) = 0, \ \zeta_2(x,0) = 0.$$

Its solution is

$$\begin{aligned}
\xi_2(x,t;C_{11},C_{12},C_{21}) &= \frac{t\sin t}{2}\Bigg[2C_{11} + (2+t-2t\cos x)C_{11}^2 + 2C_{12}\\
&\quad + (2t\cos x - 2t + 6t\cos 2x)C_{11}C_{21}\Bigg],\\
\zeta_2(x,t;C_{11},C_{21},C_{22}) &= \frac{1}{2}\Bigg[(2t\sin x - 8t\cos x\sin x)C_{21} - 2t^2\cos x\sin x C_{11}C_{21} + (2t\sin x + 7t^2\sin x\\
&\quad - 8t\cos x\sin x - 22t^2\cos x\sin x + 18t^2 2\cos 2x\sin x)C_{21}^2\\
&\quad + (2t\sin x - 8t\cos x\sin x)C_{22}\Bigg].
\end{aligned} \tag{3.5}$$

Adding Eqs. (3.3), (3.4), and (3.5), we obtain

$$
\begin{aligned}
\xi(x,t;C_{11},C_{12},C_{21}) &= \sin x + C_{11}t\sin x + \frac{t\sin t}{2}\Bigg[2C_{11} + (2+t-2t\cos x)C_{11}^2 + 2C_{12} \\
&\qquad + (2t\cos x - 2t + 6t\cos 2x)C_{11}C_{21}\Bigg], \\
\zeta(x,t;C_{11},C_{21},C_{22}) &= \Bigg[\sin x + C_{21}t(\sin x - 5\cos x\sin x) \\
&\qquad + \frac{1}{2}\Bigg\{(2t\sin x - 8t\cos x\sin x)C_{21} - 2t^2\cos x\sin x C_{11}C_{21} \\
&\qquad + (2t\sin x + 7t^2\sin x - 8t\cos x\sin x \\
&\qquad - 22t^2\cos x\sin x + 18t^2\cos 2x\sin x)C_{21}^2 + (2t\sin x - 8t\cos x\sin x)C_{22}\Bigg\}\Bigg].
\end{aligned}
\tag{3.6}
$$

For the computation of the constants C_{11}, C_{12}, C_{21}, and C_{22} using (3.6) in (3.1) and applying the technique mentioned in (3.7)-(3.10), we get

$$
\begin{aligned}
C_{11} &= 2.7729652214966792 \times 10^{-12}, \\
C_{12} &= -0.937519284833785, \\
C_{21} &= -5.975690719999961 \times 10^{-13}, \\
C_{22} &= 1.0135423977443716,
\end{aligned}
$$

$$
\begin{aligned}
\xi(x,t) &= \Bigg[3.65926 \times 10^{-24}(-3.62784 \times 10^{23} + t)(-0.753285 + t) \\
&\qquad + t^2(-6.02014 \times 10^{-24}\cos x + 3.8951 \times 10^{-24}\cos 2x)\Bigg]\sin x, \\
\zeta(x,t) &= \Bigg[1.24956 \times 10^{-24}(-8.1112 \times 10^{23} + t)(-0.986639 + t) \\
&\qquad - 2.62882 \times 10^{-24}(-1.5422 \times 10^{-24} + t)t\cos x + 3.21315 \times 10^{-24}t^2\cos 2x\Bigg]\sin x.
\end{aligned}
\tag{3.7}
$$

Table 1: Absolute error of EOHAM solution $\xi(x,t)$ corresponding to the exact solution.

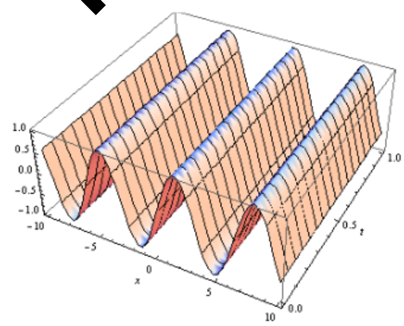
x	t	Exact	HPM	EOHAM
-2	0.01	-0.900249	-0.900249	-0.900249
-2	0.02	-0.891292	-0.891292	-0.891292
-2	0.02	-0.882423	-0.882423	-0.882423
5	0.01	-0.949382	-0.949382	-0.949382
5	0.02	-0.939936	-0.939936	-0.939936
5	0.02	-0.930583	-0.930583	-0.930583
10	0.01	-0.538608	-0.538608	-0.538608
10	0.02	-0.527942	-0.527942	-0.527942

Table 2: Absolute error of EOHAM solution $\xi(x,t)$ corresponding to the exact solution.

