Existence of nonoscillatory solutions of nonlinear neutral differential equation of second order

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Abstract

In this paper, some necessary and sufficient conditions have been obtained to ensure the existence of nonoscillatory solutions which are bounded below and above by bounded functions. These conditions are more applicable than some known results in the references. An example is included to illustrate the results obtained.

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1. Introduction

This paper is concerned with the existence of a positive solution of the neutral differential equations of the form

$$\frac{d^2}{dt^2}(x(t) - a(t)x(t - \tau)) - p(t)f(t, x(t), x(t - \sigma), x'(t), x'(t - \sigma)) = 0,$$  \hfill (1.1)

where \(t \geq t_0, \ t_0 \in \mathbb{R}, \ \tau > 0, \ \sigma \geq 0, \ a, \ p \in C([t_0, \infty); \mathbb{R}), \ f \in C([t_0, \infty) \times \mathbb{R}^4; \mathbb{R}), \) \(f\) is bounded function, and \(x(t) f(t, x(t), x(t - \sigma), x'(t), x'(t - \sigma)) > 0, \ \ x \neq 0.\) By a solution of (1.1) we mean a function \(x \in C([t_1 - m, \infty), \mathbb{R}), \ m = \max(\tau, \sigma), \) for some \(t_1 \geq t_0,\) such that \(x(t) + a(t)x(t - \tau)\) is two times continuously differentiable on \([t_1, \infty),\) and \(x(t)\) satisfies (1.1) for \(t \geq t_1.\) The problem of the existence of the solutions of neutral differential equations has been studied by several authors in the recent years. For related results we refer the reader to [10, 11, 12, 8, 15, 16], and the references cited therein. Most authors have discussed the existence of a bounded solutions by constant, however there are a few authors who have discussed and gave a conception which guarantees the existence of positive solutions of second order neutral differential equation which are bounded below and above by positive functions. Tanaka [13]...
obtained sufficient conditions for the existence of positive solutions of higher order nonlinear neutral differential equations. Weiming et al. [14] studied existence of nonoscillatory solution of second order neutral differential equations. Culakov et al. [2] studied the existence of nonoscillatory solutions of second order nonlinear neutral differential equations. Olach et al. [6] obtained some sufficient conditions for the existence of a positive solution which is bounded with exponential functions. Olach et al. [5] and [3] obtained some sufficient conditions for the existence of positive solutions which are bounded below and above by positive functions for the nonlinear neutral differential equations of higher order. Olach et al. [4] studied the existence of uncountably many positive solutions which are bounded below and above by positive functions for the first-order nonlinear neutral differential equations. Candan [1] in his paper presented the existence of positive periodic solutions for first order neutral differential equation above by positive functions for the first-order nonlinear neutral differential equations. Olach et al. [5] and [6] obtained some sufficient conditions for the existence of positive solutions which are bounded below for the existence of a positive solution which is bounded with exponential functions. Olach et al. [3] obtained some sufficient conditions for the existence of positive solutions which are bounded below for the existence of bounded solution of (1.1) is investigated. In the following results, some necessary and sufficient conditions are obtained to ensure the existence of positive solution for (1.1), which is bounded by two positive functions.

**Theorem 2.1.** Suppose \( p(t) \geq 0 \) and there exist bounded functions \( u, v \in C^1([t_0, \infty), [0, \infty)) \), a constant \( \delta > 0 \) such that

\[
\frac{u(t)}{u(t-\tau)+\delta} \leq a(t) \leq \frac{v(t)}{v(t-\tau)}, \quad a(t) < 1, \quad v(t-\tau) > 0, \quad t \geq t_1 \geq t_0 + m,
\]

and there exists \( k \in (t, t_1) \) for \( t \in [t_0, t_1] \) such that

\[
(v(t) - v(t_1))(1 - \frac{u'(k)}{v'(k)}) \geq 0,
\]

\[
\int_{t_0}^{\infty} \int_{s}^{\infty} p(\xi) \, d\xi \, ds < \infty, \quad s \geq t_0.
\]

Then (1.1) has a positive solution which is bounded by the functions \( u \) and \( v \).

