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On a q-analogue degenerate Carlitz's type Daehee polynomials and numbers

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Abstract

Studies on degenerate versions of Stirling, Bernoulli and Eulerian numbers started by [L. Carlitz, Utilitas Math., **15** (1979), 51–88]. In recent years, many mathematicians have studied degenerate version of various special polynomials and numbers. In this paper, we introduce the q-analogue degenerate Carlitz's type Daehee and higher-order degenerate Carlitz's type Daehee polynomials and numbers. Also, we study some explicit identities and properties for the q-analogue degenerate Carlitz's type Daehee polynomials and numbers and higher-order q-analogue degenerate Carlitz's type Daehee polynomials and numbers arising from p-adic invariant q-integral on $\mathbb{Z}_p$.

Keywords: p-Adic q-integral of f on $\mathbb{Z}_p$, degenerate Carlitz's type Daehee polynomials and numbers, degenerate q-Bernoulli polynomials.

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1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let q be in $\mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and x in $\mathbb{Z}_p$. Then *the* q*-analogue of* x is defined to be $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

For $f \in UD(\mathbb{Z}_p)$, the space of uniformly differentiable functions on $\mathbb{Z}_p$, *the* p*-adic* q*-integral of* f *on* $\mathbb{Z}_p$ is defined by Kim to be (see [1, 4, 6, 7, 15–17])

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{n=0}^{p^N-1} f(x)q^x. \tag{1.1}$$

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From (1.1), we can drive the following integral identity.

$$qI_q(f_1) - I_q(f) = \frac{q-1}{\log q} f'(0) + (q-1)f(0), \tag{1.2}$$

where $f_1(x) = f(x+1)$. Also, the *Carlitz's q-Bernoulli polynomials* $\beta_{n,q}(x)$ are defined to be

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} \beta_{l,q},$$

where

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [y]_q^n d\mu_q(y), \ (n \geqslant 0), \quad \text{(see [2, 3, 8, 12])}.$$

Indeed,

$$[x+y]_q^n = \left(\frac{1-q^x + q^x(1-q^y)}{1-q} \right)^n = ([x]_q + q^x[y]_q)^n = \sum_{l=0}^{n} \binom{n}{l} q^{lx} [y]_q^l [x]_q^{n-l}. \tag{1.3}$$

By (1.2) and (1.3), the Carlitz's q-Bernoulli polynomials can be represented by p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} \beta_{l,q} = \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} \int_{\mathbb{Z}_p} [y]_q^l du_q(y) = \int_{\mathbb{Z}_p} [x+y]_q^n du_q(y).$$

The Carlitz's type degenerate q-Bernoulli polynomials can be represented by p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,q}(x|\lambda) \frac{t^n}{n!}, \quad \text{(see [9])}. \tag{1.4}$$

The *Daehee polynomials* are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \quad \text{(see [4, 9–11, 13, 14, 19])}.$$

In the special case $x = 0$, $D_n = D_n(0)$ are called the *Daehee numbers*.

The *degenerate Daehee polynomials* are given by the generating function to be

$$\frac{\lambda \log\left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right)}{\log(1+\lambda t)} \left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [14])}.$$

For $x = 0$, $D_{n,\lambda} = D_{n,\lambda}(0)$ are called the *degenerate Daehee numbers*. We observe here that $D_{n,\lambda} \longrightarrow D_n$ as $\lambda \longrightarrow 0$.

The *Stirling numbers of the first kind* are given by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l \ (x \geqslant 0), \ \text{(see [3, 20])},$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ is the n falling factorial, and the *Stirling numbers of the first kind* are defined by the generating function to be

$$(\log(x+1))^n = n! \sum_{l=n}^{\infty} S_1(l,n) \frac{x^l}{l!}, \ (n \geqslant 0), (\text{ see [5, 18]}).$$

Recently, several researchers have studied degenerate type special polynomials related to Bernoulli, Euler, Daehee, and Changhee polynomials. In this paper, we introduce the q-analogue degenerate Carlitz's type Daehee polynomials and numbers and q-analogue higher-order degenerate Carlitz's type Daehee polynomials. Also, we study some explicit identities and properties for those polynomials arising from p-adic invariant q-integral on $\mathbb{Z}_p$.

2. The q-analogue degenerate Carlitz's type Daehee polynomials and numbers

In this section, we assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$ and $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$.