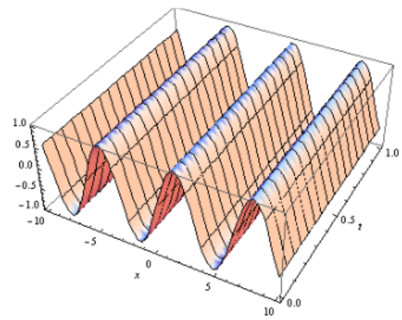
x	$t=0.2$	$t=0.1$	$t=0.01$	$t=0.001$	$t=0.0001$
-10	9.86143×10^{-2}	7.67425×10^{-4}	4.55899×10^{-5}	5.04593×10^{-6}	5.04590×10^{-7}
-8	1.79340×10^{-2}	1.39564×10^{-4}	8.29100×10^{-5}	9.17654×10^{-6}	9.17649×10^{-7}
-6	5.06494×10^{-2}	3.94158×10^{-4}	2.34155×10^{-5}	2.59165×10^{-6}	2.59165×10^{-7}
-4	1.37185×10^{-2}	1.06759×10^{-4}	6.34214×10^{-5}	7.01953×10^{-6}	7.01949×10^{-7}
-2	1.64828×10^{-2}	1.28270×10^{-4}	7.62007×10^{-5}	8.43395×10^{-6}	8.43391×10^{-7}
0	0	0	0	0	0
2	1.64828×10^{-2}	1.28270×10^{-4}	7.62007×10^{-5}	8.43395×10^{-6}	8.43391×10^{-7}
4	1.37185×10^{-2}	1.06759×10^{-4}	6.34214×10^{-5}	7.01953×10^{-6}	7.01949×10^{-7}
6	5.06494×10^{-2}	3.94158×10^{-4}	2.34155×10^{-5}	2.59165×10^{-5}	2.59165×10^{-7}
8	1.79340×10^{-2}	1.39564×10^{-4}	8.29100×10^{-5}	9.17654×10^{-5}	9.17649×10^{-7}
10	9.86143×10^{-2}	7.67425×10^{-4}	4.55899×10^{-5}	5.04593×10^{-5}	5.04590×10^{-7}

Table 3: Absolute error of EOHAM solution $\zeta(x,t)$ corresponding to the exact solution.

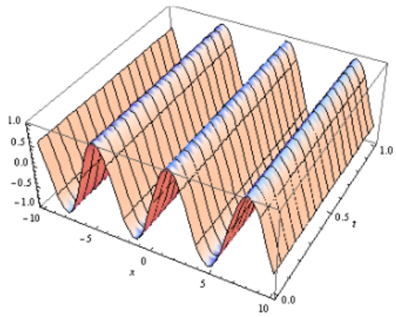
x	$t=0.2$	$t=0.1$	$t=0.01$	$t=0.001$	$t=0.0001$
-10	9.86143×10^{-2}	2.21631×10^{-4}	4.63795×10^{-5}	5.12489×10^{-6}	4.55628×10^{-7}
-8	1.79340×10^{-2}	5.99444×10^{-4}	2.43612×10^{-5}	3.32166×10^{-6}	8.28607×10^{-7}
-6	5.06494×10^{-2}	4.39363×10^{-4}	2.01266×10^{-5}	1.76256×10^{-6}	2.34016×10^{-7}
-4	1.37185×10^{-2}	1.22366×10^{-4}	5.22524×10^{-5}	5.90263×10^{-6}	6.33837×10^{-7}
-2	1.64828×10^{-2}	3.35804×10^{-4}	4.39030×10^{-5}	5.20418×10^{-6}	7.61554×10^{-7}
0	0	0	0	0	0
2	1.64828×10^{-2}	3.35804×10^{-4}	4.39030×10^{-5}	5.20418×10^{-6}	6.33837×10^{-7}
4	1.37185×10^{-2}	1.22366×10^{-4}	5.22524×10^{-5}	5.90263×10^{-6}	6.33837×10^{-7}
6	5.06494×10^{-2}	4.39363×10^{-4}	2.01266×10^{-5}	1.76256×10^{-6}	2.34016×10^{-7}
8	1.79340×10^{-2}	5.99444×10^{-4}	2.43612×10^{-5}	3.32166×10^{-6}	8.28607×10^{-7}
10	9.86143×10^{-2}	2.21631×10^{-4}	4.63795×10^{-5}	5.12489×10^{-6}	4.55628×10^{-7}



