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# On $(\alpha, p)$-convex contraction and asymptotic regularity

M. S. Khan[a], Y. Mahendra Singh[b], Georgeta Maniu[c], Mihai Postolache[d,e,*]

[a]Department of Mathematics and Statistics, Sultan Qaboos University, P. O. Box 36, PCode 123 Al-Khod, Muscat, Sultanate of Oman, Oman.

[b]Department of Humanities and Basic Sciences, Manipur Institute of Technology, Takyelpat-795001, India.

[c]Department of Computer Science, Information Technology, Mathematics and Physics, Petroleum-Gas University of Ploieşti, Bucureşti Bvd., No. 39, 100680 Ploieşti, Romania.

[d]China Medical University, Taichung, Taiwan.

[e]University Politehnica of Bucharest, Bucharest, Romania.

## Abstract

In this paper, we present the notions of $(\alpha, p)$-convex contraction (resp. $(\alpha, p)$-contraction) and asymptotically $T^2$-regular (resp. $(T, T^2)$-regular) sequences, and prove fixed point theorems in the setting of metric spaces.

**Keywords:** Approximate fixed point, fixed point, $(\alpha, p)$-convex contraction, asymptotically regular sequence, asymptotically T (resp. $T^2$ and $(T, T^2)$)-regular sequences.

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## 1. Introduction and preliminaries

Let $(X, d)$ be a metric space, and C a nonempty set of X. A mapping $T: C \to C$ is called nonexpansive if $d(Tx, Ty) \leqslant d(x, y)$ for all $x, y \in C$. In 2007, Goebel and Japón Pineda [8] introduced the class of mean nonexpansive mappings, an extension for the class of nonexpansive mappings. A mapping $T: C \to C$ is called mean nonexpansive (or $\alpha$-nonexpansive) if, for some $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\sum_{i=1}^{n} \alpha_i = 1$, $a_i \geqslant 0$ for all $i$, and $\alpha_1, \alpha_n > 0$, we have

$$\sum_{i=1}^{n} \alpha_i d(T^i x, T^i y) \leqslant d(x, y)$$

for all $x, y \in C$. Further, Goebel and Japón Pineda [8] introduced the class of $(\alpha, p)$-nonexpansive

*Corresponding author
Email addresses: mohammad@squ.edu.om (M. S. Khan), ymahenmit@rediffmail.com (Y. Mahendra Singh), maniugeorgeta@gmail.com (Georgeta Maniu), emscolar@yahoo.com (Mihai Postolache)

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mappings. A mapping $T\colon C \to C$ is called $(\alpha, p)$-nonexpansive, if for some $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geqslant 0$ for all $i$, and $\alpha_1, \alpha_n > 0$, and for some $p \in [1, \infty)$, we have

$$\sum_{i=1}^{n} \alpha_i d^p(T^i x, T^i y) \leqslant d^p(x, y)$$

for all $x, y \in C$. In particular, for $n = 2$, the above inequality reduces to

$$\alpha_1 d^p(Tx, Ty) + \alpha_2 d^p(T^2 x, T^2 y) \leqslant d^p(x, y)$$

for all $x, y \in C$, we say that $T$ is $((\alpha_1, \alpha_2), p)$-nonexpansive.

**Example 1.1.** Let $X = [0, \infty) \subset \mathbb{R}$ with usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define a translation function $T\colon X \to X$ by the formula $Tx = x + a$ for any fixed $a > 0$. Now, setting $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $p \geqslant 1$, we have

$$|Tx - Ty|^p + |T^2 x - T^2 y|^p = 2|x - y|^p,$$

that is,

$$\frac{1}{2}|Tx - Ty|^p + \frac{1}{2}|Tx - Ty|^p = |x - y|^p.$$

Therefore, $T$ is $((\alpha_1, \alpha_2), p)$-nonexpansive mapping.

**Example 1.2.** Let $X = \{0, 1, 2\}$ with usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define the mapping

$$T\colon X \to X, \quad Tx = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Setting $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$, for any $p \geqslant 1$, we have

$$\alpha_1 |Tx - Ty|^p + \alpha_2 |T^2 x - T^2 y|^p \leqslant |x - y|^p.$$

Therefore, $T$ is $((\alpha_1, \alpha_2), p)$-nonexpansive mapping.

In 1982, Istrăţescu [10] introduced the class of convex contraction mappings in the setting of metric space and generalized the well known Banach's contraction principle [2]. Some works have appeared recently on generalization of such class of mappings in the setting of metric, ordered metric, and cone metric, b-metric and 2-metric spaces (for example, Alghamdi et al. [1], Ghorbanian et al. [7], Miandaragh et al. [14], Miculescu and Mihail [15], Khan et al. [12], etc.).

Let $(X, d)$ be a metric space and $T\colon X \to X$ be a mapping. Given $\varepsilon > 0$, then $x_0 \in X$ is said to be an $\varepsilon$-fixed point of $T$ on $X$, whenever $d(x_0, Tx_0) < \varepsilon$. Note that every fixed point is $\varepsilon$-fixed point but the converse need not be true. We denote the set of all $\varepsilon$- fixed points of $T$ for a given $\varepsilon > 0$ by $F_\varepsilon(T) = \{x \in X | d(Tx, x) < \varepsilon\}$ and $Fix(T)$, the set of all fixed points of $T$.

We say that $T$ has the approximate fixed point property (AFPP) if for all $\varepsilon > 0$, there exists an $\varepsilon$-fixed point of $T$ i.e., for all $\varepsilon$, $F_\varepsilon(T) \neq \emptyset$, or equivalently, $\inf_{x \in X} d(Tx, x) = 0$.

For details we refer to Berinde [3], Kohlenbach and Leuştean [13], Reich and Zaslavski [16], Tijs et al. [17].

**Example 1.3** ([12])**.** If $X = [0, \infty)$, let $T\colon X \to X$, $Tx = x + \frac{1}{2x+1}$ for all $x \in X$. Setting $0 < \varepsilon < \frac{1}{2}$ and taking $x_0 \in X$ such that $x_0 > \frac{1-\varepsilon}{2\varepsilon}$, we obtain,

$$d(Tx_0, x_0) = |Tx_0 - x_0| = \left|\frac{1}{2x_0 + 1}\right| < \varepsilon.$$

This shows that $T$ has an $\varepsilon$-fixed point, so $F_\varepsilon(T) \neq \emptyset$. Note that $T$ has no fixed point in $X$.

**Definition 1.4** ([4]). A self mapping $T$ on $X$ is said to be asymptotically regular at a point $x \in X$ if $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$.

