On asymptotically lacunary statistical equivalent functions via ideals

Ekrem Savaş
Department of Mathematics, Uşak University, Uşak, Turkey.

Abstract

The goal of this paper is to introduce $I_\theta$-asymptotically statistical equivalent by taking nonnegative two real-valued Lebesgue measurable functions $\gamma(\nu)$ and $\mu(\nu)$ in the interval $(1, \infty)$ instead of sequences and we establish some inclusion relations.

Keywords: Asymptotical equivalent functions, ideal convergence, lacunary sequence, $I$-statistical convergence.

2010 MSC: 40A99, 40A05.

1. Introduction and preliminaries

Quite recently, Das et al. [1] studied well known summability methods by using ideal and introduced new notions, namely ideal statistical convergence and ideal lacunary statistical convergence.


The notion of $I$-convergence was studied by Kostyrko et al. [4]. Some works on ideals can be found in [3, 9, 12, 13].

The main objective is to present $I_\theta$-asymptotically statistical equivalent and $I$-asymptotically statistical equivalent by taking nonnegative two real-valued Lebesgue measurable functions in the interval $(1, \infty)$. Furthermore we prove some interesting theorems.

Definition 1.1 ([5, Marouf]). Let $\gamma = (\gamma_i)$ and $\mu = (\mu_i)$ be two nonnegative sequences. If

$$\lim_{i} \frac{\gamma_i}{\mu_i} = 1,$$

then we say that $\gamma = (\gamma_i)$ and $\mu = (\mu_i)$ are asymptotically equivalent and it is denoted by $x \sim y$. 

Email address: ekremsavas@yahoo.com (Ekrem Savaş) 
doi: 10.22436/jmcs.019.01.05

Received: 2015-05-12 Revised: 2016-04-17 Accepted: 2018-09-13
Definition 1.2 ([2, Fridy]). Let $\gamma = (\gamma_i)$ be a sequence, if for every $\varphi > 0$,
\[
\lim_{n} \frac{1}{n} \{ \text{the number of } i \leq n : |\gamma_i - \beta| \geq \varphi \} = 0,
\]
then we say that $\gamma = (\gamma_i)$ is statistically convergent to $\beta$.

The next definition is natural combination of Definitions 1.1 and 1.2.

Definition 1.3 ([6, Patterson]). Let $\gamma = (\gamma_i)$ and $\mu = (\mu_i)$ be two nonnegative sequences. If for every $\varphi > 0$,
\[
\lim_{n} \frac{1}{n} \left\{ \text{the number of } i \leq n : \left| \frac{\gamma_i}{\mu_i} - \beta \right| \geq \varphi \right\} = 0,
\]
then we say that $\gamma = (\gamma_i)$ and $\mu = (\mu_i)$ are asymptotically statistical equivalent of multiple $\beta$ and it is denoted by $\gamma \sim_{\beta} \mu$ and simply asymptotically statistical equivalent if $\beta = 1$.

The following definitions and notions will be needed.

Definition 1.4 ([4]). A non-empty family $\mathcal{J} \subset 2^Y$ of subsets of a nonempty set $Y$ is said to be an ideal in $Y$ if the following conditions hold:

(i). $R, S \in \mathcal{J}$ implies $R \cup S \in \mathcal{J}$;

(ii). $R \in \mathcal{J}$, $S \subset R$ implies $S \in \mathcal{J}$.

Definition 1.5 ([3]). A non-empty family $\mathcal{F} \subset 2^\mathbb{N}$ is said to be a filter of $\mathbb{N}$ if the following conditions hold:

(i). $\emptyset \notin \mathcal{F}$;

(ii). $R, S \in \mathcal{F}$ implies $R \cap S \in \mathcal{F}$;

(iii). $R \in \mathcal{F}$, $S \subset R$ imply $S \in \mathcal{F}$.

If $\mathcal{J}$ is proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{J}$), then the family of sets $\mathcal{F}(\mathcal{J}) = \{ K \subset \mathbb{N} : \exists R \in \mathcal{J} : K = \mathbb{N} \setminus R \}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 1.6 ([3, 4]). A proper ideal $\mathcal{J}$ is said to be admissible if $\{n\} \in \mathcal{J}$ for each $n \in \mathbb{N}$.