In the viewpoint of (1.2), we define the q-analogue degenerate Carlitz's type Daehee polynomials which are given by the generating function to be

$$\int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{[x+y]_q}{\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} D_{n,q,\lambda}(x)\frac{t^n}{n!}, \quad (n \geqslant 0). \tag{2.1}$$

When $x = 0$, $D_{n,q,\lambda} = D_{n,q,\lambda}(0)$ are called the q-analogue degenerate Carlitz's type Daehee numbers.

From (2.1), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} D_{n,q,\lambda}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{[x+y]_q}{\lambda}} d\mu_q(y) \\
&= \int_{\mathbb{Z}_p} \sum_{k=0}^{\infty} \binom{\frac{[x+y]_q}{\lambda}}{k} \lambda^k (\log(1+t))^k d\mu_q(y) \\
&= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda}\right)_k \lambda^k \frac{1}{k!} (\log(1+t))^k d\mu_q(y) \\
&= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} \left([x+y]_q\right)_{k,\lambda} \sum_{n=k}^{\infty} S_1(n,k) d\mu_q(y) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \sum_{k=0}^{n} \left([x+y]_q\right)_{k,\lambda} S_1(n,k) d\mu_q(y) \frac{t^n}{n!},
\end{aligned} \tag{2.2}$$

where $\left([x+y]_q\right)_{k,\lambda} = [x+y]_q([x+y]_q-\lambda)\cdots([x+y]_q-(k-1)\lambda)$.

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of (2.2), we have the following theorem.

Theorem 2.1. *For* $n \geqslant 0$, *we have*

$$D_{n,q,\lambda}(x) = \sum_{k=0}^{n} \int_{\mathbb{Z}_p} \left([x+y]_q\right)_{k,\lambda} S_1(n,k) d\mu_q(y).$$

By replacing t by $e^t - 1$ in (2.1), we obtain

$$\sum_{m=0}^{\infty} D_{m,q,\lambda}(x)\frac{1}{m!}\left(e^t-1\right)^m = \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_q(y) = \sum_{m=0}^{\infty} B_{m,q}(x|\lambda)\frac{t^n}{n!}. \tag{2.3}$$

On the other hand, we observe the left hand side of the previous equation.

$$\sum_{k=0}^{\infty} D_{k,q,\lambda}(x)\frac{1}{k!}\left(e^t-1\right)^k = \sum_{k=0}^{\infty} D_{k,q,\lambda}(x) \sum_{m=k}^{\infty} S_2(m,k)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m} D_{k,q,\lambda}(x) S_2(m,k)\right)\frac{t^m}{m!}. \tag{2.4}$$

Comparing the coefficients of $\frac{t^m}{m!}$ on the both sides of (2.3) and (2.4), we arrive at the following theorem.

Theorem 2.2. *For* $n \geqslant 0$, *we have*

$$B_{m,q}(x|\lambda) = \sum_{k=0}^{m} D_{k,q,\lambda}(x) S_2(m,k).$$

Now, we consider the inversion formula of Theorem 2.2. By replacing t by $\log(1+t)$ in (1.4), we obtain

$$\begin{aligned}\int_{\mathbb{Z}_p}\Big(1+\lambda\log(1+t)\Big)^{\frac{[x+y]_q}{\lambda}}d\mu_q(y) &= \sum_{m=0}^{\infty}B_{m,q}(x|\lambda)\frac{1}{m!}\Big(\log(1+t)\Big)^m\\ &= \sum_{m=0}^{\infty}B_{m,q}(x|\lambda)\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_{m,q}(x|\lambda)S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned}$$

On the other hand, we observe the left hand side of the previous equation.

$$\int_{\mathbb{Z}_p}\Big(1+\lambda\log(1+t)\Big)^{\frac{[x+y]_q}{\lambda}}d\mu_q(y) = \sum_{m=0}^{\infty}D_{m,q,\lambda}(x)\frac{t^m}{m!}.$$

Theorem 2.3. *For* $m \geqslant 0$, *we have*

$$D_{m,q,\lambda}(x) = \sum_{m=0}^{n}B_{m,q}(x|\lambda)S_1(n,m).$$

Now, we observe that

$$\begin{aligned}(1+\lambda\log(1+t))^{\frac{[x+y]_q}{\lambda}} &= e^{\frac{[x+y]_q}{\lambda}\log(1+\lambda\log(1+t))}\\ &= \sum_{n=0}^{\infty}\left(\frac{[x+y]_q}{\lambda}\right)^n\frac{1}{n!}(\log(1+\lambda\log(1+t)))^n\\ &= \sum_{n=0}^{\infty}\left(\frac{[x+y]_q}{\lambda}\right)^n\sum_{m=n}^{\infty}S_1(m,n)\lambda^m\frac{(\log(1+t))^m}{m!}\\ &= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\sum_{n=0}^{m}[x+y]_q^n\lambda^{m-n}S_1(m,n)S_1(k,m)\right)\frac{t^k}{k!}.\end{aligned} \tag{2.5}$$