**Definition 1.5** ([5]). A sequence $\{x_n\}$ in $X$ is called an asymptotically $T$-regular, if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

**Lemma 1.6** ([3]). *If $(X, d)$ is a metric space and $T$ is an asymptotically regular self mapping on $X$, that is $d(T^m x, T^{m+1}x) \to 0$ for all $x \in X$, then $T$ has the AFPP.*

In the next section, we discuss the notions of $(\alpha, p)$-convex contraction (resp. $(\alpha, p)$-contraction) and asymptotically $T^2$-regular (resp. $(T, T^2)$-regular) sequences. Further, we show with examples that the notions of asymptotically T-regular and $T^2$-regular sequences are independent to each other.

## 2. $(\alpha, p)$-convex contraction and asymptotic regularity

Let $T$ be a self mapping on a metric space $(X, d)$.

**Definition 2.1.** A self mapping $T$ on $X$ is said to be $(\alpha, p)$-contraction, if for some $\alpha \in (0, 1)$ and $p \geqslant 1$, there exists $0 \leqslant k < 1$ satisfying the following inequality

$$\alpha d^p(Tx, Ty) + (1-\alpha) d^p(T^2x, T^2y) \leqslant k d^p(x, y) \tag{2.1}$$

for all $x, y \in X$.

Note that if we set $\alpha = \alpha_1$, $\alpha_2 = 1-\alpha$, and $k = 1$ in the inequality (2.1), then $T$ reduces to $((\alpha_1, \alpha_2), p)$-nonexpansive (see [8]). Further, if $p = 1$ and $k < 1$ (resp. $k = 1$) in the inequality (2.1), then $T$ reduces to $\alpha$-contraction (resp. $\alpha$-nonexpansive) with multi-index length 2 (see [9]).

**Definition 2.2.** A self mapping $T$ on $X$ is said to be $(\alpha, p)$-convex contraction, if for some $\alpha \in (0, 1)$ and $p \geqslant 1$, there exist $k_i \geqslant 0$ for all $i \in \{1, 2, \dots, 5\}$ such that $\sum_{i=1}^{i=5} k_i < 1$ satisfying the following inequality

$$\begin{aligned}\alpha d^p(Tx, Ty) + (1-\alpha) d^p(T^2x, T^2y) &\leqslant k_1 d^p(x, y) + k_2 d^p(x, Tx) \\ &\quad + k_3 d^p(Tx, T^2x) + k_4 d^p(y, Ty) + k_5 d^p(Ty, T^2y)\end{aligned} \tag{2.2}$$

for all $x, y \in X$.

Obviously, if $k_i = 0$ for all $i \in \{2, 3, 4, 5\}$, then the inequality (2.2) reduces to $(\alpha, p)$-contraction. We shall call $\alpha$-contraction and $\alpha$-convex contraction, if $p = 1$ in the inequalities (2.1) and (2.2). If $\alpha = k_1 = 0$ and $p = 1$ in (2.2), then it reduces to two-sided convex contraction [10].

**Example 2.3.** On $X = [0, 1]$, consider $T\colon X \to X$, endowed with usual metric $d(x, y) = |x - y|$. We define $Tx = \frac{1-x^2}{2}$, for all $x \in X$. Then, we obtain $T^2x = \frac{3+2x^2-x^4}{8}$ . Now, we have

$$|Tx - Ty| = \frac{1}{2}|x^2 - y^2| = \frac{(x+y)}{2}|x - y| \leqslant |x - y|.$$

Also,

$$|T^2x - T^2y| = \frac{1}{8}|(2x^2 - x^4) - (2y^2 - y^4)| \leqslant \frac{1}{4}|x^2 - y^2| + \frac{1}{8}|x^4 - y^4| \leqslant |x - y|.$$

Therefore, for $\alpha = \frac{1}{2}$ and $p = 1$, we obtain

$$\alpha|Tx - Ty| + (1-\alpha)|T^2x - T^2y| \leqslant |x - y|.$$

This shows that $T$ is nonexpansive and $\alpha$-nonexpansive for $p = 1$.

Further, for $p = 2$ and $\alpha = \frac{1}{2}$, we obtain

$$\alpha|Tx - Ty|^2 + (1-\alpha)|T^2x - T^2y|^2 \leqslant \frac{1}{2}|x - y|^2 + \frac{1}{8}|x - y|^2 = \frac{5}{8}|x - y|^2.$$

This shows that $T$ is $(\alpha, p)$-contraction for $p = 2 > 1$.

In [6], Gallagher mentioned that all nonexpansive mappings are mean nonexpansive, but the converse is not true. That is, there exists a mean nonexpansive mapping which is not nonexpansive (see [6, Examples 2.3 and 2.4]). However, it may happen that a nonexpansive mapping need not necessarily be a mean nonexpansive.

**Example 2.4.** Let $T\colon X \to X$, where $X = [0, 1]$ with usual metric $d(x, y) = |x - y|$. We define $Tx = \frac{x^2}{2}$ for all $x \in X$. Setting $\alpha = \frac{1}{2}$ and $p = 1$. Now, we have

$$|Tx - Ty| = \frac{1}{2}|x^2 - y^2| \leqslant |x - y|.$$

Also, we have,

$$|T^2x - T^2y| = \frac{1}{8}|x^4 - y^4| = \frac{(x^2 + y^2)(x + y)}{8}|x - y| \leqslant \frac{1}{2}|x - y|.$$

Therefore,

$$\frac{1}{2}|Tx - Ty| + \frac{1}{2}|T^2x - T^2y| \leqslant \frac{3}{4}|x - y|,$$

where, $k = \frac{3}{4}, \alpha = \frac{1}{2}$. This shows that T is nonexpansive but not mean nonexpansive.

Now, we introduce the notions of asymptotically $T^2$-regular (resp. $(T, T^2)$-regular) sequences.

**Definition 2.5.** A sequence $\{x_n\}$ is called an asymptotically $T^2$-regular, if $\lim_{n\to\infty} d(x_n, T^2x_n) = 0$.

**Example 2.6.** Let $X = \mathbb{R}$ endowed with usual metric $d(x, y) = |x - y|$. We define

$$T\colon X \to X, \quad Tx = \begin{cases} 1 - x^2, & x \neq 1, \\ 2, & x = 1. \end{cases}$$

Choose a sequence $\{x_n\}$ in X such that $x_n \to 1$ as $n \to \infty$, except the constant sequence $x_n = 1$. Then, $Tx_n = (1 - x_n^2) \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} |Tx_n - x_n| = 1 \neq 0$. Also, we have $T^2x_n = T(Tx_n) = T(1 - {x_n}^2) = [1 - (1 - {x_n}^2)^2] \to 1$. Consequently, $|x_n - T^2x_n| \to 0$. Therefore, $\{x_n\}$ is asymptotically $T^2$-regular sequence but not asymptotically T-regular sequence in X.

