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Coincidence best proximity points for geraghty type proximal cyclic contractions

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Abstract

In this paper, we study the notions of generalized Geraghty proximal cyclic contractions for non-self mapping and obtain coincidence best proximity point theorems in the framework of complete metric spaces. Some examples are given to show the validity of our results. Our results extended and unify many existing results in the literature.

Keywords: α-Geraghty proximal contraction of first and second kind, α-proximal cyclic contraction, α-proximal admissible maps.

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1. Introduction

Several problems can be formulated as equations of the form $Tx = x$ where T is a self-mapping in some appropriate framework. In fact, fixed point theory delves into the existence of a solution to such generic equations and brings out the iterative algorithms to compute a solution to such equations. Banach contractive principle [7] was the key principle in the development of metric fixed point theory. Due to importance of the contractive principle in nonlinear analysis, a number of authors have improved, generalized, and extended this basic result either by defining a new contractive mapping in the context of a complete metric space or by investigating the existing contractive mappings in various abstract spaces; see, e.g., [4, 6, 9, 12–15, 19, 31] and references therein.

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On contrary, in the case that T is not a self-mapping, it is probable that the equation $Tx = x$ possesses no solution, for a solution of the preceding equation necessitates the equality between an element in the domain and an element in the co-domain of the mapping. In such scenarios, it is worthwhile to determine an approximate solution that is optimal in the sense that the error due to approximation is minimum. That is, if $T : A \to B$ is a non-self-mapping in the framework of a metric space, one desires to compute an approximate solution $x \in A$ such that the error $d(x, Tx)$ is minimum. Precisely, a solution to the non-linear programming problems. $\min_{x\in A} d(x, Tx)$ is basically an ideal optimal approximate solution to the equation $Tx = x$ which is unlikely to have a solution when T is supposed to be a non-self mapping. Considering the fact that $d(x, Tx)$ is at least $d(A, B)$ for all $x \in A$, a solution x to the aforementioned non-linear programming problem becomes an approximate solution with the lowest possible error to the corresponding equation $Tx = x$ if it satisfies the condition that $d(x, Tx) = d(A, B)$. Indeed, such a solution x is known as a best proximity point of the mapping and the results that investigate the existence of best proximity points for non-self mappings are called best proximity point theorems. Best proximity point theorems for several types of non-self mappings have been derived in [1–3, 20–24, 26]. In 1973, Geraghty [12] obtained a generalization of the Banach contraction principle in the setting of complete metric spaces by considering an auxiliary function. Later, Amini-Harandi and Emami [6] characterized the result of Geraghty in the context of a partially ordered complete metric space. This result is of particular interest since many real world problems can be identified in a partially ordered complete metric space. Cabellero et al. [8] discussed the existence of a best proximity point of Geraghty contraction. The references [5, 10, 11, 25, 27–30] furnish some appealing best proximity point theorems for various contractions while Basha [23] and Shahzad [26] explore some common best proximity point theorems for certain contractions.

The main aim of this paper is to derive a best proximity point theorem for α-Geraghty proximal cyclic contraction and for α-Geraghty proximal contractions of the first as well as of second kind in the setting of complete metric space. We present some examples to prove the validity of our results.

2. Preliminaries

Let A and B be two nonempty subsets of a metric space (X, d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{a \in A : \text{there exists some } b \in B \text{ such that } d(a, b) = d(A, B)\},$$
$$B_0 = \{b \in B : \text{there exists some } a \in A \text{ such that } d(a, b) = d(A, B)\},$$

where

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

In [16], the authors present sufficient conditions which determine when the sets A_0 and B_0 are nonempty.

Definition 2.1 ([21]). A mapping $T : A \to B$ is called proximal contraction of the first kind if there exists $k \in [0, 1)$ such that

$$d(u, Tx) = d(A, B), \quad d(v, Ty) = d(A, B)$$

implies

$$d(u, v) \leqslant kd(x, y)$$

for all $u, v, x, y \in A$. It is easy to see that a self-mapping is a contraction of the first kind is precisely a contraction. However a non self proximal contraction is not necessarily a contraction.

Definition 2.2 ([21]). A mapping $T : A \to B$ is called proximal contraction of the second kind if there exists $k \in [0, 1)$ such that

$$d(u, Tx) = d(A, B), \quad d(v, Ty) = d(A, B)$$

implies

$$d(Tu, Tv) \leqslant kd(Tx, Ty)$$

for all $u, v, x, y \in A$.

Definition 2.3 ([21]). Let (X, d) be a complete metric space. A mapping $g : X \to X$ is called an isometry if $d(gx, gy) = d(x, y)$ for $x, y \in X$.

Definition 2.4 ([21]). Let $S : A \to B$ and $g : A \to A$ be an isometry. The mapping S is said to preserve the isometric distance with respect to g if $d(Sgx, Sgy) = d(Sx, Sy)$ for all $x, y \in A$.

Definition 2.5 ([21]). Consider the non-self-mappings $S : A \to B$ and $T : B \to A$, the pair (S, T) is said to form a proximal cyclic contraction if there exists a non-negative number $\alpha < 1$ such that

$$d(u, Sx) = d(A, B), \quad d(v, Ty) = d(A, B)$$

implies

$$d(u, v) \leqslant \alpha d(x, y) + (1 - \alpha)d(A, B)$$

for all $u, v, x, y \in A$.

Definition 2.6 ([17]). Let $T : A \to B$ be a map and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. Then T is said to be α-proximal admissible if

$$\alpha(x, y) \geqslant 1, d(u, Tx) = d(A, B), d(v, Ty) = d(A, B)$$

implies that $\alpha(u, v) \geqslant 1$.