Given $\mathcal{J} \subset 2^\mathbb{N}$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(\gamma_i)$ is said to be $\mathcal{J}$-convergent to $\beta$, if for each $\varphi > 0$ the set $\Lambda(\varphi) = \{ n \in \mathbb{N} : |\gamma_i - \beta| \geq \varphi \}$ belongs to $\mathcal{J}$ (see, [3, 4]). Following these results we introduce two new notions $\mathcal{J}_{\theta}$-asymptotically statistical equivalent of multiple $\beta$ and strong $\mathcal{J}_{\theta}$-asymptotically equivalent of multiple $\beta$.

By a lacunary $\theta = (l_s); s = 0, 1, 2, \ldots$, where $l_0 = 0$, we shall mean an increasing sequence of nonnegative integers with $\tau_s = l_s - l_{s-1} \to \infty$ as $s \to \infty$. The intervals determined by $\theta$ will be denoted by $J_s = [l_{s-1}, l_s]$ and the ratio $\frac{\tau_s}{l_{s-1}}$ will be denoted by $q_s$.

Patterson and Savaş [7] introduced the following definition.

Definition 1.7. Let $\theta = (l_s)$ be a lacunary sequence, two nonnegative sequences $\gamma = (\gamma_i)$ and $\mu = (\mu_i)$ are said to be asymptotically lacunary statistical equivalent of multiple $\beta$ provided that for every $\varphi > 0$
\[
\lim_{s} \frac{1}{\tau_s} \left\{ i \in J_s : \left| \frac{\gamma_i}{\mu_i} - \beta \right| \geq \varphi \right\} = 0,
\]
where the vertical bars indicate the number elements in the enclose set.

The following definitions are given in [1].
Definition 1.8. A sequence $\gamma = (\gamma_i)$ is said to be $J$-statistically convergent to $\beta$ or $S(J)$-convergent to $\beta$ if, for any $\varphi > 0$ and $\psi > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ s \leq n : |\gamma_i - \beta| \geq \varphi \} \right| \geq \psi \right\} \in J.$$

In this case, we write $\gamma_i \to \beta(S(J))$. The class of all $J$-statistically convergent sequences will be denoted by $S(J)$.

Definition 1.9. Let $\theta$ be a lacunary sequence. A sequence $\gamma = (\gamma_i)$ is said to be $J$-lacunary statistically convergent to $\beta$ or $S_\theta(J)$-convergent to $\beta$ if, for any $\varphi > 0$ and $\psi > 0$,

$$\left\{ s \in \mathbb{N} : \frac{1}{\tau_s} \left| \{ i \in J_s : |\gamma_i - \beta| \geq \varphi \} \right| \geq \psi \right\} \in J.$$

In this case, we write $\gamma_i \to \beta(S_\theta(J))$. The class of all $J$-lacunary statistically convergent sequences will be denoted by $S_\theta(J)$.

Definition 1.10. Let $\theta$ be a lacunary sequence. A sequence $\gamma = (\gamma_i)$ is said to be strong $J$-lacunary convergent to $\beta$ or $N_\theta(J)$-convergent to $\beta$ if, for any $\varphi > 0$

$$\left\{ s \in \mathbb{N} : \frac{1}{\tau_s} \left\{ \sum_{i \in J_s} |\gamma_i - \beta| \geq \varphi \right\} \in J. \right\}$$

In this case, we write $\gamma_i \to \beta(N_\theta(J))$. The class of all strong $J$-lacunary statistically convergent sequences will be denoted by $N_\theta(J)$.

We now introduce the following definitions.

Definition 1.11. Let $\theta$ be a lacunary sequence and $\gamma(\nu)$ be a nonnegative real-valued Lebesgue measurable function in the interval $(1, \infty)$ if

$$J - \lim_{s \to \infty} \frac{1}{\tau_s} \int_{\gamma(\nu) \in J_s} |\gamma(\nu) - \beta| \, d\nu = 0.$$

Then we say that the function $\gamma(\nu)$ is $N_\theta(J)$-summable to $\beta$. If $J = J_{\text{fin}} = \{ L \subseteq \mathbb{N} : L \text{ is a finite subset} \}$, $N_\theta(J)$-summability becomes $N_\theta$-summability, which is defined as

$$\lim_{s \to \infty} \frac{1}{\tau_s} \int_{\gamma(\nu) \in J_s} |\gamma(\nu) - \beta| \, d\nu = 0.$$

Definition 1.12. A nonnegative real-valued Lebesgue measurable function $\gamma(\nu)$ is said to be $J_\theta$-statistically convergent or $S_\theta(J)$ convergent to $\beta$, if for every $\varphi > 0$ and $\psi > 0$,

$$\left\{ s \in \mathbb{N} : \frac{1}{\tau_s} \left\{ |\nu \in J_s : |\gamma(\nu) - \beta| \geq \varphi \} \right\} \geq \psi \right\} \in J.$$

In this case, we write $S_\theta(J) - \lim \gamma(\nu) = \beta$ or $\gamma(\nu) \to \beta(S_\theta(J))$. If we take $J = J_{\text{fin}}$, then $S_\theta(J)$-convergence reduces to lacunary statistical convergence.