**Example 2.7.** Let $T\colon X \to X$, where $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Define

$$Tx = \begin{cases} \frac{x^2}{2}, & x < 2, \\ 0, & x = 2, \\ 2, & x > 2. \end{cases}$$

Consider a sequence $\{x_n\}$ in X such that $x_n \to 2$ as $n \to \infty$, except the constant sequence $x_n = 2$. Then, $Tx_n \to 2$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} |Tx_n - x_n| = 0$. Further, we have $T^2x_n = T(Tx_n) \to 2$ or 0, according as $x_n \to 2$ from left or right. So, $\lim_{n\to\infty} T^2x_n$ does not exist. Therefore, $|x_n - T^2x_n|$ does not tend to 0 as $n \to 0$. It shows that $\{x_n\}$ is asymptotically T-regular sequence, but not asymptotically $T^2$-regular sequence in X.

It may be observed from Examples 2.6 and 2.7, that the notions of asymptotically T-regular and $T^2$ -regular sequences are independent to each other.

**Definition 2.8.** A sequence $\{x_n\}$ in X is called an asymptotically $(T, T^2)$-regular, if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\lim_{n\to\infty} d(x_n, T^2x_n) = 0$.

Obviously, if $\{x_n\}$ is an asymptotically $(T, T^2)$-regular sequence, then it satisfies both asymptotically T and $T^2$-regular conditions.

**Example 2.9.** Let $T\colon X \to X$, where $X = \mathbb{R}$ with usual metric $d(x, y) = |x - y|$. Define

$$Tx = \begin{cases} 4 - x, & x < 2, \\ 0, & x = 2, \\ \frac{x^2}{2}, & x > 2. \end{cases}$$

Consider a sequence $\{x_n\}$ in X such that $x_n \to 2$ as $n \to \infty$, except the constant sequence $x_n = 2$. Then, $Tx_n \to 2$ as $n \to \infty$ and $T^2x_n = T(Tx_n) \to 2$. Therefore, $|x_n - Tx_n| \to 0$ and $|x_n - T^2x_n| \to 0$ as $n \to \infty$. So, $\{x_n\}$ is both asymptotically T-regular and $T^2$-regular sequence in X. Therefore, $\{x_n\}$ is asymptotically $(T, T^2)$-regular sequence in X.

**Lemma 2.10.** *If a sequence $\{x_n\}$ in X is asymptotically $(T, T^2)$-regular in X, then*

$$\lim_{n\to\infty} d(Tx_n, T^2x_n) = 0.$$

*Proof.* By the triangle inequality, we obtain

$$d(Tx_n, T^2x_n) \leqslant d(Tx_n, x_n) + d(x_n, T^2x_n).$$

Hence, $d(Tx_n, T^2x_n) \to 0$ as $n \to \infty$. □

The converse of Lemma 2.10 is not true. In support of this, we have the following example.

**Example 2.11.** Let $T\colon X \to X$, where $X = \mathbb{R}$ with usual metric $d(x, y) = |x - y|$. We consider

$$Tx = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Choose a sequence $\{x_n\}$ in X such that $x_n \to 0$ as $n \to \infty$. Then, $Tx_n$ and $T^2x_n$ converge to 1 as $n \to \infty$. Therefore, $|Tx_n - x_n| \to 1 \neq 0$ and $|x_n - T^2x_n| \to 1 \neq 0$ as $n \to \infty$. It shows that $d(Tx_n, T^2x_n) \to 0$ as $n \to \infty$, but the sequence $\{x_n\}$ is neither asymptotically T-regular nor asymptotically $T^2$-regular in X. Therefore, the sequence $\{x_n\}$ is not asymptotically $(T, T^2)$-regular.

## 3. Fixed point results

**Theorem 3.1.** *Let $(X, d)$ be a metric space and $T\colon X \to X$ be a $(\alpha, p)$-contraction such that $k + \alpha < 1$. Then, T has the AFPP. Further, if $(X, d)$ is a complete metric space, then T has a unique fixed point.*

*Proof.* Let $x_0 \in X$. Now, we define a sequence $\{x_n\}$ by $x_{n+1} = T^{n+1}x_0$ for all $n \geqslant 0$. If $x_n = x_{n+1}$ i.e., $T^n x_0 = T(T^n x_0)$ for some $n$, then the conclusion follows immediately. Without lost of generality, we assume that $x_n \neq x_{n+1}$ for all $n \geqslant 0$. Setting $\nu = d(x_0, Tx_0) + d(Tx_0, T^2x_0)$ we have $d(x_0, Tx_0) \leqslant \nu$ and $d(Tx_0, T^2x_0) \leqslant \nu$. Taking $x = x_0$ and $y = Tx_0$ in the inequality (2.1), we obtain

$$\begin{aligned}(1 - \alpha)d^p(T^2x_0, T^3x_0) &\leqslant \alpha d^p(Tx_0, T^2x_0) + (1 - \alpha)d^p(T^2x_0, T^3x_0) \\ &\leqslant kd^p(x_0, Tx_0) = k\nu^p \Rightarrow d^p(T^2x_0, T^3x_0) \leqslant \frac{k}{1 - \alpha}\nu^p \Rightarrow d(T^2x_0, T^3x_0) \leqslant h\nu,\end{aligned}$$

where $h^p = \frac{k}{1-\alpha}$, and since $k + \alpha < 1 \Rightarrow h^p < 1$.

Again, taking $x = Tx_0$ and $y = T^2x_0$ in relation (2.1), we obtain

$$(1 - \alpha)d^p(T^3x_0, T^4x_0) \leqslant \alpha d^p(T^2x_0, T^3x_0) + (1 - \alpha)d^p(T^3x_0, T^4x_0)$$

$$\leqslant kd^p(Tx_0, T^2x_0) \Rightarrow d^p(T^3x_0, T^4x_0) \leqslant h^p\nu^p \Rightarrow d(T^3x_0, T^4x_0) \leqslant h\nu.$$

And

$$(1-\alpha)d^p(T^4x_0, T^5x_0) \leqslant \alpha d^p(T^3x_0, T^4x_0) + (1-\alpha)d^p(T^4x_0, T^5x_0) \leqslant kd^p(T^2x_0, T^3x_0) \Rightarrow d(T^4x_0, T^5x_0) \leqslant h^2\nu.$$

Also, we obtain

$$d(T^5x_0, T^6x_0) \leqslant h^2\nu.$$

Following similar arguments as in ([12, 14]), we obtain $d(T^mx_0, T^{m+1}x_0) \leqslant h^l\nu$, whenever $m = 2l$ or $m = 2l+1$. Therefore, $d(T^mx_0, T^{m+1}x_0) \to 0$ as $m \to \infty$, i.e., T is asymptotically regular at $x_0$. By Lemma 1.6, T has an approximate fixed point. Now, suppose that T is continuous and $(X, d)$ is a complete metric space. In order to show that $\{x_n\}$ is a Cauchy sequence in X, fix a nonzero positive integer $m$.