(a) 3D approximate solution of $\xi(x,t)$ at $t=0.1$.

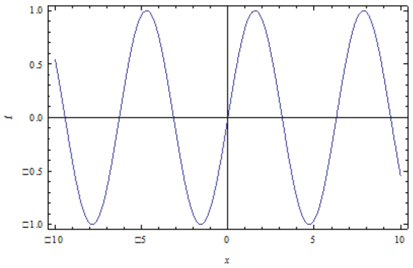


(b) 3D approximate solution of $\zeta(x,t)$ at $t=0.1$.

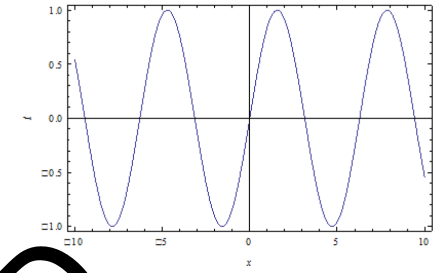


(c) 3D exact solution of $\xi(x,t)$ and $\zeta(x,t)$ at $t=0.1$.

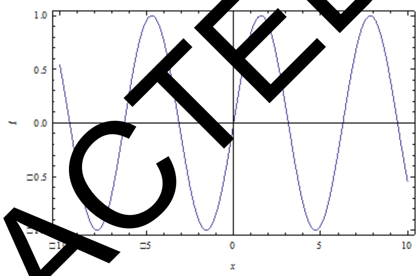
Figure 1: 3D approximate and exact solutions for eq. (3.1).



(a) 2D approximate solution of $\xi(x,t)$ at $t = 0.1$.

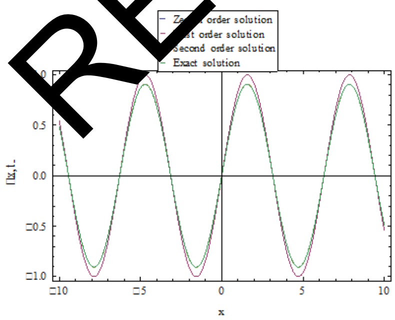


(b) 2D approximate solution of $\zeta(x,t)$ at $t = 0.1$.

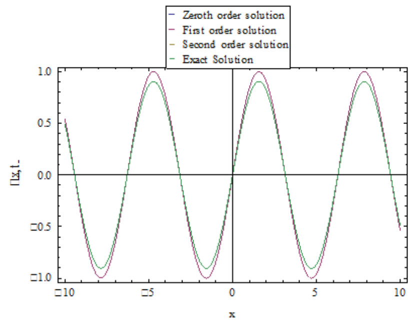


(c) 2D exact solution of eq. (3.1) at $t = 0.1$.

Figure 2: 2D approximate and exact Solutions for eq. (3.1).

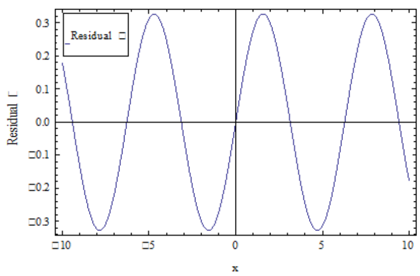


(a) 2D zeroth, first, and second order and exact solutions of $\xi(x,t)$ at $t = 0.1$.