Theorem 2.7 ([21]). *Let (X, d) be a complete metric space, let A, B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $S : A \to B, T : B \to A$, and $g : A \bigcup B \to A \bigcup B$ satisfy the following conditions:*

1. *S and T are proximal contractions of the first kind;*
2. *g is an isometry;*
3. *the pair (S, T) is a proximal cyclic contraction;*
4. *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*
5. *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

Then there exists point $x \in A$ and there exists point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

converges to the element x. For any fixed $x_0 \in A_0$, the sequence $\{y_n\}$ defined by

$$d(gy_{n+1}, Ty_n) = d(A, B)$$

converges to the element y. On the other hand, a sequence $\{u_n\}$ in A converges to x if there exists a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n\to\infty} \epsilon_n = 0,$$

$d(u_{n+1}, z_{n+1}) \leqslant \epsilon_n$*, where $z_{n+1} \in A$ satisfying the condition that* $d(gz_{n+1}, Su_n) = d(A, B)$.

Now, we introduce the class $\mathcal{F}$ of those functions $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0.$$

Definition 2.8 ([18]). A mapping $T : A \to B$ is called Geraghty's proximal contraction of first kind if there exists $\beta \in \mathcal{F}$ such that

$$d(u, Tx) = d(A, B), \quad d(v, Ty) = d(A, B)$$

implies

$$d(u, v) \leqslant \beta(d(x, y))d(x, y)$$

for all $u, v, x, y \in A$.

Definition 2.9 ([18]). A mapping $T : A \to B$ is called Geraghty's proximal contraction of the second kind if there exists $\beta \in \mathcal{F}$ such that

$$d(u, Tx) = d(A, B), \quad d(v, Ty) = d(A, B)$$

implies

$$d(u, v) \leqslant \beta(d(Tx, Ty))d(Tx, Ty).$$

for all $u, v, x, y \in A$.

Theorem 2.10 ([18]). *Let (X, d) be a complete metric space, let A, B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $S : A \to B, T : B \to A$, and $g : A \bigcup B \to A \bigcup B$ satisfy the following conditions:*

1. *S and T are Geraghty proximal contraction of the first kind;*
2. *g is an isometry;*
3. *the pair (S, T) is a proximal cyclic contraction;*
4. *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*
5. *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$;*
6. *S and T are proximal admissible maps.*

Then there exists point $x \in A$ and there exists point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

converges to the element x. For any fixed $x_0 \in A_0$, the sequence $\{y_n\}$ defined by

$$d(gy_{n+1}, Ty_n) = d(A, B)$$

converges to the element y. On the other hand, a sequence $\{u_n\}$ in A converges to x if there exists a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n\to\infty} \epsilon_n = 0,$$

$d(u_{n+1}, z_{n+1}) \leqslant \epsilon_n$, *where $z_{n+1} \in A$ satisfying the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.*

Proposition 2.11 ([18]). *Let $f : [0, \infty) \to [0, \infty)$ be a function defined by $f(t) = \ln(1 + t)$. Then we have the following inequality*

$$f(a) - f(b) \leqslant f(|a - b|)$$

for all $a, b \in [0, \infty)$.

Proposition 2.12 ([18]). *For each $x, y \in \mathbb{R}$, the following inequality holds:*

$$\frac{1}{(1 + |x|)(1 + |y|)} \leqslant \frac{1}{1 + |x - y|}.$$

3. Coincidence best proximity point results

Motivated by Basha [21], Mongkolkeha et al. [18] introduced the new classes of proximal contractions, which are more general than the class of proximal contractions of the first and second kinds. In this section we generalized the notions of Mongkolkeha et al. [18] and obtain coincidence best proximity point results.

Definition 3.1. Let $S : A \to B$ and $T : B \to A$ be mappings. The pair (S, T) is called α-proximal cyclic contraction if there exists $k \in [0, 1)$ such that

$$d(a, Sx) = d(A, B), \quad d(b, Ty) = d(A, B)$$

implies

$$\alpha(x, y)d(a, b) \leqslant kd(x, y) + (1 - k)d(A, B)$$

for all $a, x \in A$ and $b, y \in B$.

Definition 3.2. A mapping $T : A \to B$ is called α-Geraghty's proximal contraction of first kind if there exists $\beta \in \mathcal{F}$ such that

$$d(u, Tx) = d(A, B), \quad d(v, Ty) = d(A, B)$$

implies

$$\alpha(x, y)d(u, v) \leqslant \beta(d(x, y))d(x, y)$$

for all $u, v, x, y \in A$.

Example 3.3. Consider $\mathbb{R}$ with Euclidean metric. Let

$$A = \{(0, x) : x \in \mathbb{R}\}, \quad B = \{(2, y) : y \in \mathbb{R}\}.$$

Then $d(A, B) = 2$. Define the mapping $T : A \to B$ as follows:

$$T((0, x)) = (2, \ln(1 + |x|)).$$

First we show that T is α-Geraghty proximal contraction of the first kind. Define $\beta : [0, \infty) \to [0, 1)$ by

$$\beta(t) = \begin{cases} 1, & t = 0, \\ \frac{\ln(1+t)}{t}, & t > 0. \end{cases}$$

Clearly $\beta \in \mathcal{F}$. Let $(0, x_1), (0, x_2), (0, a_1)$, and $(0, a_2)$ be elements in A satisfying

$$d((0, a_1), S(0, x_1)) = d((0, a_1), (2, \ln(1 + x_1))) = d(A, B) = 2,$$

$$d((0, a_2), S(0, x_2)) = d((0, a_2), (2, \ln(1 + x_2))) = d(A, B) = 2.$$

Then $a_i = \ln(1 + |x_i|)$ for $i = 1, 2$.