From (2.5), we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}D_{k,q,\lambda}(x)\frac{t^k}{k!} &= \int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{[x+y]_q}{\lambda}}d\mu_q(y)\\ &= \int_{\mathbb{Z}_p}\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\sum_{n=0}^{m}[x+y]_q^n\lambda^{m-n}S_1(m,n)S_1(k,m)\right)d\mu_q(y)\frac{t^k}{k!}\\ &= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\sum_{n=0}^{m}\lambda^{m-n}S_1(m,n)S_1(k,m)B_{n,q}(x)\right)\frac{t^k}{k!}.\end{aligned} \tag{2.6}$$

Comparing the coefficients of $\frac{t^k}{k!}$ on the both sides of (2.6), we arrive at the following theorem.

Theorem 2.4. *For* $k \geqslant 0$, *we have*

$$D_{k,q,\lambda}(x) = \sum_{m=0}^{k}\sum_{n=0}^{m}\lambda^{m-n}S_1(m,n)S_1(k,m)B_{n,q}(x).$$

Note that

$$[x+y]_q = \frac{1-q^{x+y}}{1-q} = \frac{1-q^x}{1-q} + \frac{q^x(1-q^y)}{1-q} = [x]_q + q^x[y]_q.$$

We observe that

$$\begin{aligned}
&(1+\lambda\log(1+t))^{\frac{[x+y]_q}{\lambda}}\\
&= (1+\lambda\log(1+t))^{\frac{[x]_q}{\lambda}}(1+\lambda\log(1+t))^{\frac{q^x[y]_q}{\lambda}}\\
&= \left(\sum_{m=0}^{\infty}\left(\frac{[x]_q}{\lambda}\right)_m \lambda^m \frac{(\log(1+t))^m}{m!}\right)\left(\sum_{l=0}^{\infty}\frac{q^{lx}[y]_q^l}{\lambda^l}\frac{(\log(1+\lambda\log(1+t))^l}{l!}\right)\\
&= \left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}([x]_q)_{m,\lambda}S_1(n,m)\right)\frac{t^k}{k!}\right)\left(\sum_{j=0}^{\infty}\left(\sum_{p=0}^{j}\sum_{l=0}^{p}\lambda^{p-l}q^{lx}[y]_q^l S_1(p,l)S_1(j,p)\right)\frac{t^j}{j!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{p=0}^{j}\sum_{l=0}^{p}\binom{n}{j}([x]_q)_{m,\lambda}\lambda^{p-l}q^{lx}[y]_q^l S_1(m,n)S_1(p,l)S_1(j,p)\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of (2.6), we arrive at the following theorem.

Theorem 2.5. *For $n \geqslant 0$, we have*

$$D_{n,q,\lambda}(x) = \sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{p=0}^{j}\sum_{l=0}^{p}\binom{n}{j}([x]_q)_{m,\lambda}\lambda^{p-l}q^{lx}[y]_q^l S_1(m,n)S_1(p,l)S_1(j,p).$$