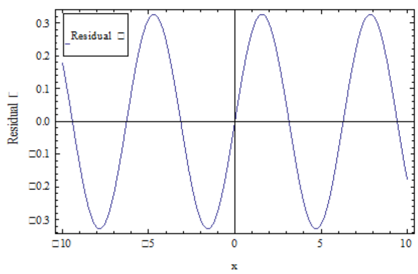


(b) 2D zeroth, first, and second order and exact solutions of $\zeta(x,t)$ at $t = 0.1$.

Figure 3: 2D zeroth, first, and second order and exact solutions for eq. (3.1).



(a) 2D residual of $\xi(x,t)$ at $t = 0.1$.



(b) 2D residual of $\xi(x,t)$ at $t = 0.1$.

Figure 4: 2D residual for eq. (3.1).

Example 3.2. The nonlinear coupled mKdV equations with boundary conditions [1]

$$\begin{aligned}\frac{\partial \xi}{\partial t} &= \frac{1}{2}\frac{\partial^3 \xi}{\partial x^3} - 3\xi^2\frac{\partial \xi}{\partial x} + \frac{3}{2}\frac{\partial^2}{\partial x^2} + 3\frac{\partial \xi}{\partial x}(\xi\zeta) - 3\lambda\frac{\partial \xi}{\partial x},\\ \frac{\partial \zeta}{\partial t} &= -\frac{\partial^3 \zeta}{\partial x^3} - 3\zeta\frac{\partial \zeta}{\partial x} - 3\frac{\partial^2 \zeta}{\partial x^2} + 3\frac{\partial \xi}{\partial x}\frac{\partial \zeta}{\partial x} + 3\xi^2\frac{\partial \zeta}{\partial x} + 3\lambda\frac{\partial \zeta}{\partial x},\end{aligned} \tag{3.8}$$

with initial conditions

$$\zeta(x,0) = \frac{b}{2k} + k\tanh(kx),\ \ \xi(x,0) = \frac{\lambda}{2}(1+\frac{k}{b}) + k\tanh(kx). \tag{3.9}$$

The exact solutions of Eq. (3.8) are

$$\zeta(x,t) = \frac{b}{2k} + k\tanh(k\tau),\ \ \xi(x,t) = \frac{\lambda}{2}(1+\frac{k}{b}) + b\tanh(k\tau), \tag{3.10}$$

where $\tau = x + \frac{1}{4}(-4k^2 - 6\lambda + \frac{6k\lambda}{b} + \frac{3\lambda^2}{k^2})t$. Applying the technique discussed in Section 2 we have

$$\begin{aligned}(1-l)\frac{\partial \xi}{\partial t} - U_1(l)\Big[\frac{\partial \xi}{\partial t} - \frac{1}{2}\frac{\partial^3 \xi}{\partial x^3} + 3\xi^2\frac{\partial \xi}{\partial x} - \frac{3}{2}\frac{\partial^2 \zeta}{\partial x^2} - 3\frac{\partial \xi}{\partial x}(\xi\zeta) + 3\lambda\frac{\partial \xi}{\partial x}\Big] &= 0,\\ (1-l)\frac{\partial \zeta}{\partial t} - U_2(l)\Big[\frac{\partial \zeta}{\partial t} - \frac{\partial^3 \zeta}{\partial x^3} + 3\zeta\frac{\partial \zeta}{\partial x} + 3\frac{\partial^2 \zeta}{\partial x^2} - 3\frac{\partial \xi}{\partial x}\frac{\partial \zeta}{\partial x} - 3\xi^2\frac{\partial \zeta}{\partial x} - 3\lambda\frac{\partial \zeta}{\partial x}\Big] &= 0.\end{aligned}$$

We consider

$$\xi = \xi_0 + l\xi_1 + l^2\xi_2,\quad \zeta = \zeta_0 + l\zeta_1 + l^2\zeta_2,\quad U_1(l) = lC_{11} + l^2C_{12},\quad U_2(l) = lC_{21} + l^2C_{22}.$$

3.4. Zeroth order system

$$\frac{\partial \xi_0}{\partial t} = 0,\ \frac{\partial \zeta_0}{\partial t} = 0,$$

with initial conditions

$$\xi_0(x,0) = \frac{b}{2k} + k\tanh(kx),\ \ \zeta_0(x,0) = \frac{\lambda}{2}(1+\frac{k}{b}) + k\tanh(kx).$$