Let $\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ be defined as follows:

$$\alpha((x_1, y_1), (x_2, y_2)) = \begin{cases} 1, & \text{if } 0 \leqslant x_1, x_2 \leqslant 2 \text{ and } y_1 \neq y_2 \geqslant 0, \\ 0, & \text{elsewhere.} \end{cases}$$

If $x_1 = x_2$, we have done. Assume that $x_1 \neq x_2$. Then by Proposition 2.11 and the fact that $f(t) = \ln(1 + t)$ is increasing, we have

$$\alpha((0, x_1), (0, x_2))d((0, a_1), (0, a_2)) = d(0, \ln(1 + |x_1|), (0, \ln(1 + |x_2|)))$$

$$\begin{aligned}
&= |\ln(1+|x_1|) - \ln(1+|x_2|)| \\
&\leqslant |\ln(1+||x_1|-|x_2||)| \\
&\leqslant |\ln(1+|x_1-x_2|)| \\
&= \frac{|\ln(1+|x_1-x_2|)|}{|x_1-x_2|}|x_1-x_2| \\
&= \beta(d(0,x_1),d(0,x_2))d((0,x_1),(0,x_2)).
\end{aligned}$$

Thus S is α-Geraghty proximal contraction of the first kind.

Next, we prove that S is not α-proximal contraction of the first kind. Suppose S is α-proximal contraction of the first kind, then for each $(0,x^*),(0,y^*),(0,a^*),(0,b^*) \in A$ satisfying

$$d((0,x^*),S(0,a^*)) = d(A,B) = 2, \quad d((0,y^*),S(0,b^*)) = d(A,B) = 2,$$

there exists $k \in [0,1)$ such that

$$\alpha((0,a^*),(0,b^*))d((0,x^*),(0,y^*)) \leqslant kd((0,a^*),(0,b^*)).$$

From Proposition 2.12, we get $x^* = \ln(1+|a^*|)$ and $y^* = \ln(1+|b^*|)$ and so

$$\begin{aligned}
\ln(1+|a^*|)-\ln(1+|b^*|) = d((0,x^*),(0,y^*)) = \alpha((0,a^*),(0,b^*))d((0,x^*),(0,y^*)) &\leqslant kd((0,a^*),(0,b^*)) \\
&= k|a^*-b^*|.
\end{aligned}$$

Letting $b^* = 0$, we get

$$1 = \lim_{|a^*|\to 0^+} \frac{|\ln(1+|a^*|)|}{|a^*|} \leqslant k < 1,$$

which is a contradiction. Thus, S is not α-proximal contraction of the first kind.

Definition 3.4. A mapping $T : A \to B$ is called α-Geraghty's proximal contraction of the second kind if there exists $\beta \in \mathcal{F}$ such that

$$d(u,Tx) = d(A,B), \quad d(v,Ty) = d(A,B)$$

implies

$$\alpha(x,y)d(u,v) \leqslant \beta(d(Tx,Ty))d(Tx,Ty)$$

for all $u,v,x,y \in A$.

Theorem 3.5. *Let (X,d) be a complete metric space, $\alpha : X\times X \to \mathbb{R}^+$ be a function and let A,B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $S : A \to B, T : B \to A$, and $g : A\bigcup B \to A\bigcup B$ satisfy the following conditions:*

1. *S and T are α-Geraghty proximal contractions of the first kind;*
2. *g is an isometry with $A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$;*
3. *the pair (S,T) is an α-proximal cyclic contraction;*
4. *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*
5. *S and T are α-proximal admissible maps;*
6. *$\alpha(x_0,x_1) \geqslant 1$ for $x_0,x_1 \in X$.*

Then there exists a point $x \in A$ and there exists a point $y \in B$ such that

$$d(gx,Sx) = d(gy,Ty) = d(x,y) = d(A,B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(gx_{n+1},Sx_n) = d(A,B)$$

converges to the element x*. For any fixed* $x_0 \in A_0$*, the sequence* $\{y_n\}$ *defined by*

$$d(gy_{n+1}, Ty_n) = d(A, B)$$

converges to the element y*. On the other hand, a sequence* $\{u_n\}$ *in* A *converges to* x *if there exists a sequence of positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\to\infty} \epsilon_n = 0,$$

$d(u_{n+1}, z_{n+1}) \leqslant \epsilon_n$*, where* $z_{n+1} \in A$ *satisfying the condition that* $d(gz_{n+1}, Su_n) = d(A, B)$*.*

Proof. Let x_0 be a fixed element in A_0. In view of the fact that $S(A_0) \subseteq B_0$, also it is given that $A_0 \subseteq g(A_0)$, there exists an element $x_1 \in A_0$ such that

$$d(gx_1, Sx_0) = d(A, B).$$

Again, since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_2 \in A_0$ such that

$$d(gx_2, Sx_1) = d(A, B).$$

Continuing in a similar fashion, we can find x_n in A_0 such that

$$d(gx_n, Sx_{n-1}) = d(A, B).$$

Inductively, there exists an element $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B).$$

Since $\alpha(x_0, x_1) \geqslant 1$ for $x_0, x_1 \in X$, and for $x_1 \in A_0$, $S(A_0) \subseteq B_0$ there exists $x_2 \in A_0$ such that

$$d(x_2, Sx_1) = d(A, B)$$

for $x_2 \in A_0$, $S(A_0) \subseteq B_0$ there exists $x_3 \in A_0$ such that

$$d(x_3, Sx_2) = d(A, B).$$

Since S is α-proximal admissible mapping, then from

$$d(x_2, Sx_1) = d(A, B), \quad d(x_3, Sx_2) = d(A, B),$$

implies that $\alpha(x_2, x_3) \geqslant 1$. Proceeding in the same manner, we have

$$\alpha(x_n, x_{n+1}) \geqslant 1$$

for $n \in \mathbb{N}$. Since S is α-Geraghty proximal contraction of the first kind, it follows that, for each $n \geqslant 1$

$$d(x_{n+1}, x_n) = d(gx_{n+1}, gx_n) \leqslant \alpha(x_n, x_{n-1}) d(gx_{n+1}, gx_n) \leqslant \beta(d(x_n, x_{n-1})) d(x_n, x_{n-1}) < d(x_n, x_{n-1}).$$