Case (i). For $m = 2l$ with $l, q \geqslant 1$, then

$$\begin{aligned} d(T^mx_0, T^{m+q}x_0) =& d(T^{2l}x_0, T^{2l+q}x_0) \\ \leqslant& d(T^{2l}x_0, T^{2l+1}x_0) + d(T^{2l+1}x_0, T^{2l+2}x_0) \\ &+ d(T^{2l+2}x_0, T^{2l+3}x_0) + d(T^{2l+3}x_0, T^{2l+4}x_0) + \cdots \\ &+ d(T^{2l+q-2}x_0, T^{2l+q-1}x_0) + d(T^{2l+q-1}x_0, T^{2l+q}x_0) \\ \leqslant& h^l\nu + h^l\nu + h^{l+1}\nu + h^{l+1}\nu + \cdots \\ \leqslant& 2h^l\Big(1 + h + h^2 + h^3 + \cdots\Big)\nu \leqslant 2h^l\frac{1}{(1-h)}\nu. \end{aligned}$$

Case (ii). Similarly, for $m = 2l+1$ with $l, q \geqslant 1$, we obtain

$$\begin{aligned} d(T^mx_0, T^{m+q}x_0) =& d(T^{2l+1}x_0, T^{2l+q+1}x_0) \\ \leqslant& d(T^{2l+1}x_0, T^{2l+2}x_0) + d(T^{2l+2}x_0, T^{2l+3}x_0) \\ &+ d(T^{2l+3}x_0, T^{2l+4}x_0) + d(T^{2l+4}x_0, T^{2l+5}x_0) + \cdots \\ &+ d(T^{2l+q-1}x_0, T^{2l+q}x_0) + d(T^{2l+q}x_0, T^{2l+q+1}x_0) \\ \leqslant& h^l\nu + h^{l+1}\nu + h^{l+1}\nu + h^{l+2}\nu + \cdots \\ \leqslant& 2h^l\Big(1 + +h + h^2 + h^3 + \cdots\Big)\nu \leqslant 2h^l\frac{1}{(1-h)}\nu. \end{aligned}$$

Taking $l \to \infty$ in all cases, since $h < 1$, we obtain, $d(T^mx_0, T^nx_0) \to 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since, X is complete, there exists a point $z \in X$ such that $x_n = T^nx_0 \to z \in X$ as $n \to \infty$. This shows that $z$ is a fixed point of T. Now, we prove that T has a unique fixed point in X. Let $z^* \in X$ be another fixed point of T. Using (2.1) for $x = z$ and $y = z^*$, we obtain

$$\alpha d^p(Tz, Tz^*) + (1-\alpha)d^p(T^2z, T^2z^*) \leqslant kd^p(z, z^*) \Rightarrow (1-k)d^p(z, z^*) \leqslant 0$$

leading to $d(z, z^*) = 0$, a contradiction. Hence, T has a unique fixed point in X. □

We have the following example for the validity of Theorem 3.1.

**Example 3.2.** Let $T\colon X \to X$, where $X = [0, 1]$ with usual metric $d(x, y) = |x-y|$. Define $Tx = \frac{1-x^2}{2}$ for all $x \in X$. Setting $\alpha = \frac{1}{6}$ and $p = 2$, we obtain

$$\alpha|Tx - Ty|^2 + (1-\alpha)|T^2x - T^2y|^2 \leqslant \alpha|x-y|^2 + \frac{(1-\alpha)}{2}|x-y|^2 = \frac{(1+\alpha)}{2}|x-y|^2 = \frac{7}{12}|x-y|^2.$$

This shows that T is $(\alpha, p)$-contraction with $\alpha + k = \frac{3}{4} < 1$. Moreover, $x = -1 + \sqrt{2}$ is the unique fixed point of T in X.

**Theorem 3.3.** *Let* $(X, d)$ *be a metric space and* $T\colon X \to X$ *be a* $(\alpha, p)$*-convex contraction such that* $\left(\sum_{i=1}^{5} k_i\right) + \alpha < 1$. *Then,* $T$ *has the* AFPP. *Further, if* $(X, d)$ *is a complete metric space, then* $T$ *has a unique fixed point.*

*Proof.* We define a sequence $\{x_n\}$ by $x_{n+1} = T^{n+1}x_0$ for all $n \geqslant 0$ and continue the same arguments as in Theorem 3.1, setting $\nu = d(x_0, Tx_0) + d(Tx_0, T^2x_0)$. Now, using (2.2) for $x = x_0$ and $y = Tx_0$, we obtain

$$\begin{aligned}(1-\alpha)d^p(T^2x_0, T^3x_0) &\leqslant + \alpha d^p(Tx_0, T^2x_0) + (1-\alpha)d^p(T^2x_0, T^3x_0)\\ &\leqslant (k_1+k_2)d^p(x_0, Tx_0) + (k_3+k_4)d^p(Tx_0, T^2x_0 + k_5 d^p(T^2x_0, T^3x_0)\\ &\leqslant (k_1+k_2+k_3+k_4)\nu^p + k_5 d^p(T^2x_0, T^3x_0).\end{aligned}$$

Therefore,

$$d^p(T^2x_0, T^3x_0) \leqslant \frac{k_1+k_2+k_3+k_4}{1-\alpha-k_5}\nu^p = h^p\nu^p \Rightarrow d(T^2x_0, T^3x_0) \leqslant h\nu$$

for $h^p = \left(\frac{k_1+k_2+k_3+k_4}{1-\alpha-k_5}\right)$; moreover, since $\left(\sum_{j=1}^{5} k_j\right) + \alpha < 1 \Rightarrow h^p < 1$.

Similarly, one can obtain

$$d(T^3x_0, T^4x_0) \leqslant h\nu, \quad \text{and} \quad d(T^4x_0, T^5x_0) \leqslant h^2\nu, \quad \text{and} \quad d(T^5x_0, T^6x_0) \leqslant h^2\nu.$$

Following similar arguments as in Theorem 3.1, we obtain $d(T^mx_0, T^{m+1}x_0) \to 0$ as $m \to \infty$, i.e., $T$ is asymptotically regular at $x_0$. By Lemma 1.4, $T$ has AFPP. Further, by assuming the continuity of $T$ and the completeness of $X$, the existence of a fixed point $z$ can be proved, using similar arguments as in Theorem 3.1.