Its solution is

$$\begin{aligned}\xi_0(x,t) &= 5 + 0.333333333333\tanh(0.333333333333x),\\ \zeta_0(x,t) &= 0.066666666667 + \tanh(0.333333333333x).\end{aligned} \tag{3.11}$$

3.5. First order system

$$\begin{aligned}\frac{\partial \xi_1}{\partial t} &= (1+C_11)\frac{\partial \xi_0}{\partial t} + 3\lambda C_{11}\frac{\partial \xi_0}{\partial x} + 3C_{11}\xi_0^2\frac{\partial \xi_0}{\partial x} - 3C_{11}\zeta_0\frac{\partial \zeta_0}{\partial x}\\ &\quad - 3C_{11}\xi_0\frac{\partial \zeta_0}{\partial x} - \frac{3}{2}C_{11}\frac{\partial^2 \zeta_0}{\partial x^2} - \frac{1}{2}C_{11}\frac{\partial^3 \xi_0}{\partial x^3},\\ \frac{\partial \zeta}{\partial t} &= (1+C_{21})\frac{\partial \zeta_0}{\partial t} - 3\lambda C_{21}\frac{\partial \zeta_0}{\partial x} - 3\lambda C_{21}\xi_0^2\frac{\partial \zeta_0}{\partial x} + 3C_{21}\zeta_0\frac{\partial \zeta_0}{\partial x}\\ &\quad + 3C_{21}\frac{\partial \zeta_0}{\partial x}\frac{\partial \xi_0}{\partial x} - C_{21}\frac{\partial^3 \zeta_0}{\partial x^3}.\end{aligned}$$

Its solution is

$$
\begin{aligned}
\xi_1(x,t;C_{11}) &= tC_{11}\Bigg[\Big(8.033333\sec h^2(0.3333333x) + 0.013456790123\sec h^4(0.3333333x)\Big) \\
&\quad + \Big(1.088888\sec h^2(0.3333333x)\tanh(0.3333333x) \\
&\quad + 0.01234567901\sec h^2(0.3333333x)\tanh^2(0.3333333x)\Big)\Bigg], \\
\zeta_1(x,t;C_{21}) &= -tC_{21}\Bigg[\Big(1.6733333\sec h^2(0.3333333x) \\
&\quad - 0.012345678901\sec h^4(0.3333333x)\Big)\Big(0.217777\sec h^2(0.3333333x)\tanh(0.3333333x) \\
&\quad + 0.01728395061\sec h^2(0.3333333x)\tanh^2(0.3333333x)\Big)\Bigg].
\end{aligned} \tag{3.12}
$$

Adding Eqs. (3.11) and (3.12), we obtain

$$
\begin{aligned}
\xi(x,t;C_{11}) &= 5 + 0.333333333333\tanh(0.333333333333x) \\
&\quad + tC_{11}\Bigg[\Big(8.033333\sec h^2(0.3333333x) + 0.013456790123\sec h^4(0.3333333x)\Big) \\
&\quad + \Big(1.088888\sec h^2(0.3333333x)\tanh(0.3333333x) \\
&\quad + 0.01234567901\sec h^2(0.3333333x)\tanh^2(0.3333333x)\Big)\Bigg], \\
\zeta(x,t;C_{21}) &= 0.066666666667 + \tanh(0.333333333333x) \\
&\quad - \Bigg[\Big(1.6733333\sec h^2(0.3333333x) - 0.012345678901\sec h^4(0.3333333x)\Big) \\
&\quad \times \Big(0.217777\sec h^2(0.3333333x)\tanh(0.3333333x) \\
&\quad + 0.01272839506\sec h^2(0.3333333x)\tanh^2(0.3333333x)\Big)\Bigg].
\end{aligned} \tag{3.13}
$$

For the computation of the constants C_{11}, C_{12} using (3.13) in (3.8) and applying the technique mentioned in (3.7)-(3.10), we get

