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Some Resultson Isomorphic Fuzzy Subgroups

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Abstract

The purpose of this paper is to give some properties of the isomorphic fuzzy subgroups and to obtain relationships among their level subgroups.

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1. Introduction

The concept of fuzzy sets was first introduced by Zadeh in 1965 [13]. The study offuzzy algebraic structures was started with the introduction of the concept of fuzzysubgroups by Rosenfeld in 1971 [10]. In 1975 Negoita and Ralescu [7], consideredgeneralization of Rosenfeld's definition in which theunit interval I = [0,1] wasreplaced by an appropriate lattice structure. In 1979 Anthony and Sherwood [2]redefined a fuzzy subgroup of a group using the concept of triangular norm. In factmany basic properties in group theory are found to be carried over to fuzzy groups.R. Ameri and H. Hedayeti studied fuzzy isomorphism and quotient of fuzzy subpolygroups [1]. Li and Wang presented (λ,α)-homomorphisms and (λ,α)-isomorphismsbetween two intuitionist fuzzy groups [7].Apart from these, many researchers have conducted various studies on fuzzy algebraic structures and fuzzy isomorphic groups[4, 6, 8, 11]. The classification of fuzzy subgroups ofa finite group of any given order is considered as a fundamental general problem.

This paper continues that study by looking at isomorphic fuzzy subgroups.The second section of this paper gives some conventions, de definitions and theoremswhich are essential later in the paper. Some of the de definitions can be found elsewhere, but we include them for the convenience of the reader. Our main is to givesome properties of the isomorphic fuzzy subgroups in section three.

2. Preliminaries

In this section we summarize the preliminary definition and results that are required later in this paper. Let us first recall some concepts related isomorphic fuzzy subgroups. For details, we refer to [2], [3] and [10].

Definition 2.1. Let μ be a fuzzy subset of a group G. Then μ is called a fuzzysubgroup of G under a t-norm T (T fuzzy subgroup)iff for all $x,\ y\ \in\ G$

(1) $\mu\ (xy) \geq T(\mu(x), \mu(y))$

(2) $\mu\ (e)\ =\ 1$, where e is the identity of G.

(3) $\mu\ (x^{-1}) \geq \mu(x)$

where the product of x and y is denoted by xy and the inverse of x byx^{-1}.

Note:μ is called a Min-fuzzy subgroup if μ satisfies conditions (1) and (3) only by replacing T with Min.

Definition 2.2. Let μ be a fuzzy subset of a group G. Thenthesubset$\{x \in G : \mu(x) \geq t\}$, $t \in [0,1]$, of G is called a t-level subset of G under μ and it is denotedbyμ_t.

Definition 2.3. For $i = 1$ and 2, let μ_i be a fuzzy subset of a group G_i. If there is a group isomorphism φ from G_1 to G_2 such that $\mu_1 = \mu_2{}^\circ\varphi$, then we say μ_1 and μ_2 are isomorphic.

Definition 2.4. Let μ be a T −fuzzy subgroup of G. Then the least positive integer n satisfying the condition $\mu(x^n) = 1$ is called the fuzzy order of x with respect to μ and we use the notation $\mu \bullet (x) = n$. If n does not exist, we say that x is of infinitive fuzzy order with respect to μ and write $\mu \bullet (x) = \infty$.

Definition 2.5. Let μ be a T −fuzzy subgroup of G. Then the least common multiple of the fuzzy orders of the elements of G with respect to μ is called the order of the fuzzy subgroup μ and it is denoted by $|\mu|_F$. If it does not exist, then $|\mu|_F = \infty$.

Theorem 2.6. Let μ be a T −fuzzy subgroup of G.Then,

1. If $\mu(x^r) = 1$, then $\mu \bullet (x)|r$, and
2. $\mu \bullet (x) < \infty$, then $\mu \bullet (x)|^\circ(x)$, where $^\circ(x)$ is the order of x.

Definition 2.7. Let μ be a T −fuzzy subgroup of G and p be prime. Then μ is called a primary fuzzy subgroup of G if for every $x$in G, there exist a natural number r such that $\mu \bullet (x) = p^r$.

Theorem 2.8. Ifφ is an isomorphism from G_1 to G_2 and μ_2 is an fuzzy subgroup of G_2under T, then $\mu_1 = \mu_2{}^\circ\varphi$ is a fuzzy subgroup of G_1 under T.