Now, we show that $T$ has a unique fixed point in $X$. Let $z^* \in X$ be another fixed point of $T$. Using (2.2) for $x = z$ and $y = z^*$, we obtain

$$\begin{aligned}\alpha d^p(Tz, Tz^*) + (1-\alpha)d^p(T^2z, T^2z^*) \leqslant& k_1 d^p(z, z^*) + k_2 d^p(z, Tz) + k_3 d^p(Tz, T^2z)\\ &+ k_4 d^p(z^*, Tz^*) + k_5 d^p(Tz^*, T^2z^*) \Rightarrow (1-k_1)d^p(z, z^*) \leqslant 0,\end{aligned}$$

which gives $d(z, z^*) = 0$, a contradiction and hence, $T$ has a unique fixed point in $X$. □

One can verify the validity of Theorem 3.3 with Example 3.2 taking with $\alpha = \frac{1}{6}, k_1 = \frac{7}{12}, k_2 = k_3 = k_4 = k_5 = 0$, and $p = 2$.

**Theorem 3.4.** *Let* $(X, d)$ *be a complete metric space and* $T\colon X \to X$ *be a* $(\alpha, p)$*-contraction such that* $0 \leqslant k < \alpha$ *or* $k + \alpha < 1$. *If* $T$ *is asymptotically regular at some point* $x_0$ *in* $X$, *then there exists a unique fixed point of* $T$.

*Proof.* Let $T$ be an asymptotically regular mapping at $x_0 \in X$. Consider a sequence $\{T^nx_0\}$ in $X$ and for any two non zero positive integers $m, n \geqslant 1$ such that $m > n$, let us analyze the following two situations:

Case(i). When $0 \leqslant k < \alpha$. Using the inequality (2.1), we obtain

$$\begin{aligned}\alpha d^p(T^mx_0, T^nx_0) \leqslant& \alpha d^p(T^mx_0, T^nx_0) + (1-\alpha)d^p(T^{m+1}x_0, T^{n+1}x_0)\\ \leqslant& k d^p(T^{m-1}x_0, T^{n-1}x_0) \leqslant k\Big[d(T^{m-1}x_0, T^mx_0) + d(T^mx_0, T^nx_0) + d(T^nx_0, T^{n-1}x_0)\Big]^p.\end{aligned}$$

Taking $n, m \to \infty$ and using the asymptotically regularity of $T$ at $x_0$, the above inequality gives

$$\alpha \lim_{n\to\infty} d^p(T^mx_0, T^nx_0) \leqslant k \lim_{n\to\infty} d^p(T^mx_0, T^nx_0),$$

that is,

$$(\alpha - k) \lim_{n\to\infty} d^p(T^mx_0, T^nx_0) \leqslant 0.$$

Since $0 \leqslant k < \alpha$, it follows $\lim_{n\to\infty} d(T^mx_0, T^nx_0) = 0$.

Case(ii). When $0 < k + \alpha < 1$. Using the inequality (2.1), we obtain

$$\begin{aligned}(1-\alpha)d^p(T^m x_0, T^n x_0) \leqslant & \alpha d^p(T^{m-1}x_0, T^{n-1}x_0) + (1-\alpha)d^p(T^m x_0, T^n x_0)\\ \leqslant & k d^p(T^{m-2}x_0, T^{n-2}x_0)\\ \leqslant & k\Big[d(T^{m-2}x_0, T^m x_0) + d(T^m x_0, T^n x_0) + d(T^n x_0, T^{n-2}x_0)\Big]^p\\ \leqslant & k\Big[d(T^{m-2}x_0, T^{m-1}x_0) + d(T^{m-1}x_0, T^m x_0)\\ & + d(T^m x_0, T^n x_0) + d(T^n x_0, T^{n-1}x_0) + d(T^{n-1}x_0, T^{n-2}x_0)\Big]^p.\end{aligned}$$

Taking $n, m \to \infty$, we find

$$(1-\alpha)\lim_{n\to\infty} d^p(T^m x_0, T^n x_0) \leqslant k \lim_{n\to\infty} d^p(T^m x_0, T^n x_0) \Rightarrow (1-\alpha-k)\lim_{n\to\infty} d^p(T^m x_0, T^n x_0) \leqslant 0.$$

Therefore, $\lim_{n\to\infty} d(T^m x_0, T^n x_0) = 0$ as $0 < k + \alpha < 1$. Consequently, $\{T^n x_0\}$ is a Cauchy sequence in X. Since X is complete, it follows $T^n x_0 \to z$ as $n \to \infty$ for some $z \in X$. Now, we show that $Tz = z$, i.e., $z$ is a fixed point of T. For this, using again the inequality (2.1), we find

$$\alpha d^p(Tz, T^n x_0) \leqslant \alpha d^p(Tz, T^n x_0) + (1-\alpha)d^p(T^2 z, T^{n+1}x_0) \leqslant k d^p(z, T^{n-1}x_0).$$

As $n \to \infty$, we obtain

$$\alpha d^p(Tz, z) \leqslant 0,$$

which leads to $d(Tz, z) = 0$, that is $Tz = z$. Therefore, $z$ is a fixed point of T. The uniqueness of the fixed point follows immediately as in Theorem 3.1. □

**Example 3.5.** Let $T\colon X \to X$, where $X = [0, 1]$ with usual metric $d(x, y) = |x - y|$. Define $Tx = \frac{1+x}{2}$ for all $x \in X$. For any arbitrary $x_0 \in X$, we have $Tx_0 = \frac{1+x_0}{2}$ and $T^n x_0 = \frac{2^n-1+x_0}{2^n}$, where $T^n$ denotes the $n^{th}$ iterate of T. Also, we have

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1}x_0) = \lim_{n\to\infty}\left|\frac{2^n - 1 + x_0}{2^n} - \frac{2^{n+1} - 1 + x_0}{2^{n+1}}\right| = 0.$$

This shows that T is asymptotically regular at all points in X. Obviously, $\{T^n x_0\}$ is a sequence in X such that $T^n x_0 \to 1 \in X$ as $n \to \infty$. Taking $\alpha = \frac{1}{3}$, $k = \frac{1}{8}$, and $p = 2$, then T is $(\alpha, p)$-contraction for all $x, y \in X$ such that $k < \alpha$ or $k + \alpha < 1$. Thus, all the conditions of Theorem 3.4 are satisfied and hence, 1 is the unique fixed point of T.

**Theorem 3.6.** *Let $(X, d)$ be a complete metric space and $T\colon X \to X$ be a $\alpha$-contraction such that $k < \alpha$. If there exists an asymptotically T-regular sequence in X, then T has a unique fixed point.*

*Proof.* Let $\{x_n\}$ be an asymptotically T-regular sequence in X. Then, for any two non zero positive integers $m, n$ such that $m > n$, we obtain

$$\begin{aligned}\alpha d(x_m, x_n) \leqslant & \alpha\Big[d(x_m, Tx_m) + d(Tx_m, Tx_n) + d(Tx_n, x_n)\Big]\\ = & \alpha\Big[d(x_m, Tx_m) + d(Tx_n, x_n)\Big] + \alpha d(Tx_m, Tx_n)\\ \leqslant & \alpha\Big[d(x_m, Tx_m) + d(Tx_n, x_n)\Big] + \alpha d(Tx_m, Tx_n) + (1-\alpha)d(T^2 x_m, T^2 x_n)\\ \leqslant & \alpha\Big[d(x_m, Tx_m) + d(Tx_n, x_n)\Big] + k d(x_m, x_n),\end{aligned}$$

that is,

$$d(x_m, x_n) \leqslant \frac{\alpha}{\alpha - k}\Big[d(x_m, Tx_m) + d(Tx_n, x_n)\Big].$$

Taking $n, m \to \infty$ and using the fact that the sequence $\{x_n\}$ is asymptotically T-regular, we obtain

$$\lim_{n\to\infty} d(x_m, x_n) = 0.$$

This shows that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z \in X$ as $n \to \infty$.

