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Bounded and sequential σ-approximate amenability of Banach algebras

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Abstract

In this paper, we study the notions of bounded σ-approximate amenability and sequential σ-approximate amenability for Banach algebras, where σ is a continuous homomorphism of the corresponding Banach algebra. Also, we discuss some hereditary properties of these concepts.

Keywords: Banach algebra, σ-derivation, bounded σ-approximately inner, bounded σ-approximate amenability, sequential σ-approximate amenability.

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1. Introduction

The concept of amenability for a Banach algebra was introduced by Johnson in 1972 [6]. Several modifications of this notion were introduced by relaxing some of the restrictions on the definition of amenability. Some of the most notables are concepts of approximate amenability [3], boundedly approximate amenability [4], and σ-amenability [7], where the former had been studied previously under different names.

Our main goal in this paper is to develop the definitions of bounded approximate amenability and sequential approximate amenability to bounded σ-approximate amenability and sequential σ-approximate amenability, respectively, where verify the relation between usual notions and σ-notions of bounded approximate amenability. Moreover, we obtain some hereditary properties of these concepts.

2. Preliminaries

Let A be a Banach algebra and E be a Banach A-bimodule such that the following statements hold

$$\|a.x\| \leqslant \|a\|\|x\| \quad \text{and} \quad \|x.a\| \leqslant \|x\|\|a\|$$

for all $a \in \mathcal{A}$ and $x \in E$.

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Then, the dual space E^* of E is a Banach $\mathcal{A}$-bimodule under the following actions

$$\langle x, \varphi.a\rangle = \langle a.x, \varphi\rangle \quad \text{and} \quad \langle x, a.\varphi\rangle = \langle x.a, \varphi\rangle$$

for all $a \in \mathcal{A}$, $x \in E$, and $\varphi \in E^*$.

Similarly, the second dual space E^{**} of E becomes a Banach $\mathcal{A}$-bimodule with the following actions

$$\langle \varphi, \Lambda.a\rangle = \langle a.\varphi, \Lambda\rangle \quad \text{and} \quad \langle \varphi, a.\Lambda\rangle = \langle \varphi.a, \Lambda\rangle$$

for all $a \in A$, $\varphi \in E^*$, and $\Lambda \in E^{**}$.

Now, let $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism. A σ-derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule E is a bounded linear map $D : \mathcal{A} \longrightarrow E$ such that for each $a, b \in \mathcal{A}$,

$$D(ab) = \sigma(a).D(b) + D(a)\sigma(b).$$

Also, the σ-derivation $\delta : \mathcal{A} \longrightarrow E$ is σ-inner if there exists $x \in E$ such that $\delta(a) = \delta_x(a) = \sigma(a).x - x.\sigma(a)$ for all $a \in \mathcal{A}$.

A σ-derivation $D : \mathcal{A} \longrightarrow E$ from Banach algebra $\mathcal{A}$ into Banach $\mathcal{A}$-bimodule E is a σ-approximately inner if there exists a net $(\xi_\alpha) \subset E$ such that for every $a \in \mathcal{A}$, $D(a) = \lim_\alpha(\sigma(a).\xi_\alpha - \xi_\alpha.\sigma(a))$. We say that $\mathcal{A}$ is called σ-approximately amenable if for any Banach $\mathcal{A}$-bimodule E, every σ-derivation $D : A \longrightarrow E^*$ be a σ-approximately inner.

In [4], the notion of bounded approximate amenability was also introduced; a Banach algebra $\mathcal{A}$ is boundedly approximately amenable if for each Banach $\mathcal{A}$-bimodule E, and every derivation $D : \mathcal{A} \longrightarrow E^*$, there exists a net $(\xi_\alpha) \subseteq E^*$ such that the net (ad_{ξ_α}) is norm bounded in $B(\mathcal{A}, E^*)$ and for $a \in \mathcal{A}$, $Da = \lim_\alpha ad_{\xi_\alpha}(a)$.

3. Bounded and sequential σ-approximate amenability

In this section, we continue the investigation of σ-approximate amenability. We study basic properties of the notion of Bounded σ-approximate amenability. Moreover, we show that any boundedly σ-approximately amenable Banach algebra which is also separable as a Banach space is sequentially σ-approximately amenable.