2. New definitions

Definition 2.1. Let $\theta$ be a lacunary sequence; and $J$ be an admissible ideal in $\mathbb{N}$ and $\gamma(\nu)$, $\mu(\nu)$ be two nonnegative real-valued Lebesgue measurable functions in the interval $(1, \infty)$. If for every $\varphi > 0$ and $\psi > 0$,

$$\left\{ s \in \mathbb{N} : \frac{1}{\tau_s} \left\{ |\nu \in J_s : |\frac{\gamma(\nu)}{\mu(\nu)} - \beta| \geq \varphi \} \right\} \geq \psi \right\} \in J,$$
then we say that the functions \( \gamma(v) \) and \( \mu(v) \) are \( \mathcal{I}_\theta \)-asymptotically equivalent of multiple \( \beta \) (denoted by \( \gamma(v) \sim^s_{\mathcal{I}_\theta} \mu(v) \)) and simply \( \mathcal{I}_\theta \)-asymptotically lacunary statistical equivalent if \( \beta = 1 \). Furthermore, let \( S^\beta_\theta(\mathcal{I}) \) denote the set of \( \gamma(v) \) and \( \mu(v) \) such that \( \gamma(v) \sim^s_{\mathcal{I}_\theta} \mu(v) \).

If we take \( \mathcal{I} = \mathcal{I}_{\text{fin}}, \mathcal{I}_\theta \)-asymptotically statistical equivalent coincides with lacunary asymptotically statistical equivalent which is given below.

**Definition 2.2.** Let \( \theta \) be a lacunary sequence; and \( \mathcal{I} \) be an admissible ideal in \( \mathbb{N} \) and \( \gamma(v), \mu(v) \) be two nonnegative real-valued functions which are measurable in the interval \( (1, \infty) \). If for every \( \varphi > 0 \)

\[
\lim_{s \to \infty} \frac{1}{\tau_s} | \left\{ v \in J_s : \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \geq \varphi \right\} | = 0,
\]

then we say that the functions \( \gamma(v) \) and \( \mu(v) \) are lacunary asymptotically statistical equivalent of multiple \( \beta \) (denoted by \( \gamma(v) \sim^s_{\theta} \mu(v) \)), and simply asymptotically statistical equivalent if \( \beta = 1 \).

**Definition 2.3.** Let \( \theta \) be a lacunary sequence; and \( \mathcal{I} \) be an admissible ideal in \( \mathbb{N} \) and \( \gamma(v), \mu(v) \) be two nonnegative real-valued Lebesgue measurable functions in the interval \( (1, \infty) \). If

\[
\left\{ s \in \mathbb{N} : \frac{1}{\tau_s} \int_{v \in J_s} \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \, dv \geq \varphi \right\} \in \mathcal{I},
\]

we say that the functions \( \gamma(v) \) and \( \mu(v) \) are strongly \( \mathcal{I}_\theta \)-asymptotically equivalent of multiple \( \beta \) (denoted by \( \gamma(v) \sim^{N_\theta}_{\mathcal{I}_\theta} \mu(v) \)) and strong simply \( \mathcal{I}_\theta \)-asymptotically lacunary equivalent if \( \beta = 1 \). Let \( N^\theta_\mathcal{I}(\mathcal{I}) \) denote the set of \( \gamma(v) \) and \( \mu(v) \) such that \( \gamma(v) \sim^{N_\theta}_{\mathcal{I}_\theta} \mu(v) \).