Now, we show that $Tz = z$, i.e., $z$ is a fixed point of T.

$$\begin{aligned}\alpha d(Tz, x_n) \leqslant& \alpha\Big[d(Tz, Tx_n) + d(Tx_n, x_n)\Big]\\ \leqslant& \alpha d(Tz, Tx_n) + (1-\alpha)d(T^2z, T^2x_n) + \alpha d(Tx_n, x_n) \leqslant kd(z, x_n) + \alpha d(Tx_n, x_n).\end{aligned}$$

As $n \to \infty$ and since $\{x_n\}$ is asymptotically T-regular, we obtain

$$\alpha d(Tz, z) \leqslant 0$$

leading to $Tz = z$. Therefore, $z$ is a fixed point of T. The uniqueness of the fixed point follows immediately. □

**Example 3.7.** Let $T\colon X \to X$, where $X = [0, 1]$ with usual metric $d(x, y) = |x - y|$. Define $Tx = \frac{x}{3}$ for all $x \in X$. Consider a sequence $\{x_n\}$ in X such that $x_n \to 0$, then $Tx_n \to 0$, i.e., $|x_n - Tx_n| \to 0$ as $n \to \infty$. So, $\{x_n\}$ is asymptotically T-regular in X. Setting $\alpha = \frac{1}{2}$, $k = \frac{2}{9}$, then T is $\alpha$-contraction for all $x, y \in X$ such that $k < \alpha$. Thus, all the conditions of Theorem 3.6 are satisfied and hence, 0 is the unique fixed point of T.

**Theorem 3.8.** *Let* $(X, d)$ *be a complete metric space and* $T\colon X \to X$ *be a* $\alpha$*-contraction such that* $k + \alpha < 1$. *If there exists an asymptotically* $T^2$*-regular sequence in* X*, then* T *has a unique fixed point.*

*Proof.* Let $\{x_n\}$ be an asymptotically $T^2$-regular sequence in X. Then, for any two non zero positive integers $m, n$ such that $m > n$, we obtain

$$\begin{aligned}(1-\alpha)d(x_m, x_n) \leqslant& (1-\alpha)\Big[d(x_m, T^2x_m) + d(T^2x_m, T^2x_n) + d(T^2x_n, x_n)\Big]\\ =& (1-\alpha)\Big[d(x_m, T^2x_m) + d(T^2x_n, x_n)\Big] + (1-\alpha)d(T^2x_m, T^2x_n)\\ \leqslant& (1-\alpha)\Big[d(x_m, T^2x_m) + d(T^2x_n, x_n)\Big] + \alpha d(Tx_m, Tx_n) + (1-\alpha)d(T^2x_m, T^2x_n)\\ \leqslant& (1-\alpha)\Big[d(x_m, T^2x_m) + d(T^2x_n, x_n)\Big] + kd(x_m, x_n),\end{aligned}$$

that is,

$$d(x_m, x_n) \leqslant \frac{1-\alpha}{1-\alpha-k}\Big[d(x_m, T^2x_m) + d(T^2x_n, x_n)\Big].$$

Since $\{x_n\}$ is asymptotically $T^2$-regular sequence, by taking $n, m \to \infty$, we obtain

$$\lim_{n\to\infty} d(x_m, x_n) = 0,$$

which proves that $\{x_n\}$ is a Cauchy sequence. Since, X is complete, there exists a point $z \in X$ such that $x_n \to z \in X$ as $n \to \infty$.

In order to show that $z$ is a fixed point of T in X, we make several steps.

First, we show that $T^2z = z$. Using inequality (2.1), we obtain

$$\begin{aligned}(1-\alpha)d(T^2z, x_n) \leqslant& (1-\alpha)\left[d(T^2z, T^2x_n) + d(T^2x_n, x_n)\right]\\ \leqslant& \alpha d(Tz, Tx_n) + (1-\alpha)d(T^2z, T^2x_n) + (1-\alpha)d(T^2x_n, x_n)\\ \leqslant& kd(z, x_n) + (1-\alpha)d(T^2x_n, x_n).\end{aligned}$$

Taking $n \to \infty$, and using the asymptotically $T^2$-regularity of the sequence $\{x_n\}$, we obtain

$$(1-\alpha)d(T^2z, z) \leqslant 0,$$

which gives $T^2z = z$. Therefore, one can obtain inductively that $T^{2n}z = z$ and $T^{2n+1}z = Tz$ for $n \geqslant 1$.

We show that $Tz = z$, i.e., $z$ is a fixed point of $T$.

Using the inequality(2.1), we obtain

$$(1-\alpha)d(z, Tz) = (1-\alpha)d(T^2z, T^3z) \leqslant \alpha d(Tz, T^2z) + (1-\alpha)d(T^2z, T^3z) \leqslant kd(z, Tz),$$

that is,

$$(1-\alpha-k)d(z, Tz) \leqslant 0$$

a contradiction, if $Tz \neq z$. Therefore, $z$ is a fixed point of $T$. Using the inequality (2.1), one can obtain the uniqueness of fixed point. □

**Example 3.9.** Let $T\colon X \to X$, where $X = \{0, 1, 2\}$ and $A = \{0, 1\} \subset X$ with usual metric $d(x, y) = |x - y|$. Define

$$Tx = \begin{cases} 1, & x \notin A,\\ 0, & x \in A.\end{cases}$$

Consider a sequence $\{x_n\}$ in $X$ such that $x_n \to 0$, then $Tx_n \to 1$ and $T^2x_n \to 0$ as $n \to \infty$. Consequently, $|x_n - T^2x_n| \to 0$ as $n \to \infty$. So, $\{x_n\}$ is asymptotically $T^2$-regular in $X$. Setting $\alpha = k = \frac{1}{3}$, then $T$ is $\alpha$-contraction for all $x, y \in X$ such that $k + \alpha < 1$. Thus, all the conditions of Theorem 3.8 are satisfied and hence, 0 is the unique fixed point of $T$.

The following Theorems 3.10 and 3.12 are motivated by Theorems 3.1 and 3.4 of Khan and Jhade [11].