The following definition extends the concept of the approximate amenability.

Definition 3.1 ([3]). Let $\mathcal{A}$ be a Banach algebra and $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism. Then a continuous σ-derivation $D : \mathcal{A} \longrightarrow E$ from $\mathcal{A}$ into Banach $\mathcal{A}$-bimodule E is boundedly σ-approximately inner if there exists a net $(\xi_\alpha) \subseteq E$ such that the net $(ad^\sigma_{\xi_\alpha})$ is norm bounded in $B(\mathcal{A}, E)$ and for $a \in \mathcal{A}$,

$$Da = \lim_\alpha ad^\sigma_{\xi_\alpha}(a).$$

If in the above definition (ξ_α) can be chosen to be a sequence, then D is called sequentially σ- approximately inner.

Definition 3.2. Let $\mathcal{A}$ be a Banach algebra and $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism. Then $\mathcal{A}$ is called boundedly (Resp. sequentially) σ-approximately amenable if for every Banach $\mathcal{A}$-bimodule E, every continuous σ-derivation $D : \mathcal{A} \longrightarrow E^*$ be a boundedly (Resp. sequentially) σ-approximately inner.

Definition 3.3. Let $\mathcal{A}$ be a Banach algebra and $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism. Then $\mathcal{A}$ is called boundedly σ-approximately contractible if for every Banach $\mathcal{A}$-bimodule E, every continuous σ-derivation $D : \mathcal{A} \longrightarrow E$ be a boundedly σ-approximately inner.

For example, the Banach algebra $\mathcal{A}$ is boundedly approximately amenable (contractible) if and only if $\mathcal{A}$ is boundedly id_A-approximately amenable (contractible).

Remark 3.4. Note that it is the net of σ-derivation $(\mathrm{ad}^{\sigma}_{\xi_\alpha})$ that is required to be bounded, not the implementing net (ξ_α). On the other hand, if $\mathcal{A}$ is σ-amenable then [5, Proposition 1] shows that $\mathcal{A}$ is boundedly σ-approximately contractible with the implementing net bounded.

Remark 3.5. Let σ be a continuous homomorphism on Banach algebra $\mathcal{A}$ and E be a Banach $\mathcal{A}$-bimodule. Then, E is a Banach $\mathcal{A}$-bimodule under the following module actions

$$a.x := \sigma(a)x \quad \text{and} \quad x.a := x\sigma(a)$$

for all $a \in \mathcal{A}$ and $x \in E$.

A standard argument shows the following proposition.

Proposition 3.6. *Let $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on Banach algebra $\mathcal{A}$. Then $\mathcal{A}$ is boundedly σ-approximately amenable if and only if there exists a constant $K > 0$ such that for each Banach $\mathcal{A}$-bimodule E and any continuous σ-derivation $D : \mathcal{A} \longrightarrow E^*$ there is a net $(\xi_\alpha) \subseteq E^*$ such that* (i) $\sup_\alpha \|\mathrm{ad}^{\sigma}_{\xi_\alpha}\| \leqslant K\|D\|$; *and* (ii) *for any $a \in \mathcal{A}$,*

$$Da = \lim_\alpha \mathrm{ad}^{\sigma}_{\xi_\alpha}(a).$$

Proof.

⇒: Assume towards a contradiction that there exists no such K, then for every integer $n \in \mathbb{N}$ there exists a module M_n with constant at least n for some norm on σ-derivation $D_n : \mathcal{A} \longrightarrow M_n^*$. Take the module $\ell^1(M_n)$ with dual $\ell^\infty(M_n^*)$. Then the σ-derivation $D = (D_n)$ into the latter has constant at least n, for each n, a contradiction.

⇐: It is clear. □

In terms of the basic characterization of σ-approximate amenability, we have the following result.