Using (2.4), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} D_{n,q,\lambda}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{[x+y]_q}{\lambda}}\,d\mu_q(y)\\
&= \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{y=0}^{p^N-1}(1+\lambda\log(1+t))^{\frac{[x+y]_q}{\lambda}}q^y\\
&= \lim_{N\to\infty}\frac{1}{[dp^N]_q}\sum_{a=0}^{d-1}\sum_{y=0}^{p^N-1}(1+\lambda\log(1+t))^{\frac{[x+a+dy]_q}{\lambda}}q^{a+dy}\\
&= \lim_{N\to\infty}\frac{1}{[d]_q}\frac{1}{[p^N]_{q^d}}\sum_{a=0}^{d-1}\sum_{y=0}^{p^N-1}(1+\lambda\log(1+t))^{\frac{[d]_q[\frac{a+x}{d}+y]_{q^d}}{\lambda}}q^a(q^d)^y\\
&= \frac{1}{[d]_q}\sum_{a=0}^{d-1}q^a\lim_{N\to\infty}\frac{1}{[p^N]_{q^d}}\sum_{y=0}^{p^N-1}\sum_{l=0}^{\infty}\binom{\frac{[d]_q[\frac{a+x}{d}+y]_{q^d}}{\lambda}}{l}\lambda^l(\log(1+t))^l(q^d)^y\\
&= \frac{1}{[d]_q}\sum_{a=0}^{d-1}(q)^a\lim_{N\to\infty}\frac{1}{[p^N]_{q^d}}\\
&\quad\times\sum_{y=0}^{p^N-1}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}S_1(l,k)S_1(n,l)[d]_q^k\left[\frac{a+x}{d}+y\right]_{q^d}^k\lambda^{n-l}\right)\frac{t^n}{n!}(q^d)^y\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}\frac{1}{[d]_q}\sum_{a=0}^{d-1}(q)^a S_1(l,k)S_1(n,l)[d]_q^k\lambda^{n-l}B_{k,q^d}\left(\frac{a+x}{d}\right)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.7}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of (2.7), we arrive at the following theorem.

Theorem 2.6. *For $n \geqslant 0$, we have*

$$D_{n,q,\lambda}(x) = \sum_{l=0}^{n}\sum_{k=0}^{l}\frac{1}{[d]_q}\sum_{a=0}^{d-1}(q)^a S_1(l,k)S_1(n,l)[d]_q^k\lambda^{n-l}B_{k,q^d}(\frac{a+x}{d}).$$

For $r \in \mathbb{N}$, the higher-order q-analogue degenerate Carlitz's type Daehee polynomials are given by the multivariate bosonic p-adic q-integral as follows:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{[x_1+\cdots+x_r+x]_q}{\lambda}}d\mu_q(x_1)\cdots d\mu_q(x_r) = \sum_{n=0}^{\infty}D_{n,q,\lambda}^{(r)}(x)\frac{t^n}{n!},\quad (n \geqslant 0). \tag{2.8}$$

Now, we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty}D_{n,q,\lambda}^{(r)}(x)\frac{t^n}{n!}\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{[x_1+\cdots+x_r+x]_q}{\lambda}}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{l=0}^{\infty}\binom{\frac{[x_1+\cdots+x_r+x]_q}{\lambda}}{l}(\lambda\log(1+t))^l d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{l=0}^{\infty}\Big([x_1+\cdots+x_r+x]_q\Big)_{l,\lambda}\frac{1}{l!}(\log(1+t))^l d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\Big([x_1+\cdots+x_r+x]_q\Big)_{l,\lambda}S_1(n,l)\right)d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}S_1(l,k)[x_1+\cdots+x_r+x]_q^k\lambda^{l-k}S_1(n,l)\right)d\mu_q(x_1)\cdots d\mu_q(x_r)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}S_1(l,k)S_1(n,l)\lambda^{l-k}B_{k,q}^{(r)}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.9}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of (2.9), we arrive at the following theorem.

Theorem 2.7. *For $n \geqslant 0$, we have*

$$D_{n,q,\lambda}^{(r)}(x) = \sum_{l=0}^{n}\sum_{k=0}^{l}S_1(l,k)S_1(n,l)\lambda^{l-k}B_{k,q}^{(r)}(x).$$

Substitute t by e^t-1 in (2.8), we get

$$\sum_{k=0}^{\infty}D_{k,q,\lambda}^{(r)}(x)\frac{1}{k!}\left(e^t-1\right)^k = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[x_1+\cdots+x_r+x]_q}{\lambda}}d\mu_q(x_1)\cdots d\mu_q(x_r) = \sum_{n=0}^{\infty}B_{k,q}^{(r)}(x|\lambda)\frac{t^n}{n!}. \tag{2.10}$$

On the other hand, we observe the left hand side of the previous equation.

$$\sum_{k=0}^{\infty}D_{k,q,\lambda}^{(r)}(x)\frac{1}{k!}\left(e^t-1\right)^k = \sum_{k=0}^{\infty}D_{k,q,\lambda}^{(r)}(x)\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}D_{k,q,\lambda}^{(r)}(x)S_2(n,k)\right)\frac{t^n}{n!}. \tag{2.11}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of (2.10) and (2.11), we arrive at the following theorem.

Theorem 2.8. *For $n \geqslant 0$, we have*

$$B_{k,q}^{(r)}(x|\lambda) = \sum_{k=0}^{n}D_{k,q,\lambda}^{(r)}(x)S_2(n,k).$$

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