This shows that $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence and bounded below. Hence there exists $r \geqslant 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. Suppose that $r > 0$. Observed that

$$\frac{d(x_{n+1}, x_n)}{d(x_n, x_{n-1})} \leqslant \beta(d(x_n, x_{n-1})).$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \beta(d(x_n, x_{n-1})) = 1.$$

Since $\beta \in \mathcal{F}$, so that $r = 0$, which is a contradiction to our supposition and hence

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = 0.$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not Cauchy sequence. Then there exists $\epsilon > 0$ and subsequences $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ such that for any positive integers $n_k > m_k \geqslant k$,

$$r_k := d(x_{m_k}, x_{n_k}) \geqslant \epsilon,$$

$d(x_{m_k}, x_{n_k-1}) < \epsilon$ for any $k \in \{1, 2, 3, \ldots\}$. For each $n \geqslant 1$, let $\gamma_n := d(x_{n+1}, x_n)$. Then we have,

$$\epsilon \leqslant r_k = d(x_{m_k}, x_{n_k}) \leqslant d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) < \epsilon + \gamma_{n_k-1}.$$

Taking limit as $k \to \infty$, we get

$$\epsilon \leqslant \lim_{k\to\infty} r_k < \epsilon + \lim_{k\to\infty} \gamma_{n_k-1}$$

implies

$$\epsilon \leqslant \lim_{k\to\infty} r_k < \epsilon + 0.$$

Thus

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon.$$

Notice also that

$$\begin{aligned}
\epsilon \leqslant r_k = d(x_{m_k}, x_{n_k}) &\leqslant d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\
&= \gamma_{m_k} + \gamma_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\
&= \gamma_{m_k} + \gamma_{n_k} + d(gx_{m_k+1}, gx_{n_k+1}) \\
&\leqslant \gamma_{m_k} + \gamma_{n_k} + \alpha(x_{m_k}, x_{n_k}) d(gx_{m_k+1}, gx_{n_k+1}) \\
&\leqslant \gamma_{m_k} + \gamma_{n_k} + \beta(d(x_{m_k}, x_{n_k})) d(x_{m_k}, x_{n_k})
\end{aligned}$$

implies

$$\frac{d(x_{m_k}, x_{n_k}) - \gamma_{m_k} - \gamma_{n_k}}{d(x_{m_k}, x_{n_k})} \leqslant \beta(d(x_{m_k}, x_{n_k})).$$

Taking limit as $k \to \infty$, we have

$$\lim_{k\to\infty} \beta(d(x_{m_k}, x_{n_k})) = 1,$$

since $\beta \in \mathcal{F}$, so

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = 0.$$

Hence $\epsilon = 0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence and converges to some element $x \in A$.

Similarly, in view of the fact that $T(B_0) \subseteq A_0$ and $A_0 \subseteq g(A_0)$, we can conclude that there exists a sequence $\{y_n\}$ such that it converges to some element $y \in B$. Since the pair (S, T) is α-proximal cyclic contraction and g is an isometry, we have for $x_{n+1} \in A, y_{n+1} \in B$,

$$d(gx_{n+1}, Sx_n) = d(A, B), \quad d(gy_{n+1}, Ty_n) = d(A, B).$$

Also S is α-Geraghty proximal contraction of the first kind and $\alpha(x_n, x_{n+1}) \geqslant 1$,

$$\begin{aligned}
\Rightarrow d(x_{n+1}, y_{n+1}) = d(gx_{n+1}, gy_{n+1}) &\leqslant \alpha(x_n, y_n) d(gx_{n+1}, gy_{n+1}) \\
&\leqslant kd(x_n, y_n) + (1-k) d(A, B) \\
&\Rightarrow d(x_{n+1}, y_{n+1}) \leqslant kd(x_n, y_n) + (1-k) d(A, B).
\end{aligned}$$

Letting $n \to \infty$, it follows that

$$d(x, y) = kd(x, y) + (1 - k)d(A, B) \Rightarrow d(x, y) = d(A, B). \tag{3.1}$$

Thus $x \in A_0$ and $y \in B_0$. Since $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$, there exists $u \in A$ and $v \in B$ such that

$$\left.\begin{array}{l} d(u, Sx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{array}\right\}$$

Since S is α-Geraghty proximal contraction of the first kind, we get from $d(u, Sx) = d(A, B)$ and $d(gx_{n+1}, Sx_n) = d(A, B)$ as

$$d(u, gx_{n+1}) \leqslant \alpha(x_n, x)d(u, gx_{n+1}) \leqslant \beta(d(x, x_n))d(x, x_n).$$

Letting $n \to \infty$ in the above inequality,

$$d(u, gx) = 0$$

and so $u = gx$. Therefore, we have

$$d(gx, Sx) = d(A, B). \tag{3.2}$$

Similarly, we have $v = gy$ and so

$$d(gy, Ty) = d(A, B). \tag{3.3}$$

From (3.1), (3.2), and (3.3) we get

$$d(x, y) = d(gx, Sx) = d(gy, Ty) = d(A, B).$$

On the other hand, let $\{u_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \epsilon_n = 0, \quad d(u_{n+1}, z_{n+1}) \leqslant \epsilon_n, \tag{3.4}$$

where $z_{n+1} \in A$ satisfies the condition that

$$d(gz_{n+1}, Su_n) = d(A, B).$$

Since S is α-Geraghty's proximal contraction of the first kind and g is an isometry, then by using $d(gx_{n+1}, Sx_n) = d(A, B)$ and $d(gz_{n+1}, Su_n) = d(A, B)$,

$$d(x_{n+1}, z_{n+1}) = d(gx_{n+1}, gz_{n+1}) \leqslant \alpha(x_n, u_n)d(gx_{n+1}, gz_{n+1}) \leqslant \beta(d(x_n, u_n))d(x_n, u_n).$$