Definition 2.9. For each $i = 1,2, \dots, n$ let G_i be a fuzzy subgroup under a minimum operation in a group X_i. The membership function of the product$G = G_1 \times G_2 \times \dots \times G_n$in $X = X_1 \times X_2 \times \dots \times X_n$is defined by

$$(G_1 \times G_2 \times \dots \times G_n(x_1, x_2, \dots, x_n) = \min\big(G_1(x_1), G_2(x_2), \dots, G_n(x_n)\big).$$

3. İsomorphic Fuzzy Subgroups

Theorem 3.1. If φ is an isomorphism from G_1onto G_2 and μ_1 is a fuzzy subgroup of $G_1$and $\mu_1 = \mu_2{}^\circ\varphi$, then μ_2 is a fuzzy subgroup of G_2.

ProofSince φ is an isomorphism from G_1onto G_2, for all $x, y \in G_2$, there are a$, b \in G_1$ such that $\varphi(a) = x$ and $\varphi(b) = y$. Using de_nition of fuzzysubgroupand$\mu_1$is a fuzzy subgroup, we have that

$$\mu_2(xy) = \mu_2\big(\varphi(a)\varphi(b)\big) = \mu_2(\varphi(ab))$$

$$= (\mu_2{}^\circ\varphi)(ab) = \mu_1(ab)$$

$$\geq min\{\mu_1(a), \mu_1(b)\ \}$$

$$= min\{(\mu_2{}^\circ\varphi)(a), (\mu_2{}^\circ\varphi)(b)\}$$

$$= min\big\{\mu_2\big(\varphi(a)\big), \mu_2(\varphi(b))\big\}$$

$$= min\{\mu_2(x), \mu_2(y)\}$$

and

$$\mu_2(x^{-1}) = \mu_2\big(\varphi(a)\big) = (\mu_2{}^\circ\varphi)(a)$$

$$= \mu_1(a) = \mu_1(a^{-1}) = (\mu_2{}^\circ\varphi)(a^{-1})$$

$$\mu_2\big(\varphi(a^{-1})\big) = \mu_2(x)$$

Theorem 3.2. The products of isomorphic fuzzy subgroups are isomorphic.

Proof. Let A_i and B_i be isomorphic fuzzy subgroup of $G_i$and H_i, respectively and $\emptyset: G = G_1 \times \dots \times G_n \to H = H_1 \times \dots \times H_n$ be an isomorphism. Then for all $x_i \in G_i$ we get that

$$A_1 \times \ldots \times A_n(x_1, \ldots, x_n) = \min((B_1 \circ \emptyset)(x_1), \ldots, (B_n \circ \emptyset)(x_n)$$

$$= \min(B_1(\emptyset(x_1)), \ldots, B_n(\emptyset(x_n)))$$

$$= (B_1 \times \ldots \times B_n)(\emptyset(x_1), \ldots, \emptyset(x_n))$$

$$= (B_1 \times \ldots \times B_n)\circ\emptyset(x_1, \ldots, x_n)$$

Proposition 3.3. Let G_1 and G_2 be groups, $\emptyset: G_1 \to G_2$ an isomorphism, μ_1 and μ_2 be fuzzy subgroups of $G_1$and G_2, respectively. Then μ_1 isomorphic to $\mu_2$iff for all$x \in Ker\emptyset$, $\mu_1(x) = 1$.

Proof. Let μ_1 and μ_2 be isomorphic fuzzy subgroups. For all $x \in Ker\emptyset$,

$$\mu_1(x) = (\mu_2 \circ \emptyset)(x) = \mu_2(e) = 1$$

Conversely, suppose for all $x \in Ker\emptyset, \mu_1(x) = 1$.Then

$$\mu_1(x) = 1 \geq \mu_2(\emptyset(x)) = (\mu_2 \circ \emptyset)(x),$$

that is $\mu_2 \circ \emptyset \subset \mu_1$. Similarly,

$$(\mu_2 \circ \emptyset)(x) = \mu_2(\emptyset(x)) = \mu_2(e) = 1 \geq \mu_1(x)$$

So we have

$$\mu_1 = \mu_2 \circ \emptyset$$

Proposition 3.4. Let G_1 and G_2 be groups, $\emptyset: G_1 \to G_2$ an isomorphism,μ_1 and μ_2 be $T-$fuzzy subgroups of $G_1$and G_2, respectively. Then $\mu_1 \bullet (x) = \mu_2 \bullet (\emptyset(x))$.