Proposition 3.7. *Let $\mathcal{A}$ be a boundedly σ-approximate amenable and let $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism. Then there exists a net $(M_\alpha) \subset (\mathcal{A}^\sharp \otimes \mathcal{A}^\sharp)^{**}$ and constant $K > 0$ such that for each $a \in \mathcal{A}^\sharp$, $\sigma(a).M_\alpha - M_\alpha.\sigma(a) \to 0$, $\pi^{**}(M_\alpha) \to e$ and $\|\sigma(a).M_\alpha - M_\alpha.\sigma(a)\| \leqslant K\|\sigma(a)\|$. Conversely, if $\mathcal{A}$ has this latter property and $(\pi^{**}(M_\alpha))$ is bounded, then $\mathcal{A}$ is boundedly σ-approximate amenable.*

Proof. The argument of [3, Theorem 2.1] suffices. □

Remark 3.8. Let $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on Banach algebra $\mathcal{A}$, then we define $\sigma^\sharp : \mathcal{A}^\sharp \longrightarrow \mathcal{A}^\sharp$ by $\sigma^\sharp(a+\alpha) = \sigma(a)+\alpha$ for each $a \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. It is routinely shown that $\sigma^\sharp$ is a continuous homomorphism on $\mathcal{A}^\sharp$. We know that $(\mathcal{A}^\sharp)^* = \mathcal{A}^* \oplus \mathbb{C}e^*$ where for each $a \in \mathcal{A}, f \in \mathcal{A}^*$ and $\alpha, \beta \in \mathbb{C}$ we have $\langle a+\beta, f+\alpha e^*\rangle = f(a)+\alpha\beta$. In particular, $e^*(a+\alpha) = \alpha$. Note that for $f \in \mathcal{A}^*$ and $\alpha \in \mathbb{C}$ we have $\|f+\alpha e^*\| = \max\{\|f\|, |\alpha|\}$. Now, for each $a, b \in \mathcal{A}$, $f \in \mathcal{A}^*$ and $\alpha, \beta \in \mathbb{C}$ we have

$$(a+\alpha).(f+\beta e^*) = af + \alpha f + (f(a)+\alpha\beta)e^*,$$
$$(f+\beta e^*).(a+\alpha) = fa + \alpha f + (f(a)+\beta\alpha)e^*.$$

Lemma 3.9. *Let $\mathcal{A}$ be a unital Banach algebra with unit e, $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on $\mathcal{A}$ such that $\overline{\sigma(\mathcal{A})} = \mathcal{A}$, E be a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \longrightarrow E^*$ be a continuous σ-derivation. Then, there exists a continuous σ-derivation $D_1 : \mathcal{A} \longrightarrow e.E^*.e$ and $\eta \in E^*$ such that*

(i) $\|\eta\| \leqslant 2C_E\|D\|$;

(ii) $D = D_1 + \delta^{\sigma}_{\eta}$.

Proof. Since $\mathcal{A}$ is a unital Banach algebra with unit e and $\overline{\sigma(\mathcal{A})} = \mathcal{A}$, it is easy to show that $\sigma(e) = e$. We follow the standard argument in [7, Lemma 2.3]. Take $Y_1 = e.E^*.e$, $Y_2 = (1-e).E^*.e$, $Y_3 = e.E^*.(1-e)$, and $Y_4 = (1-e).E^*.(1-e)$, which are Banach $\mathcal{A}$-bimodules. Note that the action on the left (Resp. right, both sides) on Y_2(Resp.Y_3, Y_4) is zero. For $f \in E^*$, we have

$$f = efe + (f - ef) + (f - ef) + (ef + fe - efe - f) = efe + (f - ef) + (f - ef) - (1 - e)f(1 - e).$$

This shows that $E^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4$. Let $D : \mathcal{A} \longrightarrow E^*$ be a continuous σ-derivation. Consider the derivations $D_i = P_i o D : \mathcal{A} \longrightarrow Y_i$ for $1 \leqslant i \leqslant 4$, where $P_i : E^* \longrightarrow Y_i$ be the associated canonical projections. Thus, $D = D_1 + D_2 + D_3 + D_4$ and $D_i : \mathcal{A} \longrightarrow Y_i$ is continuous σ-derivation for $1 \leqslant i \leqslant 4$. Now, it can be easily shown that $D_2 = \delta_{(-D_2 e)}$, $D_3 = ad_{D_3 e}$, $D_4 = 0$. Therefore, D is inner if and only if D_1 is inner. Consider $\eta = D_3 e - D_2 e = e.De.(1 - e) - (1 - e).De.e = e.De - De.e$. Then, we have $\|\eta\| \leqslant 2C_E\|D\|$. □