For any $\epsilon > 0$, choose a positive integer N such that $\epsilon_n \leqslant \epsilon$ for all $n \geqslant N$. Observe that

$$d(x_{n+1}, u_{n+1}) \leqslant d(x_{n+1}, z_{n+1}) + d(z_{n+1}, u_{n+1}) \leqslant \beta(d(x_n, u_n))d(x_n, u_n) + \epsilon_n < d(x_n, u_n) + \epsilon,$$

where $\beta(d(x_n, u_n)) \in [0, 1)$. Since $\epsilon > 0$ is arbitrary, we can conclude that for all $n \geqslant N$, the sequence $\{d(x_n, u_n)\}$ is non-increasing and bounded below and hence converges to some nonnegative real number $\acute{r}$. Since the sequence $\{x_n\}$ converges to x, we get

$$\lim_{n\to\infty} d(u_n, x) = \lim_{n\to\infty} d(u_n, x_n) = \acute{r}. \tag{3.5}$$

Suppose that $\acute{r} > 0$. Since

$$d(u_{n+1}, x) \leqslant d(u_{n+1}, x_{n+1}) + d(x_{n+1}, x) \leqslant \beta(d(x_n, u_n))d(x_n, u_n) + \epsilon_n + d(x_{n+1}, x). \tag{3.6}$$

It follows from (3.4), (3.5), and (3.6) that

$$\frac{d(u_{n+1}, x) - \epsilon_n - d(x_{n+1}, x)}{d(x_n, u_n)} \leqslant \beta(d(u_n, x_n)) < 1, \tag{3.7}$$

which implies that $\beta(d(u_n, x_n)) \to 1$ and so $d(u_n, x_n) \to 0$, i.e.,

$$\lim_{n\to\infty} d(u_n, x) = \lim_{n\to\infty} d(u_n, x_n) = 0,$$

which is a contradiction. Thus $\acute{r} = 0$ and hence $\{u_n\}$ is convergent to the point x. $\square$

If g is the identity mapping in Theorem 3.5, we obtain the following best proximity point result.

Theorem 3.6. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}^+$ be a function, and let A, B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $S : A \to B, T : B \to A$ satisfy the following conditions:*

1. *S and T are α-Geraghty proximal contractions of the first kind;*
2. *the pair (S, T) is an α-proximal cyclic contraction;*
3. *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*
4. *S and T are α-proximal admissible maps;*
5. *$\alpha(x_0, x_1) \geqslant 1$ for $x_0, x_1 \in X$.*

Then there exists point $x \in A$ and there exists point $y \in B$ such that

$$d(x, Sx) = d(y, Ty) = d(x, y) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Sx_n) = d(A, B)$$

converges to the element x. For any fixed $x_0 \in A_0$, the sequence $\{y_n\}$ defined by

$$d(y_{n+1}, Ty_n) = d(A, B)$$

converges to the element y. On the other hand, a sequence $\{u_n\}$ in A converges to x if there exists a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n\to\infty} \epsilon_n = 0,$$

$d(u_{n+1}, z_{n+1}) \leqslant \epsilon_n$, where $z_{n+1} \in A$ satisfying the condition that $d(z_{n+1}, Su_n) = d(A, B)$.

If we take $\alpha(x_0, x_1) = 1$ in Theorem 3.5, we obtain the following main result of [18].

Corollary 3.7. *Let (X, d) be a complete metric space and let A, B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $S : A \to B, T : B \to A$ and $g : A \bigcup B \to A \bigcup B$ satisfy the following conditions:*

1. *S and T are Geraghty proximal contractions of the first kind;*
2. *g is an isometry with $A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$;*
3. *the pair (S, T) is an proximal cyclic contraction;*
4. *$S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;*
5. *S and T are proximal admissible maps.*

Then there exists point $x \in A$ and there exists point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

converges to the element x. For any fixed $x_0 \in A_0$, the sequence $\{y_n\}$ defined by

$$d(gy_{n+1}, Ty_n) = d(A, B)$$

converges to the element y. On the other hand, a sequence $\{u_n\}$ in A converges to x if there exists a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n\to\infty} \epsilon_n = 0,$$

$d(u_{n+1}, z_{n+1}) \leqslant \epsilon_n$, where $z_{n+1} \in A$ satisfying the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.

If we take $\alpha(x_0, x_1) = 1$ in Theorem 3.6, we obtain the following.

Corollary 3.8. *Let* (X, d) *be a complete metric space and let A, B be nonempty closed subsets of* X *such that* A_0 *and* B_0 *are nonempty. Let* $S : A \to B$ *and* $T : B \to A$ *satisfy the following conditions:*

1. S *and* T *are Geraghty proximal contraction of the first kind;*
2. *the pair* (S, T) *is a proximal cyclic contraction;*
3. $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$.

Then there exists a unique point $x \in A$ *and* $y \in B$ *such that* $d(x, Sx) = d(y, Ty) = d(x, y) = d(A, B)$.

If we take g as identity map in Corollary 3.7, we have the following corollary.