**Proof.** Let \( C([t_0, \infty), \mathbb{R}) \) be the set of all continuous bounded functions with the norm \( \|x\| = \sup_{t \geq t_0} |x(t)| \). Then \( C([t_0, \infty), \mathbb{R}) \) is a Banach space. We define a closed, bounded, and convex subset \( \Omega \) of \( C([t_0, \infty), \mathbb{R}) \) as follows

\[
\Omega = \{x(t) \in C([t_0, \infty), \mathbb{R}) : u(t) \leq x(t) \leq v(t), \quad t \geq t_0\}.
\]
By (2.4) it follows that
\[ \lim_{t \to +\infty} \int_{t}^{\infty} \int_{s}^{\infty} p(\xi) \, d\xi \, ds = 0, \quad s \geq t. \]

It follows that for every \( \varepsilon > 0 \), there exists \( t_1 \geq t_0 \) such that
\[ \int_{t}^{\infty} \int_{s}^{\infty} p(\xi) \, d\xi \, ds < \varepsilon, \quad s \geq t \geq t_1. \]

For simplicity let \( f(t,x(t)) = f(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma)) \), we now define maps \( S_1, S_2 \in C([t_0,\infty), \mathbb{R}) \) as follows
\[
\begin{align*}
(S_1x)(t) &= \begin{cases} 
(S_1x)(t_1), & t_0 \leq t \leq t_1, \\
a(t)x(t-\tau), & t \geq t_1,
\end{cases} \\
(S_2x)(t) &= \begin{cases} 
(S_2x)(t_1) + v(t) - v(t_1), & t_0 \leq t \leq t_1, \\
-\int_{t}^{\infty} \int_{s}^{\infty} p(\xi) f(\xi, x(\xi)) \, d\xi \, ds, & t \geq t_1.
\end{cases}
\end{align*}
\]

We will show that for any \( x, y \in \Omega \) we have \( S_1x + S_2y \in \Omega \). Let \( x, y \in \Omega \) and \( t \geq t_1 \) we obtain
\[
(S_1x)(t) + (S_2y)(t) = a(t)x(t-\tau) - \int_{t}^{\infty} \int_{s}^{\infty} p(\xi) f(\xi, x(\xi)) \, d\xi \, ds
\leq a(t)v(t-\tau)
\leq v(t).
\]

For \( t \in [t_0, t_1] \), we have
\[
(S_1x)(t) + (S_2y)(t) = (S_1x)(t_1) + (S_2y)(t_1) + v(t) - v(t_1)
\leq v(t_1) + v(t) - v(t_1)
= v(t).
\]

Since \( f(t,x(t)) \) is bounded so we get
\[
\lim_{t \to +\infty} \int_{t}^{\infty} \int_{s}^{\infty} p(\xi) f(\xi, x(\xi)) \, d\xi \, ds = 0, \quad s \geq t.
\]

Hence, for \( t \geq t_1 \) we get
\[
(S_1x)(t) + (S_2y)(t) = a(t)x(t-\tau) - \int_{t}^{\infty} \int_{s}^{\infty} p(\xi) f(\xi, x(\xi)) \, d\xi \, ds
\geq a(t)(u(t-\tau) - (1 - a(t)) \varepsilon)
\geq a(t)(u(t-\tau + \varepsilon)),
\]
by choosing \( \delta = \varepsilon \), and using condition (2.1) it follows from (2.5)
\[
(S_1x)(t) + (S_2y)(t) \geq u(t), \quad t \geq t_1.
\]