Remark 3.10. A Banach $\mathcal{A}$-bimodule E is said to be neo-unital if

$$E = A.E.A = \{a.x.b : a, b \in \mathcal{A}, x \in E\}.$$

Proposition 3.11. *Let $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on Banach algebra $\mathcal{A}$ such that $\overline{\sigma(\mathcal{A})} = \mathcal{A}$, then $\mathcal{A}$ is boundedly σ-approximately amenable if and only if $\mathcal{A}^\sharp$ is boundedly $\sigma^\sharp$-approximately amenable.*

Proof.
⇒: Since $\overline{\sigma(\mathcal{A})} = \mathcal{A}$, it is clear $\overline{\sigma^\sharp(\mathcal{A})} = \mathcal{A}^\sharp$. Let E be a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A}^\sharp \longrightarrow E^*$ be a continuous σ-derivation. By Lemma 3.9, there exists a continuous σ-derivation $D_1 : \mathcal{A}^\sharp \longrightarrow e.E^*.e$ and $\eta \in E^*$ such that $D = D_1 + \delta^\sigma_\eta$, where $\sigma(e) = e$ is the identity of $\mathcal{A}^\sharp$. Thus $D_1(e) = 0$. Hence without loss of generality, we may suppose that E^* is neo-unital and so $D(e) = 0$. By assumption, there exists a net $(\xi_\alpha)_\alpha \subset E^*$ and a constant $K > 0$ such that for all $a \in \mathcal{A}$,

$$D(a) = \lim_\alpha(\xi_\alpha.\sigma(a) - \xi_\alpha.\sigma(a)).$$

Moreover, $\|\xi_\alpha.\sigma(a) - \xi_\alpha.\sigma(a)\| \leqslant K\|\sigma(a)\|$ for all α. Since $D(e) = 0$ and $e.x = x.e$ for all $x \in E^*$, it follows that

$$D(a + \lambda e) = \lim_\alpha((\sigma(a) + \lambda e).\xi_\alpha - \xi_\alpha.(\sigma(a) + \lambda e))$$

and

$$\|(\sigma(a) + \lambda e).\xi_\alpha - \xi_\alpha.(\sigma(a) + \lambda e)\| \leqslant K\|\sigma(a)\| \leqslant K\|\sigma(a) + \lambda e\|$$

for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Therefore $\mathcal{A}^\sharp$ is boundedly $\sigma^\sharp$-approximately amenable.
⇐: Let $D : \mathcal{A} \longrightarrow E^*$ be a continuous σ-derivation for Banach $\mathcal{A}$-bimodule E^*. Set $\tilde{D} : \mathcal{A}^\sharp \longrightarrow E^*$ by $\tilde{D}(a + \lambda e) = D(a)$ for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Setting $e.x = x.e = x$ (for all $e \in \mathcal{A}^\sharp$ and $x \in E$), makes E^* into a Banach algebra $\mathcal{A}^\sharp$-bimodule and $\tilde{D}$ is a continuous σ-derivation. So $\tilde{D}$ is σ-approximately inner, therefore there exists a net $(\xi_\alpha)_\alpha \subset E^*$ and a constant $K > 0$ such that for all $a \in \mathcal{A}$,

$$D(a) = \lim_\alpha(\xi_\alpha.\sigma(a) - \xi_\alpha.\sigma(a))$$

with

$$\|\xi_\alpha.\sigma(a) - \xi_\alpha.\sigma(a)\| \leqslant K\|\sigma(a)\|$$

for all α. □

Proposition 3.12. *Let $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on Banach algebra $\mathcal{A}$. If $\mathcal{A}$ is boundedly approximately amenable, then $\mathcal{A}$ is boundedly σ-approximately amenable.*