Corollary 3.9. *Let* (X, d) *be a complete metric space and let* A, B *be nonempty closed subsets of* X *such that* A_0 *and* B_0 *are nonempty. Let* $S : A \to B, T : B \to A$ *satisfy the following conditions:*

1. S *and* T *are Geraghty proximal contractions of the first kind;*
2. *the pair* (S, T) *is an proximal cyclic contraction;*
3. $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;
4. S *and* T *are proximal admissible maps.*

Then there exists point $x \in A$ *and there exists point* $y \in B$ *such that*

$$d(x, Sx) = d(y, Ty) = d(x, y) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$*, the sequence* $\{x_n\}$ *defined by*

$$d(x_{n+1}, Sx_n) = d(A, B)$$

converges to the element x*. For any fixed* $x_0 \in A_0$*, the sequence* $\{y_n\}$ *defined by*

$$d(y_{n+1}, Ty_n) = d(A, B)$$

converges to the element y*. On the other hand, a sequence* $\{u_n\}$ *in* A *converges to* x *if there exists a sequence of positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\to\infty} \epsilon_n = 0,$$

$d(u_{n+1}, z_{n+1}) \leqslant \epsilon_n$*, where* $z_{n+1} \in A$ *satisfying the condition that* $d(z_{n+1}, Su_n) = d(A, B)$.

Remark 3.10. If we take $\beta(t) = k$, where $k \in [0, 1)$, in Theorems 3.5, 3.6, and Corollaries 3.7, 3.8, 3.9 we have the corresponding coincidence best proximity and best proximity point results which are given in [21] for proximal cyclic contraction of first kind.

The following is the best proximity point theorem for non self-mappings which are α-Geraghty's proximal contractions of the first and second kind:

Theorem 3.11. *Let* (X, d) *be a complete metric space,* $\alpha : X \times X \to \mathbb{R}^+$ *be a function and let* A, B *be nonempty closed subsets of* X *such that* A_0 *and* B_0 *are nonempty. Let* $S : A \to B$ *and* $g : A \to A$ *satisfy the following conditions:*

1. S *is* α*-Geraghty proximal contraction of the first and second kind;*
2. g *is an isometry;*
3. S *preserves the isometric distance with respect to* g*;*
4. $S(A_0) \subseteq B_0$;
5. $A_0 \subseteq g(A_0)$;
6. S *is* α*-proximal admissible map.*

Then there exists a point $x \in A$ *such that*
$$d(gx, Sx) = d(A, B).$$
Moreover, for any fixed $x_0 \in A_0$*, the sequence* $\{x_n\}$ *defined by*
$$d(gx_{n+1}, Sx_n) = d(A, B)$$
converges to the element x*. On the other hand, a sequence* $\{u_n\}$ *in A converges to* x *if there exists a sequence of positive numbers* $\{\epsilon_n\}$ *such that*
$$\lim_{n\to\infty} \epsilon_n = 0,$$
$d(u_{n+1}, z_{n+1}) \leqslant \epsilon_n$*, where* $z_{n+1} \in A$ *satisfying the condition that* $d(gz_{n+1}, Su_n) = d(A, B)$.

Proof. Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, as in the proof of Theorem 3.5, we can construct a sequence x_n in A_0 such that
$$d(gx_n, Sx_{n-1}) = d(A, B)$$
for each $n \geqslant 1$. So, inductively, $x_{n+1} \in A_0$ such that
$$d(gx_{n+1}, Sx_n) = d(A, B) \tag{3.8}$$
for each $n \geqslant 1$. Also $\alpha(x_n, x_{n+1}) \geqslant 1$ for $n \in \mathbb{N}$. Since g is an isometry and S is α-Geraghty proximal contraction of the first kind, we have
$$d(x_n, x_{n+1}) = d(gx_n, gx_{n+1}) \leqslant \alpha(x_n, x_{n-1})d(gx_n, gx_{n+1}) \leqslant \beta(d(x_n, x_{n-1}))d(x_n, x_{n-1})$$
for all $n \geqslant 1$. Again, similarly, we can show that the sequence $\{x_n\}$ is a Cauchy sequence and so it converges to some $x \in A$. Since S is α-Geraghty proximal contraction of the second kind and preserves the isometric distance with respect to g,
$$\begin{aligned} d(Sx_n, Sx_{n+1}) = d(Sgx_n, Sgx_{n+1}) &\leqslant \alpha(Sx_n, Sx_{n+1})d(Sgx_n, Sgx_{n+1}) \\ &\leqslant \beta(d(Sx_{n-1}, Sx_n))d(Sx_{n-1}, Sx_n) < d(Sx_{n-1}, Sx_n). \end{aligned}$$
This shows that $\{d(Sx_{n+1}, Sx_n)\}$ is a decreasing sequence and bounded below. Hence there exists $r \geqslant 0$ such that $\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = r$. Suppose that $r > 0$. Observe that
$$\frac{d(Sx_n, Sx_{n+1})}{d(Sx_{n-1}, Sx_n)} \leqslant \beta(d(Sx_{n-1}, Sx_n)).$$
Taking limit as $n \to \infty$,
$$\lim_{n\to\infty} \beta(d(Sx_{n-1}, Sx_n)) = 1.$$
Since $\beta \in \mathcal{F}$, so we get $r = 0$, which is a contradiction to our supposition and hence
$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0. \tag{3.9}$$
Now we claim that $\{Sx_n\}$ is a Cauchy sequence. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and subsequences $\{Sx_{m_k}\}, \{Sx_{n_k}\}$ of $\{Sx_n\}$ such that for any $n_k > m_k \geqslant k$
$$r_k := d(Sx_{m_k}, Sx_{n_k}) \geqslant \epsilon, \quad d(Sx_{m_k}, Sx_{n_k-1}) < \epsilon$$
for any $k \in \{1, 2, 3, \dots\}$. For each $n \geqslant 1$, let $\gamma_n := d(Sx_{n+1}, Sx_n)$. Then we have,
$$\epsilon \leqslant r_k = d(Sx_{m_k}, Sx_{n_k}) \leqslant d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k}) < \epsilon + \gamma_{n_k-1} \tag{3.10}$$

and by taking limit $k \to \infty$, we get as

$$\epsilon \leqslant \lim_{k\to\infty} r_k < \epsilon + \lim_{k\to\infty} \gamma_{n_k-1} < \epsilon + 0.$$