Let \( t \in [t_0, t_1] \), then by Cauchy mean value theorem in virtue of (2.3), for \( t \in [t_0, t_1] \) there exists \( k \in (t, t_1) \) such that
\[
\begin{align*}
u'(k)(v(t_1) - v(t)) &= v'(k)((u(t_1) - u(t))
\end{align*}
\]
\[
\begin{align*}
u'(k) u(t) &= v'(k) u(t_1) + u'(k)(v(t) - v(t_1))
\end{align*}
\]
\[
\begin{align*}u(t) &= u(t_1) + \frac{u'(k)}{v'(k)} (v(t) - v(t_1))
\end{align*}
\]
\[
\begin{align*}u(t) \leq u(t_1) + v(t) - v(t_1), \quad t_0 \geq t \geq t_1.
\end{align*}
\]
Then for \( t \in [t_0, t_1] \) and for any \( x, y \in \Omega \), we obtain
\[
(S_1 x)(t) + (S_2 y)(t) = (S_1 x)(t_1) + (S_2 y)(t_1) + v(t) - v(t_1) \\
\geq u(t_1) + v(t) - v(t_1) \geq u(t).
\]

Thus we have proved that \( S_1 x + S_2 y \in \Omega \) for any \( x, y \in \Omega \). We will show that \( S_1 \) is a contraction mapping on \( \Omega \). For \( x, y \in \Omega \) and \( t \geq t_1 \), we have
\[
\| S_1 x - S_1 y \| = \sup_{t \geq t_1} |a(t)x(t - \tau) - a(t)y(t - \tau)| \\
= \sup_{t \geq t_1} a(t)|x(t - \tau) - y(t - \tau)| \\
\leq c_1 \sup_{t \geq t_1} |x(t - \tau) - y(t - \tau)|, \quad \text{(where } a(t) \leq c_1 < 1) \\
= c_1 \| x - y \|.
\]

Also for \( t \in [t_0, t_1] \), we get
\[
\| S_1 x - S_1 y \| = \sup_{t_0 \leq t \leq t_1} \| (S_1 x)(t) - (S_1 y)(t) \| \\
= |a(t_1)x(t_1 - \tau) - a(t_1)y(t_1 - \tau)| \\
= a(t_1)|x(t_1 - \tau) - y(t_1 - \tau)| \\
\leq a(t_1) \sup_{t_0 \leq t \leq t_1} |x(t - \tau) - y(t - \tau)| \\
= a(t_1) \| x - y \|.
\]

We conclude that \( S_1 \) is a contraction mapping on \( \Omega \). We now show that \( S_2 \) is completely continuous. First we will show that \( S_2 \) is continuous. Let \( x_k = x_k(t) \in \Omega \) be such that \( \lim_{k \to \infty} x_k(t) = x(t) \). Since \( \Omega \) is closed, we conclude that \( x(t) \in \Omega \). For \( t \geq t_1 \), we have
\[
\| (S_2 x_k)(t) - (S_2 x)(t) \| = \sup_{t \geq t_1} \| (S_2 x_k)(t) - (S_2 x)(t) \| \\
\leq \int_t^{t_1} \int_s^\infty \| p(\xi) (f(\xi, x_k(\xi)) - f(\xi, x(\xi))) \| d\xi \, ds \\
\leq \int_t^{t_1} \int_s^\infty |p(\xi) (f(\xi, x_k(\xi)) - f(\xi, x(\xi)))| d\xi \, ds.
\]

According to (2.5), we get
\[
\int_{t_1}^{t} \int_s^\infty \| p(\xi) f(\xi, x(\xi), x(\xi - \sigma), x'(\xi), x'(\xi - \sigma)) \| d\xi \, ds < \infty. \tag{2.6}
\]

Since
\[
\lim_{k \to \infty} f(t, x_k(t)) = f(t, x(t)),
\]
then by applying the Lebesgue dominated convergence Theorem 1.2, we obtain
\[
\lim_{k \to \infty} \| (S_2 x_k)(t) - (S_2 x)(t) \| = 0.
\]