Proof. Let E be a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \longrightarrow E^*$ be a continuous σ-derivation. Consider the following module actions on E

$$a.x = \sigma(a).x \text{ and } x.a = x\sigma(a)$$

for all $a \in \mathcal{A}$ and $x \in E$. Then E with this product is a Banach A-bimodule and since $D : \mathcal{A} \longrightarrow E^*$ is a

σ-derivation, then

$$D(ab) = D(a)\sigma(b) + \sigma(a)D(b) = D(a).b + a.D(b)$$

for each $a, b \in \mathcal{A}$. Thus, D is a derivation and so, by bounded approximately amenability of $\mathcal{A}$, there exists a net $(\xi_\alpha) \subset E^*$ and a constant $K > 0$ such that for each $a \in \mathcal{A}$

$$D(a) = \lim_\alpha(\xi_\alpha.a - a.\xi_\alpha)$$

and moreover $\|\xi_\alpha.a - a.\xi_\alpha\| < K\|a\|$ for all α. Hence for each $a \in \mathcal{A}$,

$$D(a) = \lim_\alpha(\xi_\alpha\sigma(a) - \sigma(a)\xi_\alpha)$$

and

$$\|\xi_\alpha\sigma(a) - \sigma(a)\xi_\alpha\| = \|\xi_\alpha.a - a.\xi_\alpha\| < K\|a\|$$

for all α. Thus, $\mathcal{A}$ is boundedly σ-approximately amenable. □

Now, in the following proposition we show that the converse of Proposition 3.12 holds, when σ be a continuous homomorphism on Banach algebra $\mathcal{A}$ with dense range.

Proposition 3.13. *Let $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on Banach algebra $\mathcal{A}$ with dense range. If $\mathcal{A}$ be a boundedly σ-approximately amenable. Then, $\mathcal{A}$ is boundedly approximately amenable.*

Proof. Given a Banach $\mathcal{A}$-bimodule E, then it is also Banach $\mathcal{A}$-bimodule with actions

$$a.x := \sigma(a).x \quad \text{and} \quad x.a := x.\sigma(a) \quad (a \in \mathcal{A}, x \in E).$$

Let $D : \mathcal{A} \longrightarrow E^*$ be a continuous derivation. Then, this is clear that $D_1 = Do\sigma$ is a σ-derivation. Therefore, the boundedly σ-approximately of $\mathcal{A}$ implies that there exists a net $(\xi_\alpha) \subset E^*$ and a constant $K > 0$ such that for each $a \in \mathcal{A}$, $D_1(a) = \lim_\alpha(\xi_\alpha.\sigma(a) - \sigma(a).\xi_\alpha)$ and $\|ad^\sigma_{\xi_\alpha}\| < K$ for α. Now, since $\overline{\sigma(\mathcal{A})} = \mathcal{A}$, for any $b \in \mathcal{A}$ there exists a net $(a_\beta) \subseteq \mathcal{A}$ such that

$$\sigma(a_\beta) \to b.$$

For an arbitrary ϵ, there exists an index β such that

$$\|D\|\|b - \sigma(a_\beta)\| + K\|b - \sigma(a_\beta)\| < 2\epsilon/3.$$

Then we may choose ϵ such that $\|D_1(a_\beta) - (\xi_\alpha.\sigma(a_\beta) - \sigma(a_\beta).\xi_\alpha)\| < \epsilon/3$. Now $\|D(b) - (\xi_\alpha.b - b.\xi_\alpha)\| < \epsilon$. Thus, $\mathcal{A}$ is boundedly approximately amenable. □

The uniform boundedness principle shows that every sequentially σ-approximately amenable Banach algebra is boundedly σ-approximately amenable.