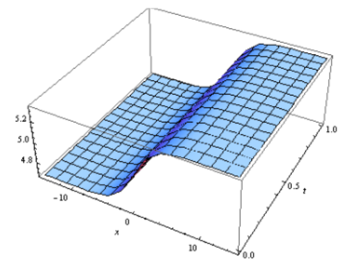
$$
\begin{aligned}
C_{11} &= 0.09188852998980887, \\
C_{12} &= -1.219991641721691, \\
\xi(x,t) &= 5 + 1.13443\times 10^{-10}\sec h^4(0.3333333x) \\
&\quad + 0.3333333\tanh(0.3333333x) \\
&\quad + 1.13443\times 10^{-10}\sec h^2(0.3333333x) \\
&\quad + \Big(0.12626 + \tanh(0.3333333x)(80.0737 + \tanh(0.3333333x))\Big), \\
\zeta(x,t) &= 0.0666667 - 1.50616\times 10^{-9}\sec h^4(0.3333333x) \\
&\quad + \tanh(0.3333333x) + \sec h^2(0.3333333x) \\
&\quad \times \Big(2.04145\times 10^{-7} + 2.05687\times 10^{-8} \\
&\quad + 2.10863\times 10^{-9}\tanh(0.3333333x)\Big)\tanh(0.3333333x).
\end{aligned}
$$

Table 4: Absolute error of EOHAM solution $\xi(x,t)$ corresponding to the exact solution.

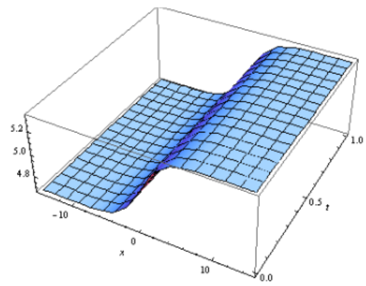
x	$t = 0.2$	$t = 0.1$	$t = 0.01$	$t = 0.001$	$t = 0.0001$
-15	1.18888×10^{-5}	1.18888×10^{-5}	6.68012×10^{-5}	1.02594×10^{-5}	1.02857×10^{-6}
-12	1.39349×10^{-4}	1.39349×10^{-4}	4.93661×10^{-5}	7.57713×10^{-5}	7.59653×10^{-6}
-9	1.01076×10^{-3}	1.01076×10^{-3}	3.65102×10^{-4}	5.57916×10^{-4}	5.59337×10^{-5}
-6	6.89963×10^{-3}	6.89963×10^{-3}	2.71230×10^{-3}	4.01754×10^{-3}	4.02739×10^{-4}
-3	2.78361×10^{-2}	2.78361×10^{-2}	1.97907×10^{-2}	2.47463×10^{-2}	2.47923×10^{-3}
0	1.11107×10^{-2}	1.11107×10^{-2}	7.63359×10^{-2}	6.66663×10^{-2}	6.66665×10^{-3}
3	2.47383×10^{-2}	2.47383×10^{-2}	4.26791×10^{-2}	3.12482×10^{-2}	3.12030×10^{-3}
6	4.88596×10^{-3}	4.88596×10^{-3}	7.6229×10^{-3}	5.40207×10^{-3}	5.39250×10^{-4}
9	6.97365×10^{-4}	6.97365×10^{-4}	1.07775×10^{-3}	7.57481×10^{-4}	7.56104×10^{-5}
12	9.50825×10^{-5}	9.50825×10^{-5}	1.41114×10^{-4}	1.03012×10^{-4}	1.02824×10^{-5}
15	1.28738×10^{-5}	1.28738×10^{-5}	1.97820×10^{-5}	1.39503×10^{-5}	1.39248×10^{-6}

Table 5: Absolute error of EOHAM solution $\zeta(x,t)$ corresponding to the exact solution.