If \( \mathcal{I} = \mathcal{I}_{\text{fin}} = \{ L \subseteq \mathbb{N} : L \text{ is a finite subset } \} \), strongly \( \mathcal{I}_\theta \)-asymptotically equivalent becomes strongly lacunary asymptotically equivalent which is defined as

\[
\lim_{s \to \infty} \frac{1}{\tau_s} \int_{v \in J_s} \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \, dv = 0.
\]

3. Main result

**Theorem 3.1.** Let \( \theta = \{ l_s \} \) be a lacunary sequence, then

1. if \( \gamma(v) \sim^{N_\theta}_{\mathcal{I}_\theta} \mu(v) \), then \( \gamma(v) \sim^s_{\mathcal{I}_\theta} \mu(v) \);
2. if \( \gamma(v) \) and \( \mu(v) \in B(X,Y) \) and \( \gamma(v) \sim^{N_\theta}_{\mathcal{I}_\theta} \mu(v) \), then \( \gamma(v) \sim^{N_\theta}_{\mathcal{I}_\theta} \mu(v) \);
3. \( \gamma(v) \sim \mu(v) \cap B(X,Y) = \gamma(v) \sim^{N_\theta}_{\mathcal{I}_\theta} \mu(v) \cap B(X,Y) \),

where \( B(X,Y) \), is set of bounded functions.

**Proof.**

Part (1): If \( \varphi > 0 \) and \( \gamma(v) \sim^{N_\theta}_{\mathcal{I}_\theta} \mu(v) \), then

\[
\int_{v \in J_s} \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \, dv \geq \int_{v \in J_s, |\gamma(v)/\mu(v) - \beta| > \varphi} \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \, dv \geq \varphi \left\{ v \in J_s : \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \geq \varphi \right\}
\]

and so

\[
\frac{1}{\tau_s} \int_{v \in J_s} \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \, dv \geq \frac{1}{\tau_s} \left\{ v \in J_s : \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \geq \varphi \right\}.
\]
Then, for any $\psi > 0$
\[
\left\{ s \in \mathbb{N} : \frac{1}{\tau_s} \left\| \frac{\gamma(v)}{\mu(v)} - \beta \right\| \geq \varphi \right\} \subseteq \left\{ s \in \mathbb{N} : \frac{1}{\tau_s} \int_{v \in J_s} \left| \frac{\gamma(v)}{\mu(v)} - L \right| \, dv \geq \varphi \psi \right\} \in \mathcal{J}.
\]
Hence we have $\gamma(v)^{S^0_\beta(\delta)} \sim \mu(v)$.

Part (2): Suppose $\gamma(v)$ and $\mu(v)$ are in $B(X, Y)$ and $\gamma^{S^0_\beta(\delta)} \sim \mu$. Then we can assume that
\[
\frac{\gamma(v)}{\mu(v)} - \beta \leq M \text{ for all } v.
\]
Given $\varphi > 0$, we have
\[
\frac{1}{\tau_s} \int_{v \in J_s} \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \, dv = \frac{1}{\tau_s} \int_{v \in J_s, |\frac{\gamma(v)}{\mu(v)} - \beta| > \varphi} \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \, dv + \frac{1}{\tau_s} \int_{v \in J_s, |\frac{\gamma(v)}{\mu(v)} - \beta| < \varphi} \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \, dv
\]
\[
\leq \frac{M}{\tau_s} \left\{ v \in J_s : \frac{\gamma(v)}{\mu(v)} - \beta \geq \frac{\varphi}{2} \right\} + \frac{\varphi}{2}.
\]
Consequently, we have
\[
\left\{ s \in \mathbb{N} : \frac{1}{\tau_s} \int_{v \in J_s} \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \, dv \geq \varphi \right\} \subseteq \left\{ s \in \mathbb{N} : \frac{1}{\tau_s} \left\{ v \in J_s : \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \geq \frac{\varphi}{2} \right\} \geq \frac{\varphi}{2M} \right\} \in \mathcal{J}.
\]
Therefore $\gamma(v)^{N^0_\delta(\delta)} \sim \mu(v)$.