This means that \( S_2 \) is continuous. We now show that \( S_2 \Omega \) is relatively compact. It is sufficient to show by Arzelá-Ascoli theorem that the family of functions \( \{ S_2 x : x \in \Omega \} \) is uniformly bounded and equicontinuous on \( [t_0, \infty) \). The uniform boundedness follows from the definition of \( \Omega \). For the equicontinuity we only need to show that for any given \( \varepsilon > 0 \) the interval \( [t_0, \infty] \) can be decomposed into finite subintervals
in such a way that on each subinterval all functions of the family have a change of amplitude less than $\epsilon$. Then with regard to (2.6), for $x \in \Omega$ and any $\epsilon > 0$, we take $t_1 \geq t_2$ large enough so that
\[ \left| \int_{t_2}^{\infty} \int_{s}^{\infty} p(\xi) f(\xi, x(\xi), x(\xi - \sigma), x'(\xi), x'(\xi - \sigma)) \, d\xi \, ds \right| < \epsilon/2. \]

Then, for $x \in \Omega$, $t_s \leq T_1 < T_2$, we have
\[ \| (S_2x)(T_2) - (S_2x)(T_1) \| \leq \left| - \int_{T_2}^{\infty} \int_{s}^{\infty} p(\xi) f(\xi, x(\xi), x(\xi - \sigma), x'(\xi), x'(\xi - \sigma)) \, d\xi \, ds \right| \]
\[ + \left| \int_{T_1}^{\infty} \int_{s}^{\infty} p(\xi) f(\xi, x(\xi)) \, d\xi \, ds \right| \]
\[ \leq \int_{T_2}^{\infty} \int_{s}^{\infty} p(\xi) f(\xi, x(\xi), x(\xi - \sigma), x'(\xi), x'(\xi - \sigma)) \, d\xi \, ds \]
\[ + \int_{T_1}^{\infty} \int_{s}^{\infty} p(\xi) f(\xi, x(\xi)) \, d\xi \, ds, \]
hence we get
\[ \| (S_2x)(T_2) - (S_2x)(T_1) \| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]

For $x \in \Omega$ and $t_1 \leq T_1 < T_2 \leq t_*,$ it yields
\[ \| (S_2x_k)(T_2) - (S_2x)(T_1) \| \leq \left| - \int_{T_2}^{\infty} \int_{s}^{\infty} p(\xi) f(\xi, x(\xi), x(\xi - \sigma), x'(\xi), x'(\xi - \sigma)) \, d\xi \, ds \right| \]
\[ + \left| \int_{T_1}^{\infty} \int_{s}^{\infty} p(\xi) f(\xi, x(\xi)) \, d\xi \, ds \right| \]
\[ \leq \max_{t_1 \leq s \leq t_*} \left\{ \int_{s}^{\infty} p(\xi) f(\xi, x(\xi), x(\xi - \sigma), x'(\xi), x'(\xi - \sigma)) \, d\xi \right\} (T_2 - T_1). \]

Thus there exists $\delta_1 = \frac{\epsilon}{M}$, where $M = \max_{t_1 \leq s \leq t_*} \left\{ \int_{s}^{\infty} p(\xi) \, d\xi \right\}$ such that
\[ \| (S_2x(T_2) - (S_2x)(T_1) \| < \epsilon, \quad \text{if} \quad 0 < T_2 - T_1 < \delta_1. \]

Finally for any $x \in \Omega$, $t_0 \leq T_1 < T_2 \leq t_1$, there exists $k_1 \in (T_1, T_2)$ and $\delta_2 = \frac{\epsilon}{|v'(k_1)|} > 0$ such that
\[ \| (S_2x)(T_2) - (S_2x)(T_1) \| = |v(T_2) - v(T_1)| \]
\[ = |v'(k_1)| (T_2 - T_1) < \epsilon, \quad \text{if} \quad 0 < T_2 - T_1 < \delta_2. \]

Then $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S_2 \Omega$ is relatively compact subset of $C([t_0, \infty), \mathbb{R})$. By Lemma 1.1 there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. We conclude that $x_0(t)$ is a positive solution of (1.1). The proof is complete. \qed