**Theorem 3.10.** *Let $(X, d)$ be a complete metric space and $T\colon X \to X$ be an $\alpha$-convex contraction such that $0 < k_1 + \alpha < 1$ and $\mu, h < 1$, where $\mu = \max\{\frac{k_3}{\alpha - k_2 - k_3}, \frac{k_5}{\alpha - k_4 - k_5}\}$ and $h = \max\{\frac{k_2 + k_3}{1 - \alpha - k_3}, \frac{k_4 + k_5}{1 - \alpha - k_5}\}$. If there exists an asymptotically $(T, T^2)$-regular sequence in $X$, then $T$ has a unique fixed point.*

*Proof.* Let $\{x_n\}$ be an asymptotically $(T, T^2)$-regular sequence in $X$. Then, for any non zero positive integers $m, n$ such that $m > n$, we obtain

$$\begin{aligned}(1-\alpha)d(x_m, x_n) \leqslant& (1-\alpha)\left[d(x_m, T^2x_m) + d(T^2x_m, T^2x_n) + d(T^2x_n, x_n)\right]\\ =& (1-\alpha)\left[d(x_m, T^2x_m) + d(T^2x_n, x_n)\right] + (1-\alpha)d(T^2x_m, T^2x_n)\\ \leqslant& (1-\alpha)\left[d(x_m, T^2x_m) + d(T^2x_n, x_n)\right] + \alpha d(Tx_m, Tx_n) + (1-\alpha)d(T^2x_m, T^2x_n)\\ \leqslant& (1-\alpha)\left[d(x_m, T^2x_m) + d(T^2x_n, x_n)\right]\\ &+ k_1d(x_m, x_n) + k_2d(x_m, Tx_m) + k_3d(Tx_m, T^2x_m) + k_4d(x_n, Tx_n) + k_5d(Tx_n, T^2x_n),\end{aligned}$$

that is,

$$\begin{aligned}(1-\alpha-k_1)d(x_m,x_n) \leqslant &(1-\alpha)\Big[d(x_m,T^2x_m)+d(T^2x_n,x_n)\Big]\\ &+k_2d(x_m,Tx_m)+k_3d(Tx_m,T^2x_m)+k_4d(x_n,Tx_n)+k_5d(Tx_n,T^2x_n).\end{aligned}$$

Since, $\{x_n\}$ is asymptotically $(T,T^2)$-regular sequence. Letting $n,m\to\infty$ and using Lemma 2.10, we obtain $\lim_{n\to\infty} d(x_n,x_m)=0$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Since, X is complete, there exists a point $z\in X$ such that $x_n\to z\in X$ as $n\to\infty$. Now, we show that $z$ is a fixed point of T in X. For this, first we show that $T^2z=z$. Using inequality (2.1), we obtain

$$\begin{aligned}(1-\alpha)d(T^2z,x_n) \leqslant &(1-\alpha)\Big[d(T^2z,T^2x_n)+d(T^2x_n,x_n)\Big]\\ \leqslant &\Big[\alpha d(Tz,Tx_n)+(1-\alpha)d(T^2z,T^2x_n)\Big]+(1-\alpha)d(T^2x_n,x_n)\\ \leqslant &k_1d(z,x_n)+k_2d(z,Tz)+k_3d(Tz,T^2z)\\ &+k_4d(x_n,Tx_n)+k_5d(Tx_n,T^2x_n)+(1-\alpha)d(T^2x_n,x_n)\\ \leqslant &k_1d(z,x_n)+k_2d(z,Tz)+k_3\Big[d(Tz,x_n)+d(T^2z,x_n)\Big]\\ &+k_4d(x_n,Tx_n)+k_5d(Tx_n,T^2x_n)+(1-\alpha)d(T^2x_n,x_n),\end{aligned}$$

that is,

$$\begin{aligned}(1-\alpha-k_3)d(T^2z,x_n) \leqslant &k_1d(z,x_n)+k_2d(z,Tz)+k_3d(Tz,x_n)\\ &+k_4d(x_n,Tx_n)+k_5d(Tx_n,T^2x_n)+(1-\alpha)d(T^2x_n,x_n).\end{aligned}$$

Taking $n\to\infty$ and using Lemma 2.10, we obtain

$$(1-\alpha-k_3)d(T^2z,z) \leqslant (k_2+k_3)d(z,Tz),$$

that is,

$$d(T^2z,z) \leqslant \frac{k_2+k_3}{1-\alpha-k_3}d(Tz,z).$$

Similarly, by symmetry of the $\alpha$-convex contraction, one can obtain

$$d(T^2z,z) \leqslant \frac{k_4+k_5}{1-\alpha-k_5}d(Tz,z).$$

Since, $h=\max\{\frac{k_2+k_3}{1-\alpha-k_3},\frac{k_4+k_5}{1-\alpha-k_5}\}<1$. This shows that $d(T^2z,z)\leqslant hd(Tz,z)$.

Now, we show that $Tz=z$, i.e., $z$ is a fixed point of T.

$$\begin{aligned}\alpha d(Tz,x_n) \leqslant &\alpha\Big[d(Tz,Tx_n)+d(Tx_n,x_n)\Big]+(1-\alpha)d(T^2z,T^2x_n)\\ = &\alpha d(Tz,Tx_n)+(1-\alpha)d(T^2z,T^2x_n)+\alpha d(Tx_n,x_n)\\ \leqslant &k_1d(z,x_n)+k_2d(z,Tz)+k_3d(Tz,T^2z)\\ &+k_4d(x_n,Tx_n)+k_5d(Tx_n,T^2x_n)+\alpha d(Tx_n,x_n)\\ \leqslant &k_1d(z,x_n)+k_2d(z,Tz)+k_3d(Tz,z)\\ &+k_3d(T^2z,z)+k_4d(x_n,Tx_n)+k_5d(Tx_n,T^2x_n)+\alpha d(Tx_n,x_n).\end{aligned}$$

As $n\to\infty$, we obtain

$$\alpha d(Tz,z) \leqslant (k_2+k_3)d(Tz,z)+k_3d(T^2z,z),$$

that is,

$$d(Tz,z) \leqslant \frac{k_3}{\alpha - k_2 - k_3} d(T^2z,z).$$

Similarly, based on the symmetry of $\alpha$-convex contractions, one can prove

$$d(Tz,z) \leqslant \frac{k_5}{\alpha - k_4 - k_5} d(T^2z,z).$$

Since $\mu = \max\{\frac{k_3}{\alpha-k_2-k_3}, \frac{k_5}{\alpha-k_4-k_5}\} < 1$, we find

$$d(Tz,z) \leqslant \mu d(T^2z,z) \leqslant h\mu d(Tz,z),$$

that is,

$$(1-h\mu)d(Tz,z) \leqslant 0$$

leading to $d(Tz,z) = 0$ as $h\mu < 1$. Therefore, $z$ is a fixed point of $T$. For uniqueness, let $z^* \in X$ be another fixed point of $T$. Using (2.1) for $x = z$ and $y = z^*$, we obtain

$$\alpha d(Tz,Tz^*) + (1-\alpha)d(T^2z,T^2z^*) \leqslant k_1 d(z,z^*) + k_2 d(z,Tz) + k_3 d(Tz,T^2z) + k_4 d(z^*,Tz^*) + k_5 d(Tz^*,T^2z^*),$$

that is,

$$(1-k_1)d(z,z^*) \leqslant 0,$$

which in turn gives $d(z,z^*) = 0$ and hence, $T$ has a unique fixed point in $X$. □