Part (3): It follows from (1) and (2). \qed

**Theorem 3.2.** Let $\mathcal{J}$ be an ideal and $\theta = \{l_s\}$ be a lacunary sequence with $\liminf q_s > 1$, then
\[
\gamma(v)^{S^\beta(\delta)} \sim \mu(v) \text{ implies } \gamma(v)^{S^0_\beta(\delta)} \sim \mu(v).
\]
**Proof.** Suppose first that $\liminf q_s > 1$, then there exists a $\delta > 0$ such that $q_s \geq 1 + \delta$ for sufficiently large $s$, which implies
\[
\frac{\tau_s}{l_s} \geq \frac{\delta}{1 + \delta}.
\]
If $x^{S^\beta(\delta)} \sim y$, then for every $\varphi > 0$ and for sufficiently large $s$, we have
\[
\frac{1}{\tau_s} \left\{ v \leq l_s : \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \geq \varphi \right\} \geq \frac{1}{l_s} \left\{ v \in J_s : \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \geq \varphi \right\} \geq \frac{\delta}{1 + \delta} \frac{1}{\tau_s} \left\{ v \in J_s : \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \geq \varepsilon \right\}.
\]
Then, for any $\psi > 0$, we get
\[
\left\{ s \in \mathbb{N} : \frac{1}{\tau_s} \left\{ v \in J_s : \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \geq \varepsilon \right\} \geq \delta \right\} \subseteq \left\{ s \in \mathbb{N} : \frac{1}{l_s} \left\{ v \leq l_s : \left| \frac{\gamma(v)}{\mu(v)} - \beta \right| \geq \varphi \right\} \geq \frac{\psi \delta}{(1 + \delta)} \right\} \in \mathcal{J}. \quad \Box
\]

For the next result we suppose that the lacunary sequence $\theta$ satisfies the condition that for any set $C \in F(\mathcal{J})$, $\bigcup\{l : l_{s-1} < v < l_s, s \in C\} \in F(\mathcal{J})$.

**Theorem 3.3.** Let $\mathcal{J}$ be an ideal and $\theta = \{l_s\}$ be a lacunary sequence with $\sup q_s < \infty$, then
\[
\gamma(v)^{S^\beta(\delta)} \sim \mu(v) \text{ implies } \gamma(v)^{S^1(\delta)} \sim \mu(v).
\]
Proof. If \( \limsup_{s} q_{s} < \infty \), then without any loss of generality we can assume that there exists a \( B \in (0, \infty) \) such that \( q_{s} < B \) for all \( s \geq 1 \). Assume that \( q = \sup_{s} q_{s} \). Write the sets

\[
C = \{ s \in \mathbb{N} : \frac{1}{s} \sum_{j=1}^{s} \left| \frac{\gamma(j)}{\mu(j)} - \beta \right| \geq \varphi \}
\]

and

\[
T = \{ n \in \mathbb{N} : \frac{1}{n} \sum_{j=1}^{n} \left| \frac{\gamma(j)}{\mu(j)} - \beta \right| \geq \varphi \},
\]

It is clear that \( C \in F(\beta) \), the filter associated with the ideal \( \beta \). Further consider that

\[
A_{j} = \frac{1}{s} \left| \{ s \in J_{j} : \frac{\gamma(s)}{\mu(s)} - \beta \geq \varphi \} \right|
\]

for all \( j \in C \). Let \( n \in \mathbb{N} \) be such that \( l_{s-1} < n < l_{s} \) for some \( s \in C \). Now

\[
\frac{1}{n} \left| \{ s \in J_{1} : \frac{\gamma(s)}{\mu(s)} - \beta \geq \varphi \} \right| \leq \frac{1}{l_{s-1}} \left| \{ l \leq s : \frac{\gamma(l)}{\mu(l)} - \beta \geq \varphi \} \right|
\]

\[
= \frac{1}{l_{s-1}} \left| \{ \nu \in J_{1} : \frac{\gamma(\nu)}{\mu(\nu)} - \beta \geq \varphi \} \right| + \cdots + \frac{1}{l_{s-1}} \left| \{ \nu \in J_{s} : \frac{\gamma(\nu)}{\mu(\nu)} - \beta \geq \varphi \} \right|
\]

\[
= \frac{1}{l_{s-1}} \left| \{ \nu \in J_{1} : \frac{\gamma(\nu)}{\mu(\nu)} - \beta \geq \varphi \} \right| + \frac{l_{2} - l_{1}}{l_{s-1}} \frac{1}{l_{s-1}} \left| \{ \mu \in J_{2} : \frac{\gamma(\mu)}{\mu(\mu)} - \beta \geq \varphi \} \right| + \cdots + \frac{l_{s} - l_{s-1}}{l_{s-1}} A_{s}
\]

\[
\leq \sup_{j \in C} A_{j} \cdot \frac{l_{s}}{l_{s-1}} < B \delta.
\]

Taking \( \delta = \frac{\delta}{\beta} \) and in view of the fact that \( \bigcup \{ n : l_{s-1} < n < l_{s}, s \in C \} \subset T \), where \( C \in F(\beta) \), it follows from our assumption on \( \theta \) that the set \( T \) also belongs to \( F(\beta) \). \( \square \)

References