Proposition 3.14. *Let $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on Banach algebra $\mathcal{A}$ with dense range and $\mathcal{A}$ be a boundedly σ-approximately amenable. If $\mathcal{A}$ is separable as a Banach space, then it is sequentially σ-approximately amenable.*

Proof. Let $\{\sigma(a_n) \quad |n \in \mathbf{N}\}$ be a countable dense subset of $\mathcal{A}$, E be Banach $\mathcal{A}$-bimodule, and $D : \mathcal{A} \longrightarrow E^*$ be a continuous derivation. Since $\mathcal{A}$ is boundedly σ-approximately amenable, then there exists a constant $K > 0$ such that for each $n \in \mathbf{N}$ there exists $\xi_n \in E^*$ such that

$$\|D(a_k) - (\sigma(a_k).\xi_n - \xi_n.\sigma(a_k))\| < 1/n \quad (k = 1, 2, \ldots, n)$$

and

$$\|\sigma(a).\xi_n - \xi_n.\sigma(a)\| \leqslant K\|a\| \quad (a \in \mathcal{A}).$$

This shows that the sequence $(\xi_n) \subseteq E^*$ satisfies

$$D(a_k) = \lim_{n\to\infty} (\sigma(a_k).\xi_n - \xi_n.\sigma(a_k)) \quad (k = 1, 2, \ldots),$$

and the sequence $(ad^{\sigma}_{\xi_n})$ is bounded in $B(\mathcal{A}, E^*)$. Now, since $\overline{\sigma(\mathcal{A})} = \mathcal{A}$, then for any $a \in \mathcal{A}$, $\lim_{k\to\infty} \sigma(a_k) = a$. Hence for all $a \in \mathcal{A}$, $D(a) = \lim_{n\to\infty}(a.\xi_n - \xi_n.a)$. Whence $\mathcal{A}$ is sequentially σ-approximately amenable. □

The same argument gives the next.

Proposition 3.15. *Let $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on Banach algebra $\mathcal{A}$ with dense range and $\mathcal{A}$ be a boundedly σ-approximately contractible. If $\mathcal{A}$ is separable as a Banach space, then it is sequentially σ-approximately contractible.*

Example 3.16. Let $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on Banach algebra $\mathcal{A}$. Hence by applying the argument of [3, Example 6.1], we have that for any net $(\mathcal{A}_i)$ of σ-amenable (contractible) Banach algebras, $C_0(\mathcal{A}_i^{\sharp})$ is boundedly σ-approximately amenable (contractible).

We should mention that the imposed condition is necessary in above proposition. The next example shows that the conclusion of the proposition is not true for non-separable Banach algebras.

Example 3.17. Let $\mathcal{A} = C_0(S)$ where S is uncountable set and $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on Banach algebra $\mathcal{A}$. Then $\mathcal{A}$ is σ-amenable and hence is boundedly σ-approximately contractible. But $\mathcal{A}$ cannot be sequentially σ-approximately contractible, for otherwise $C_0(S)$ would have a sequential approximate identity, which is impossible. So, without separability Proposition 3.15 is not true.

Proposition 3.18. *Let $\mathcal{A}$ be a boundedly σ-approximately amenable Banach algebra and $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on $\mathcal{A}$. Then, $\mathcal{A}$ has left and right approximate identities for $\sigma(\mathcal{A})$; that is, there exists a net $(e_\alpha) \subseteq \mathcal{A}$ such that for each $a \in \mathcal{A}$*

$$\sigma(a)e_\alpha \longrightarrow \sigma(a) \ \ \textit{and} \ \ e_\alpha\sigma(a) \longrightarrow \sigma(a).$$

Proof. Let $E = \mathcal{A}^*$. Clearly, E is a Banach $\mathcal{A}$-bimodule with module actions induced via a.f(b)= f(ab) and f.a=0 for all $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Thus, $E^* = \mathcal{A}^{**}$ is a Banach $\mathcal{A}$-bimodule with right action F.a(f)=F(a.f) and zero left action a.F=0 for all $f \in E$, $F \in E^*$, and $a \in \mathcal{A}$. Now, the map $D : \mathcal{A} \longrightarrow E^*$ defined by $D(a) = \widehat{\sigma(a)}$ is a σ-derivation. Since $\mathcal{A}$ is σ-approximately amenable, then there exists $(F_\alpha)_\alpha \subset \mathcal{A}^{**}$ and a constant $K > 0$ such that $\|ad^{\sigma}_{F_\alpha}\| < K$ for all $a \in \mathcal{A}$, $D(a) = \lim_\alpha(F_\alpha.\sigma(a) - F_\alpha.\sigma(a))$ and since $a.F_\alpha = 0$ and $D(a) = \widehat{\sigma(a)}$, then $\widehat{\sigma(a)} = \lim_\alpha(F_\alpha.\sigma(a))$. Take $K \subset \sigma(\mathcal{A})$ and $\chi \subset \mathcal{A}^*$ be two non-empty finite sets and $\varepsilon > 0$. Therefore, there exists $\alpha_0 = \alpha_0(K, \chi, \varepsilon)$ where the family of such pairs (K, χ, ϵ) is a directed set for the partial order $\leqslant$ given by $(K_1, \chi, \epsilon_1) \leqslant (K_2, \chi, \epsilon_2)$ if $K_1 \subseteq K_2$ and $\epsilon_1 \geqslant \epsilon_2$, such that for $\sigma(a) \in K$, $\|\sigma(\hat{a}) - F_{\alpha_0}.\sigma(a)\| < \varepsilon/2M$, where