Proof. Let $\mu_1 \bullet (x) = n$and $\mu_2 \bullet (\emptyset(x)) = m$. Then $\mu_1(x^n) = 1$. Since μ_1 and μ_2 are isomorphic, we get

$$\mu_1(x^n) = (\mu_2 \circ \emptyset)(x^n) = \mu_2((\emptyset(x))^n) = 1.$$

By Theorem 2.5, $m|n$. On the other hand,

$$\mu_2((\emptyset(x))^m) = (\mu_2 \circ \emptyset)(x^m) = \mu_1(x^m) = 1$$

and hence $n|m$. Therefore we have $n = m$.

Proposition 3.5. Let G_1 and G_2 be groups, $\emptyset: G_1 \to G_2$ an isomorphism, μ_1 and μ_2 be $T-$fuzzy subgroups of $G_1$and G_2, respectively. If μ_1 primary fuzzy subgroup of G_1, then μ_2 is a primary fuzzy subgroup of G_2.

Proof. Ifμ_1 is primary fuzzy subgroup of G_1, then for every x in G_1, there exists a natural number r such that $\mu_1 \bullet (x) = p^r$. Hence $\mu_1\left(x^{p^r}\right) = 1$. Since μ_1 and μ_2 are isomorphic fuzzy subgroups, there exists $\emptyset$ isomorphism between G_1 and G_2 such that $\mu_1 = \mu_2 \circ \emptyset$. That is

$$\mu_1\left(x^{p^r}\right) = (\mu_2 \circ \emptyset)\left(x^{p^r}\right) = \mu_2\left(\left(\emptyset(x)\right)^{p^r}\right) = 1.$$

From Theorem 2.5, we can write $\mu_1 \bullet (\emptyset(x))|p^r$. There are two cases. Either $\mu_2 \bullet \left(\emptyset(x)\right) = 1$for every $\emptyset(x) \in G_2$ or$\mu_2 \bullet \left(\emptyset(x)\right) = p^s$ where $s \leq r$ and r, s are natural numbers. If $\mu_2 \bullet \left(\emptyset(x)\right) = 1$, then we get $\mu_2\left(\emptyset(x)\right) = 1$, that is,

$$(\mu_2 \circ \emptyset)(x) = \mu_1(x) = 1.$$

But it is not true. Thus $\mu_2 \bullet \left(\emptyset(x)\right) = p^s$. So μ_2 is a primary fuzzy subgroup of G_2.

Proposition 3.6. If μ_1 and μ_2 are isomorphic primary fuzzy subgroups, then $|\mu_1|_F = |\mu_2|_F$.

Proof. From Propotion3.2 $\mu_1 \bullet (x) = \mu_2 \bullet \emptyset(x)$ for all $x \in G_1$ Using definition of fuzzy group, we can easily see that $|\mu_1|_F = |\mu_2|_F$.

Proposition 3.7. If μ_1 and μ_2 are isomorphic primary fuzzy subgroups, then μ_{1_t} and μ_{2_t} are $t-$level subgroups of $\mu_1$and μ_2, respectively, are isomorphic.

Proof. For all $x \in \mu_{1_t}$,

$$\mu_1(x) = (\mu_2 \circ \emptyset)(x) = \mu_2\left(\emptyset(x)\right) \geq t,$$

That is, $\emptyset(x) \in \mu_{2_t}$ on the other hand for all $\emptyset(x) \in \mu_{2_t}$

$$\mu_2\left(\emptyset(x)\right) = (\mu_2 \circ \emptyset)(x) = \mu_1(x) \geq t.$$

Thus $x \in \mu_{1_t}$. Therefore, there is one-to-one correspondence among elements of the $\mu_{1_t}$and μ_{2_t}. Since $\emptyset$ is an isomorphism, μ_{1_t} and μ_{2_t} are isomorphic subgroups.

Theorem 3.8. Let μ^1 and μ^2 fuzzy subgroup of finite groups $G_1$and G_2, respectively. Ifμ^1 and μ^2 are isomorphic, then $(\mu^1) = Im(\mu^2)$.

Proof. Let $t \in Im(\mu^1)$. Since μ^1 and μ^2 are isomorphic, there exists isomorphism $\emptyset: G_1 \to G_2$ such that $\mu^1(x) = (\mu^2 \circ \emptyset)(x)$. Then $\mu^1(x) = \mu^2(\emptyset(x)) = t$and $t \in Im(\mu^2)$. On the other hand, let $t \in Im(\mu^2)$. It is clear that $\mu^2(y) = \mu^2(\emptyset(x)) = (\mu^2 \circ \emptyset)(x) = \mu^1(x) = t$. That is $t \in Im(\mu^1)$. This complete the proof.