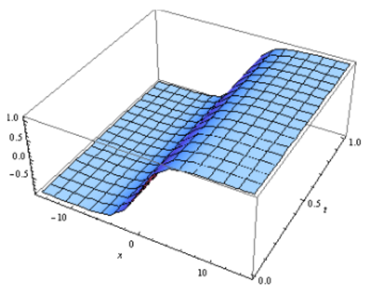
x	$t = 0.2$	$t = 0.1$	$t = 0.01$	$t = 0.001$	$t = 0.0001$
-15	6.10573×10^{-5}	6.10573×10^{-5}	1.56495×10^{-5}	2.85825×10^{-5}	2.86617×10^{-6}
-12	4.50488×10^{-4}	4.50488×10^{-4}	1.15656×10^{-4}	2.11093×10^{-4}	2.11675×10^{-5}
-9	3.29250×10^{-3}	3.29250×10^{-3}	8.55714×10^{-4}	1.55395×10^{-3}	1.55822×10^{-4}
-6	2.24608×10^{-3}	2.24608×10^{-3}	6.37501×10^{-3}	1.11717×10^{-2}	1.11201×10^{-3}
-3	9.56286×10^{-2}	9.56286×10^{-2}	4.72517×10^{-2}	6.81789×10^{-2}	6.83167×10^{-3}
0	4.98356×10^{-2}	4.98356×10^{-2}	1.90703×10^{-1}	1.80846×10^{-1}	1.80843×10^{-2}
3	5.76831×10^{-2}	5.76831×10^{-2}	1.11505×10^{-1}	8.54788×10^{-2}	8.53431×10^{-3}
6	1.19566×10^{-2}	1.19566×10^{-2}	2.01656×10^{-2}	1.48555×10^{-2}	1.48268×10^{-3}
9	1.71650×10^{-3}	1.71650×10^{-3}	2.84625×10^{-3}	2.08494×10^{-3}	2.08081×10^{-4}
12	2.34172×10^{-4}	2.34172×10^{-4}	3.87416×10^{-4}	2.83573×10^{-4}	2.83009×10^{-5}
15	3.17261×10^{-5}	3.17261×10^{-5}	5.24719×10^{-5}	3.84033×10^{-5}	3.83269×10^{-6}



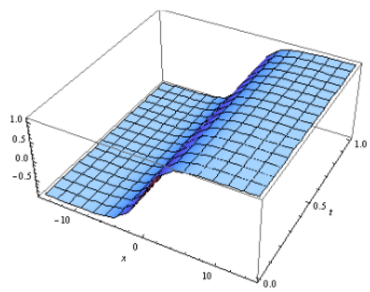
(a) 3D approximate solution of $\xi(x,t)$ at $t = 0.1$.



(b) 3D exact solution of $\xi(x,t)$ at $t = 0.1$.

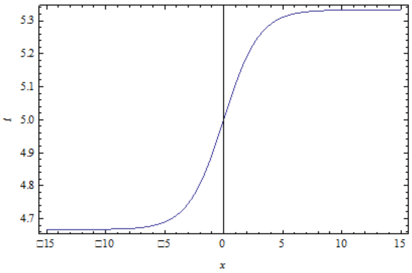


(c) 3D approximate solution of $\zeta(x,t)$ and $\zeta(x,t)$ at $t = 0.1$.

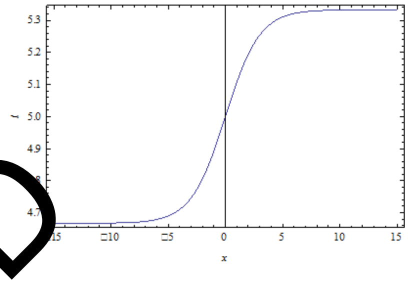


(d) 3D exact solution of $\zeta(x,t)$ at $t = 0.1$.

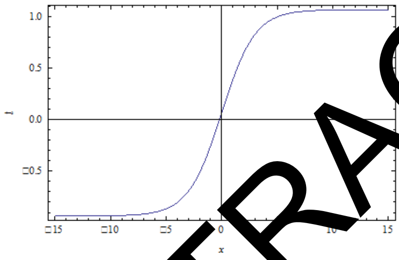
Figure 5: 3D approximate and exact solutions for eq. (3.8).



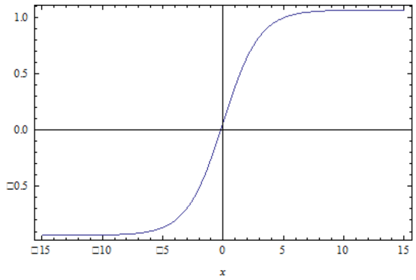
(a) 2D approximate solution of $\xi(x,t)$ at $t = 0.1$.