**Corollary 2.2.** Suppose that there exist bounded functions $u, v \in C^1([t_0, \infty), [0, \infty))$, a constant $\delta > 0$ and a positive integer $n$ such that
\[ \frac{u(t)}{u(t - \tau) + \delta} + n - 1 \leq u(a(t)) \leq \frac{v(t)}{v(t - \tau) + \delta} + (n - 1)a(t), \quad a(t) < 1, \]
\[ u(t) > 0, \quad v(t) > 0, \quad t \geq t_1 > t_0 + m, \]
in addition to the conditions (2.2), (2.3), (2.4) hold. Then (1.1) has a positive solution which is bounded by the functions $u$ and $v$.

**Proof.** We claim that condition (2.7) implies condition (2.1), we will use the induction to show it when
n = 1 there is nothing to prove, when n = 2 then it follows from condition (2.7)
\[
\frac{u(t)}{u(t-\tau)+\delta} + 1 \leq 2a(t) \leq \frac{v(t)}{v(t-\tau)} + a(t),
\]
\[
\frac{u(t)}{u(t-\tau)+\delta} \leq \frac{1}{2}(\frac{u(t)}{u(t-\tau)+\delta} + a(t)) \leq \frac{1}{2}(\frac{u(t)}{u(t-\tau)+\delta} + 1) \leq a(t) \leq \frac{1}{2}(\frac{v(t)}{v(t-\tau)} + a(t)) \leq \frac{v(t)}{v(t-\tau)},
\]
we can treat similarly to show it when n = 3, 4, \ldots. Hence all conditions of Theorem 2.1 hold, then
according to Theorem 2.1, equation (1.1) has a positive solution which is bounded by the functions u and v.

\[\Box\]

**Example 2.3.** Consider the following non-linear neutral differential equation
\[
\frac{d^2}{dt^2} (x(t) - a(t) x(t-1)) + p(t) (e^{-t} + 2e^{-t+\frac{3}{2}}) = 0, \quad t \geq \frac{1}{2}, \tag{2.8}
\]
where \(\frac{1}{4} \leq a(t) < 1, \ p(t) = \frac{9}{4} e^{-\frac{t}{2}}, \ f(t, x(t)) = e^{-t} + 2e^{-t+\frac{3}{2}}.\)

**Solution:** Let \(u(t) = e^{-t} + e^{-2t}\) and \(v(t) = \frac{3}{2} - e^{-t}, \ t \geq \frac{1}{2}.\) The condition (2.2) clearly satisfies, now we will show that the condition (2.1) holds, choose \(\delta = \frac{1}{2}\) then it follows that
\[
\frac{e^{-4} + e^{-2t}}{e^{-4} + e^{-2(t-1)} + \frac{1}{2}} < a(t) < \frac{\frac{3}{2} - e^{-t}}{\frac{3}{2} - e^{-(t-1)}}, \quad \text{for all } t \geq \frac{3}{2}.
\]
To see condition (2.3) consider that
\[
\left(\frac{3}{2} - e^{-t} - (\frac{3}{2} - e^{-1})\right)(1 - \frac{2e^{-2k}}{e^{-k}}) \geq \frac{1}{e}(1 + 2e^{-k}) > 0, \quad \text{for all } k.
\]
It remains to show that condition (2.4) holds
\[
\int_{t}^{\infty} \int_{s}^{\infty} p(t) \, dt \, ds = \int_{\frac{1}{2}}^{\infty} \int_{s}^{\frac{9}{4}} e^{-\frac{t}{2}} \, dt \, ds = 9e^{-\frac{1}{2}}.
\]
Then all conditions of Theorem 2.1 (or Corollary 2.2) are satisfies. According to Theorem 2.1 (or Corollary 2.2), equation (2.8) has a positive solution which is bounded by the functions u and v.