Theorem 3.9. Let μ^1 and μ^2 fuzzy subgroup of finite groups$G$and H, respectively. If level subgroups of μ^1 have a chain as

$$\mu^1_{t_0} < \mu^1_{t_1} < \cdots < \mu^1_{t_r} = G$$

for the numbers $t_0, t_1, \dots\ t_r$ which belong to $Im(\mu^1)$ with $t_0 > t_1 > \ \dots > t_r$, then level subgroups of μ^2 has a chain as

$$\mu^2_{t_0} < \mu^2_{t_1} < \cdots < \mu^2_{t_s} = H$$

with same length.

Proof. Since μ^1 isomorphic toμ^2, there is an one to one corresponding between level subgroup $\mu^1_{t_i}$, and level subgroup $\mu^2_{t_i}$ for all $i = 1,2, \dots, n$. Notice that

$$\mu^1(x) = (\mu^2 \circ \emptyset)(x) = \mu^2(\emptyset(x)) = \mu^2(y) \geq t_i$$

Because each level subset is a subgroup of H in the usual sense, let we prove that $\mu^2_{t_i}$ is a subset of $\mu^2_{t_{i+1}}$. For all $y \in \mu^2_{t_i}$, $\mu^2(y) \geq t$. We get

$$\mu^2(\emptyset(x)) = (\mu^2 \circ \emptyset)(x) = \mu^1(x) \geq t_i$$

because of being isomorphic of μ^1 and μ^2. As $\mu^1_{t_i} < \mu^1_{t_{i+1}}, x \in \mu^1_{t_{i+1}}$. So

$$\mu^1(x) = (\mu^2 \circ \emptyset)(x) = \mu^2(\emptyset(x)) = \mu^2(y) \geq t_{i+1}$$

and $y \in \mu^2_{t_{i+1}}$. Therefore$\mu^2_{t_i}$ is subset of $\mu^2_{t_{i+1}}$. From Theorem3.8, $Im(\mu^1) = Im(\mu^2)$ so level subgroups of μ^1 and μ^2 has a chain with same length.

References:

[1] R. Ameri, Fuzzy isomorphism and quotient of fuzzy sub polygroups, Quasi-groups and Related System, 13, (2005), 175-184.

[2] J.M. Antony and H. Sherwood, Fuzzy groups redefined, J. Math. Anal. Appl. 69,(1979), 124-130.

[3] M.Atif, A. Mishref, Primary fuzzy subgroups, Fuzzy Sets and Systems, (2000), 112 313-318.

[4] E. Hendukolaie, M. Aliakbarnia. Omran, Y. Nasabi, On fuzzy isomorphisms theorems of Γ −Hypernear-rings, Journal of mathematics and computer Science, 7 (2013) 80–88.

[5] X.D. Li and G.S. Wang, (λ,α)-Homomorphism of intuitionistic fuzzy groups, HacettepeJournal of Mathematics and Statistics, 40(5),(2011), 663-672.

[6] S.H. Nasseri, F.Taleshian, Z. Alizadeh and J. Vahidi, A New Method for Ordering *LR* Fuzzy Number, Journal of mathematics and computer Science, Vol. 4 No. 3 (2012) 283 – 294.

[7] C.V.Negoita and D.A. Ralescu, Applications of fuzzy sets to system Analysis (Wiley,New York,)(1975).

[8] E. Ranjbar-Yanehsari, M. Asghari-Larimi, Union and intersection fuzzy subhypergroups,Journal of mathematics and computer Science, Vol .5 No. 2 (2012) 82-90.

[9] S. Ray, Isomorphic fuzzy groups, Fuzzy Sets and Systems, 50, 2, (1992),201-207.

[10] A. Rosenfeld, Fuzzy groups, J.Math. Appl. 35, (1971),512-517.

[11] A. Taleshian, F. Khaniani, A. A. Hosseinzadeh, Investigate of fuzzy semi-simple Lie algebra, Journal of mathematics and computer Science,Vol .1 No.4 (2010) 287-292.

[12] H. Sherwood,(1993) Product of fuzzy subgroups, Fuzzy Sets and Systems, 11, 79-89. R.Suryansu, Generated cyclic fuzzy groups, Inform. Sci.69, (1983),185-200.

[13] L. A. Zadeh, Fuzzy sets, Inform. and control, 8, (1965),338-353.