Thus

$$\lim_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) = \epsilon.$$

So, it follows from (3.9) and (3.10) that $\lim\limits_{k\to\infty} r_k = \epsilon$. Notice also that

$$\begin{aligned}
\epsilon \leqslant r_k = d(Sx_{m_k}, Sx_{n_k}) &\leqslant d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) + d(Sx_{m_k+1}, Sx_{n_k+1}) \\
&= \gamma_{m_k} + \gamma_{n_k} + d(Sx_{m_k+1}, Sx_{n_k+1}) \\
&= \gamma_{m_k} + \gamma_{n_k} + d(Sgx_{m_k+1}, Sgx_{n_k+1}) \\
&\leqslant \gamma_{m_k} + \gamma_{n_k} + \alpha(Sx_{m_k}, Sx_{n_k}) d(Sgx_{m_k+1}, Sgx_{n_k+1}) \\
&\leqslant \gamma_{m_k} + \gamma_{n_k} + \alpha(x_{m_k}, x_{n_k}) d(Sx_{m_k}, Sx_{n_k}) \\
&\leqslant \gamma_{m_k} + \gamma_{n_k} + \beta(d(Sx_{m_k}, Sx_{n_k})) d(Sx_{m_k}, Sx_{n_k}) \\
&\Rightarrow \frac{d(Sx_{m_k}, Sx_{n_k}) - \gamma_{m_k} - \gamma_{n_k}}{d(Sx_{m_k}, Sx_{n_k})} \leqslant \beta(d(Sx_{m_k}, Sx_{n_k})).
\end{aligned}$$

Taking limit as $k \to \infty$,

$$\lim_{k\to\infty} \beta(d(Sx_{m_k}, Sx_{n_k})) = 1,$$

since $\beta \in \mathcal{F}$, so

$$\lim_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) = 0.$$

Hence $\epsilon = 0$, which is a contradiction. So $\{Sx_n\}$ is a Cauchy sequence and converges to some element $y \in B$. Therefore, we can conclude that $d(gx, y) = \lim\limits_{n\to\infty} d(gx_{n+1,Sx_n}) = d(A, B)$, which implies that $gx \in A_0$. Since $A_0 \subseteq g(A_0)$, we have $gx = gz$ for some $z \in_0 A$ and then $d(gx, gz) = 0$. By the fact that g is an isometry, we have $d(x, z) = d(gx, gz) = 0$. Hence $x = z$ and so $x \in A_0$. Since $S(A_0) \subseteq B_0$, there exists $u \in A$ such that

$$d(u, Sx) = d(A, B). \tag{3.11}$$

Since S is α-Geraghty proximal contraction of the first kind, it follows from (3.8) and (3.11) that

$$d(u, gx_{n+1}) \leqslant \alpha(x, x_n) d(u, gx_{n+1}) \leqslant \beta(d(x, x_n)) d(x, x_n)$$

for $n \geqslant 1$. Taking $n \to \infty$ in last inequality, it follows that the sequence $\{gx_n\}$ converges to a point u. Since g is continuous and $\lim_{n\to\infty} x_n = x$, we have $gx_n \to gx$ as $n \to \infty$. By the uniqueness of limit, we conclude that $u = gx$. Therefore, it follows that $d(gx, Sx) = d(u, Sx) = d(A, B)$. □

If g is identity mapping in Theorem 3.11, we obtain the following.

Theorem 3.12. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}^+$ be a function and let A, B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $S : A \to B$ satisfy the following conditions:*

1. *S is α-Geraghty proximal contraction of the first and second kind;*
2. *S preserves the isometric distance;*
3. $S(A_0) \subseteq B_0$;
4. *S is α-proximal admissible map.*

Then there exists a point $x \in A$ such that

$$d(x, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Sx_n) = d(A, B)$$

converges to the element x. On the other hand, a sequence $\{u_n\}$ in A converges to x if there exists a sequence of *positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\to\infty} \epsilon_n = 0,$$

$d(u_{n+1}, z_{n+1}) \leqslant \epsilon_n$, *where* $z_{n+1} \in A$ *satisfying the condition that* $d(z_{n+1}, Su_n) = d(A, B)$.

If $\alpha(x_0, x_1) = 1$ in Theorem 3.11, we obtain the following main result of [18].

Corollary 3.13. *Let* (X, d) *be a complete metric space and let* A, B *be nonempty closed subsets of* X *such that* A_0 *and* B_0 *are nonempty. Let* $S : A \to B$ *and* $g : A \to A$ *satisfy the following conditions:*

1. *S is Geraghty proximal contraction of the first and second kind;*
2. g *is an isometry;*
3. S *preserves the isometric distance with respect to* g;
4. $S(A_0) \subseteq B_0$;
5. $A_0 \subseteq g(A_0)$.

Then there exists a point $x \in A$ *such that*

$$d(gx, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, *the sequence* $\{x_n\}$ *defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B)$$

converges to the element x. *On the other hand, a sequence* $\{u_n\}$ *in A converges to* x *if there exists a sequence of positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\to\infty} \epsilon_n = 0,$$

$d(u_{n+1}, z_{n+1}) \leqslant \epsilon_n$, *where* $z_{n+1} \in A$ *satisfying the condition that* $d(gz_{n+1}, Su_n) = d(A, B)$.

If $\alpha(x_0, x_1) = 1$ in Theorem 3.12, we have the following corollary.

Corollary 3.14. *Let* (X, d) *be a complete metric space and let* A, B *be nonempty closed subsets of* X *such that* A_0 *and* B_0 *are nonempty. Let* $S : A \to B$ *satisfies the following conditions:*

1. S *is proximal contraction of the first and second kind;*
2. $S(A_0) \subseteq B_0$.

Then there exists a unique point $x \in A$ *such that*

$$d(x, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, *the sequence* $\{x_n\}$ *defined by*

$$d(x_{n+1}, Sx_n) = d(A, B)$$

converges to best proximity point x *of* S.