$$M = \max\{\|\varphi.\sigma(a)\|, \|\psi\|; \sigma(a) \in K, \varphi, \psi \in \chi\}.$$

Since $F_{\alpha_0} \in \mathcal{A}^{**}$ by Goldestine's theorem, there exists a net $(e_\alpha)_\alpha \subset \mathcal{A}$ such that $F_{\alpha_0} = w^* - \lim_\alpha e_\alpha$. Choose $e_\beta \in \mathcal{A}$ such that

$$|\langle \sigma(a).\varphi, F_{\alpha_0}\rangle - \langle e_\beta, \sigma(a).\varphi\rangle| < \varepsilon/2(\varphi \in \chi, \sigma(a) \in K).$$

Then,

$$|\langle e_\beta.\sigma(a), \varphi\rangle - \langle \sigma(a), \varphi\rangle| \leqslant |\langle e_\beta.\sigma(a), \varphi\rangle - \langle \varphi, F_{\alpha_0}.\sigma(a)\rangle| + |\langle \varphi, F_{\alpha_0}.\sigma(a) - \sigma(\hat{a})\rangle| \leqslant \varepsilon/2 + \|\varphi\|.\varepsilon/2M \leqslant \varepsilon.$$

Thus, $(e_\beta)_{(K,\chi,\varepsilon)}$ is a left approximate identity in weak topology for $\sigma(\mathcal{A})$. Therefore, by [2, Proposition 4] $\mathcal{A}$ has left approximate identity for $\sigma(\mathcal{A})$. By applying a similar argument, $\mathcal{A}$ has a right approximate identity for $\sigma(\mathcal{A})$. □

Recall that a Banach algebra $\mathcal{A}$ is essential if $\overline{\mathcal{A}^2} = \mathcal{A}$. We have the following corollary.

Corollary 3.19. *Let $\mathcal{A}$ be a σ-approximately amenable Banach algebra and $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous homomorphism on $\mathcal{A}$ such that $\sigma(\mathcal{A})$ is dense in $\mathcal{A}$. Then, $\mathcal{A}$ has a approximate identity. In particular, $\mathcal{A}$ is essential.*

Proof. Immediate from Proposition 3.18. □

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References

[1] F. F. Bonsall, J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York, (1973).
[2] J. B. Conway, *A course in functional analysis*, Springer, New York, (2013). 3
[3] F. Ghahramani, R. J. Loy, *Generalized notions of amenability*, J. Funct. Anal., **208** (2004), 229–260. 1, 3.1, 3, 3.16
[4] F. Ghahramani, R. J. Loy, Y. Zhang, *Generalized notions of amenability.II*, J. Funct. Anal., **254** (2008), 1776–1810. 1, 2
[5] F. Gourdeau, *Amenability of Lipschitz algebra*, Math. Proc. Combridge Philos. Soc., **112** (1992), 581–588. 3.4
[6] B. E. Johnson, *Cohomology in Banach algebras*, American Mathematical Society, Providence, (1972). 1
[7] M. S. Moslehian, A. N. Motlagh, *Some notes on (σ, τ)-amenability of Banach algebras*, Stud. Univ. Babeş-Bolyai Math., **53** (2008), 57–68. 1, 3