**Theorem 2.4.** Suppose \(p(t) \geq 0\) and there exist bounded functions \(u, v \in C^1([t_0, \infty), [0, \infty)),\) a constant \(c > 0\) such that
\[
u(t) \leq v(t), \quad t \geq t_0, \tag{2.9}
\]
\[
u(t-\tau) - \nu(t_1 - \tau) - u(t - \tau) + u(t_1 - \tau) \geq 0, \quad t \in [t_0, t_1], \tag{2.10}
\]
\[
\frac{1}{u(t-\tau)} \left(\int_{t}^{\infty} \int_{s}^{\infty} p(\xi)f(\xi, v(\xi), v(\xi - \tau), v'(s), v'(\xi - \tau)) \, d\xi \, ds\right) \leq a(t) - 1 \leq \frac{1}{v(t-\tau)} \left(\int_{t}^{\infty} \int_{s}^{\infty} p(\xi)f(\xi, u(\xi), u(\xi - \tau), u'(s), u'(\xi - \tau)) \, d\xi \, ds\right) \leq c < 1, \quad t \geq t_1. \tag{2.11}
\]
Then (1.1) has a positive solution which is bounded by the functions u and v.
Then with regard to (2.12), it follows that

Hence all conditions of Theorem 2.4 are satisfied.

Since

\( H(1.1) \)

Then

We only need to prove that condition (2.12) implies (2.10). Let \( t \in [t_0, t_1] \) and set

\[ H(t) = v(t - \tau) - v(t_1 - \tau) - u(t - \tau) + u(t_1 - \tau). \]

Then with regard to (2.12), it follows that

\[ H'(t) = v'(t - \tau) - u'(t - \tau). \]

Since \( H(t_1) = 0 \) and \( H'(t) \leq 0 \) for \( t \in [t_0, t_1] \), this implies that

\[ H(t) = v(t - \tau) - v(t_1 - \tau) - u(t - \tau) + u(t_1 - \tau) \geq 0, \quad t_0 \leq t \leq t_1. \]

Hence all conditions of Theorem 2.4 are satisfied.

Corollary 2.5. Suppose that there exist bounded functions \( u, v \in C^1([t_0, \infty), [0, \infty)) \), a constant \( c > 0 \) such that (2.9), (2.11) hold in addition to the condition

\[ v'(t - \tau) \leq u'(t - \tau), \quad t \in [t_0, t_1]. \]  

(2.12)

Then (1.1) has a positive solution which is bounded by the functions \( u \) and \( v \).

Proof. We only need to prove that condition (2.12) implies (2.10). Let \( t \in [t_0, t_1] \) and set

\[ H(t) = v(t - \tau) - v(t_1 - \tau) - u(t - \tau) + u(t_1 - \tau). \]

Then with regard to (2.12), it follows that

\[ H'(t) = v'(t - \tau) - u'(t - \tau). \]

Since \( H(t_1) = 0 \) and \( H'(t) \leq 0 \) for \( t \in [t_0, t_1] \), this implies that

\[ H(t) = v(t - \tau) - v(t_1 - \tau) - u(t - \tau) + u(t_1 - \tau) \geq 0, \quad t_0 \leq t \leq t_1. \]

Hence all conditions of Theorem 2.4 are satisfied.

Corollary 2.6. Suppose that there exists a function \( v \in C^1([t_0, \infty), [0, \infty)) \), a constant \( c > 0 \) and \( t_1 \geq t_0 + m \) such that

\[ a(t) = 1 + \frac{1}{v(t - \tau)} \int_{t_1}^{\infty} p(s)f(s,v(s),v(s - \tau),v'(s),v'(s - \tau)) \, ds \leq \delta < 1, \quad t \geq t_1. \]

Then (1.1) has a solution \( x(t) = v(t - \tau), \quad t \geq t_1. \)

Proof. We put \( u(t - \tau) = v(t - \tau) \) and apply Theorem 2.4.

Conclusion

This paper is concerned with establishing some sufficient conditions to ensure the existence of a positive solution which is bounded by two bounded functions. Two main results were obtained with corollaries.

References