**Example 3.11.** Let $T\colon X \to X$, where $X = [0,1]$. Define $Tx = \frac{1+x}{4}$ for all $x \in X$. Consider a sequence $\{x_n\}$ in $X$ such that $x_n \to \frac{1}{3}$ as $n \to \infty$. Consequently, $Tx_n, T^2x_n \to \frac{1}{3}$ as $n \to \infty$. Therefore, the sequence $\{x_n\}$ is asymptotically $(T,T^2)$-regular in $X$. Setting $\alpha = \frac{1}{2}, k_1 = \frac{5}{32}, k_2 = k_3 = k_4 = k_5 = 0$, then $T$ is $\alpha$-convex contraction such that $k_1 + \alpha < 1$, $\mu = 0 < 1$ and $h = 0 < 1$. Thus, all the conditions of Theorem 3.10 are satisfied and hence, $\frac{1}{3}$ is the unique fixed point of $T$.

**Theorem 3.12.** *Let $(X,d)$ be a complete metric space and $T : X \to X$ be a $\alpha$-convex contraction such that $k_1 < \alpha$ or, $0 < k_1 + \alpha < 1$ and $\mu, h < 1$, where $\mu = \max\{\frac{k_3}{\alpha-k_2-k_3}, \frac{k_5}{\alpha-k_4-k_5}\}$ and $h = \max\{\frac{k_2+k_3}{1-\alpha-k_3}, \frac{k_4+k_5}{1-\alpha-k_5}\}$. If $T$ is asymptotically regular at some point $x_0$ in $X$, then there exists a unique fixed point of $T$.*

*Proof.* Let $T$ be an asymptotically regular mapping at $x_0 \in X$. Consider a sequence $\{T^nx_0\}$ and for any two non zero positive integers $m, n \geqslant 1$ such that $m > n$.

We analyze the following cases.

Case (i). When $k_1 < \alpha$. We obtain

$$\begin{aligned}
\alpha d(T^mx_0,T^nx_0) \leqslant& \alpha d(T^mx_0,T^nx_0) + (1-\alpha)d(T^{m+1}x_0,T^{n+1}x_0)\\
\leqslant& k_1 d(T^{m-1}x_0,T^{n-1}x_0) + k_2 d(T^{m-1}x_0,T^mx_0)\\
&+ k_3 d(T^mx_0,T^{m+1}x_0) + k_4 d(T^{n-1}x_0,T^nx_0) + k_5 d(T^nx_0,T^{n+1}x_0)\\
\leqslant& k_1 \Big[d(T^{m-1}x_0,T^mx_0) + d(T^mx_0,T^nx_0)\\
&+ d(T^nx_0,T^{n-1}x_0)\Big] + k_2 d(T^{m-1}x_0,T^mx_0)\\
&+ k_3 d(T^mx_0,T^{m+1}x_0) + k_4 d(T^{n-1}x_0,T^nx_0) + k_5 d(T^nx_0,T^{n+1}x_0),
\end{aligned}$$

that is,

$$\begin{aligned}(\alpha - k_1)d(T^m x_0, T^n x_0) \leqslant &(k_1 + k_2)d(T^{m-1}x_0, T^m x_0)\\ &+ (k_1 + k_4)d(T^{n-1}x_0, T^n x_0) + k_3 d(T^m x_0, T^{m+1}x_0) + k_5 d(T^n x_0, T^{n+1}x_0).\end{aligned}$$

Taking $n, m \to \infty$ and using the asymptotically regularity of $T$ at $x_0$, we obtain

$$\lim_{n\to\infty} d(T^m x_0, T^n x_0) = 0.$$

Case (ii). When $0 < k_1 + \alpha < 1$, we obtain

$$\begin{aligned}(1-\alpha)d(T^m x_0, T^n x_0) \leqslant &\alpha d(T^{m-1}x_0, T^{n-1}x_0) + (1-\alpha)d(T^m m x_0, T^n x_0)\\ \leqslant &k_1 d(T^{m-2}x_0, T^{n-2}x_0) + k_2 d(T^{m-2}x_0, T^{m-1}x_0)\\ &+ k_3 d(T^{m-1}x_0, T^m x_0) + k_4 d(T^{n-2}x_0, T^{n-1}x_0) + k_5 d(T^{n-1}x_0, T^n x_0)\\ \leqslant &k_1 \Big[ d(T^{m-2}x_0, T^{m-1}x_0) + d(T^{m-1}x_0, T^m x_0)\\ &+ d(T^m x_0, T^n x_0) + d(T^n x_0, T^{n-1}x_0)\\ &+ d(T^{n-1}x_0, T^{n-2}x_0)\Big] + k_2 d(T^{m-1}x_0, T^m x_0)\\ &+ k_3 d(T^m x_0, T^{m+1}x_0) + k_4 d(T^{n-1}x_0, T^n x_0) + k_5 d(T^n x_0, T^{n+1}x_0).\end{aligned}$$

Taking $n, m \to \infty$, we obtain

$$(1 - \alpha - k_1) \lim_{n\to\infty} d(T^{m+1}x_0, T^{n+1}x_0) \leqslant 0,$$

which gives $\lim_{n\to\infty} d(T^{m+1}x_0, T^{n+1}x_0) = 0$.

In both cases it follows that $\{T^n x_0\}$ is a Cauchy sequence in $X$. Since $X$ is complete, so $T^n x_0 \to z$ as $n \to \infty$ for some $z \in X$. Thus, by following the same argument as in Theorem 3.10, one can obtain the unique fixed point of $T$. □

One can check the validity of Theorem 3.12 with Example 3.5 setting with $\alpha = \frac{2}{5}, k_1 = \frac{7}{20}, k_2 = k_3 = k_4 = k_5 = 0$, and $p = 1$.

**Corollary 3.13.** *Let $(X, d)$ be a metric space and $T: X \to X$ be a two-sided convex contraction mapping. Then, $T$ has AFPP. Further, if $(X, d)$ is a complete metric space, then $T$ has a unique fixed point.*

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