If g is identity map in Corollary 3.13, we obtain the following corollary.

Corollary 3.15. *Let* (X, d) *be a complete metric space and let* A, B *be nonempty closed subsets of* X *such that* A_0 *and* B_0 *are nonempty. Let* $S : A \to B$ *satisfies the following conditions:*

1. S *is Geraghty proximal contraction of the first and second kind;*
2. S *preserves the isometric distance;*
3. $S(A_0) \subseteq B_0$.

Then there exists a point $x \in A$ *such that*

$$d(x, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$*, the sequence* $\{x_n\}$ *defined by*

$$d(x_{n+1}, Sx_n) = d(A, B)$$

converges to the element x*. On the other hand, a sequence* $\{u_n\}$ *in A converges to* x *if there exists a sequence of positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\to\infty} \epsilon_n = 0,$$

$d(u_{n+1}, z_{n+1}) \leqslant \epsilon_n$*, where* $z_{n+1} \in A$ *satisfying the condition that* $d(z_{n+1}, Su_n) = d(A, B)$*.*

Remark 3.16. If we take $\beta(t) = k$, where $k \in [0, 1)$, in Theorems 3.11, 3.12, and Corollaries 3.13, 3.14, 3.15 we have the corresponding coincidence best proximity and best proximity point results which are given in [21] for proximal cyclic contraction of first kind.

Example 3.17. Consider the complete metric space $\mathbb{R}^2$ with metric defined by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Let

$$A = \{(0, x) : x \in \mathbb{R}\}, \quad B = \{(2, y) : y \in \mathbb{R}\}.$$

Define the mappings $S : A \to B, T : B \to A$ and $g : A \cup B \to A \cup B$ as follows:

$$S((0, x)) = (2, \frac{|x|}{2(1 + |x|)}), \quad T((2, y)) = (0, \frac{|y|}{2(1 + |y|)}), \quad g((x, y)) = (x, -y).$$

Then $d(A, B) = 2, A_0 = A, B_0 = B$, and mapping g is an isometry. Next, we show that S and T are α-Geraghty proximal contractions of the first kind with $\beta \in \mathbb{F}$ defined by

$$\beta(t) = \frac{1}{1 + t}$$

for all $t \geqslant 0$. Let $\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ be defined as follows:

$$\alpha((x_1, y_1), (x_2, y_2)) = \begin{cases} 1, & \text{if } 0 \leqslant x_1, x_2 \leqslant 2 \text{ and } y_1 \neq y_2 \geqslant 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $(0, x_1), (0, x_2), (0, a_1), (0, a_2) \in A$ satisfying

$$d((0, a_1), S(0, x_1)) = d(A, B) = 2, \quad d((0, a_2), S(0, x_2)) = d(A, B) = 2.$$

Then we have

$$a^i = \frac{|x_i|}{2(1 + |x_i|)}$$

for $i = 1, 2$. If $x_1 = x_2$, we have done. Assume that $x_1 \neq x_2$, then, by Proposition 2.12, we have

$$\begin{aligned} \alpha((0, x_1), (0, x_2))d((0, a_1), (0, a_2)) &= d((0, \frac{|x_1|}{2(1 + x_1)}), (0, \frac{|x_2|}{2(1 + |x_2|)})) \\ &= |\frac{|x_1|}{2(1 + x_1)} - \frac{|x_2|}{2(1 + x_2)}| \end{aligned}$$

$$
\begin{aligned}
&= \left|\frac{|x_1| - |x_2|}{2(1+x_1)(1+|x_2|)}\right| \\
&\leqslant \left|\frac{x_1 - x_2}{(1+|x_1|)(1+|x_2|)}\right| \\
&\leqslant \frac{1}{1+|x_1-x_2|}|x_1 - x_2| \\
&= \beta(d((0,x_1),(0,x_2)))d((0,x_1),(0,x_2)).
\end{aligned}
$$

Thus S is α-Geraghty proximal contraction of the first kind. Similarly we can show that T is α-Geraghty proximal contraction of the first kind.

Next, we show that the pair (S,T) is α proximal cyclic contraction. Let $(0,u),(0,x) \in A$ and $(2,v),(2,y) \in B$ be such that

$$d((0,u),S(0,x)) = d(A,B) = 2, \quad d((2,v),T(2,y)) = d(A,B) = 2.$$

Then we get

$$u = \frac{|x|}{2(1+|x|)}, \quad v = \frac{|y|}{2(1+|y|)}.$$

The case $x = y$ is clear. Suppose that $x \neq y$, then we have

$$
\begin{aligned}
\alpha((0,x),(2,y))d((0,u),(2,v)) &= |u - v| + 2 \\
&= \left|\frac{|x|}{2(1+|x|)} - \frac{|y|}{2(1+|y|)}\right| + 2 \\
&= \left|\frac{|x| - |y|}{2(1+|x|)(1+|y|)}\right| + 2 \\
&\leqslant \frac{|x-y|}{2(1+|x|)(1+|y|)} + 2 \\
&\leqslant \frac{1}{2}|x-y| + 2 \\
&\leqslant k(|x-y|) + (1-k)2 \\
&= kd((0,x),(2,y)) + (1-k)d(A,B),
\end{aligned}
$$

where $k = [\frac{1}{2}, 1)$. Hence the pair (S,T) is a α-proximal cyclic contraction. Therefore, all hypothesis of Theorem 3.5 are satisfied. Thus, $(0,0) \in A$ and $(2,0) \in B$ are elements such that

$$d(g(0,0),S(0,0)) = d(g(2,0),T(2,0)) = d((0,0),(2,0)) = d(A,B).$$

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