(b) 2D exact solution of $\xi(x,t)$ at $t = 0.1$.

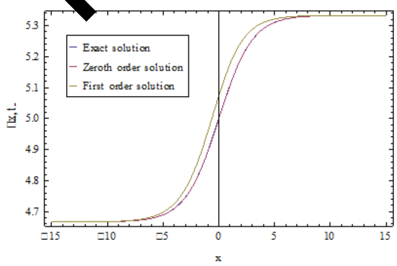


(c) 2D approximate solution of $\zeta(x,t)$ at $t = 0.1$.

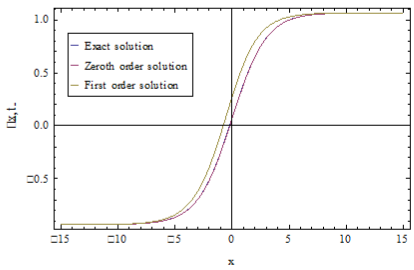


(d) 2D exact solution of $\zeta(x,t)$ at $t = 0.1$.

Figure 6: 2D approximate and exact solutions obtained for eq. (3.8).

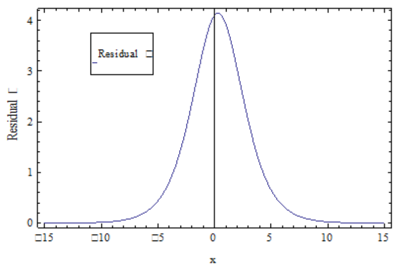


(a) 2D zeroth, and first order and exact solutions of $\xi(x,t)$ at $t = 0.1$.

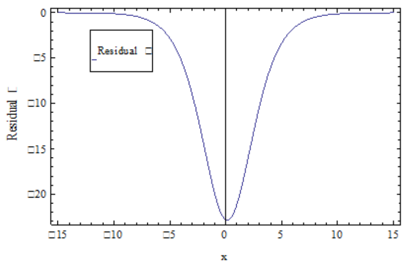


(b) 2D zeroth, and first order and exact solutions of $\zeta(x,t)$ at $t = 0.1$.

Figure 7: 2D zeroth, and first order and exact solutions for eq. (3.8).



(a) 2D residual of $\xi(x,t)$ at $t = 0.1$.



(b) 2D residual of $\zeta(x,t)$ at $t = 0.1$.

Figure 8: 2D residual for eq. (3.8).

4. Results and discussions

The mathematical theory given in Section 2 gives highly accurate solutions for the BVP presented in Section 3. We have used Mathematica 7 for most of our computational work. In Table 1, the EOHAM results for the coupled burgers equations are compared with exact and HPM solutions for different values of x at $t = 0.01, 0.02, 0.03$. In Table 2, 3, 4, and 5 the absolute errors of the coupled burgers equations and coupled mKdV equations corresponding to exact solutions are given at different values of x for $t = 0.2, 0.01, 0.001, 0.0001$. Figures 1 and 5 show the 3D and Figures 2 asd 6 show the 2D comparison of the approximate solutions $\xi(x, t)$ and $\zeta(x, t)$ with exact solutions for coupled burgers and coupled mKdV equations respectively. The convergence of OHAM is presented in Figures 3 and 7 and the residuals are given in Figures 4 and 8 for the coupled burgers and coupled mKdV equations respectively at $t = 0.1$. Here we observed that the OHAM solution converges rapidly with the increase in the order of approximation. From these Tables and plots it is evident that the OHAM results are nearly identical to the exact solutions. Here the results are very consistent with the decreasing time.

5. Conclusion

In this paper, we have seen the effectiveness of EOHAM to coupled burgers and coupled mKdV equations. By applying the basic idea of EOHAM to coupled burgers and coupled mKdV equations, we found it simpler in applicability, more convenient to control convergence and involved less computational overhead. Therefore, EOHAM shows its validity and great potential for the solution of nonlinear coupled PDEs problems in science and engineering.

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