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A note on Furuta type operator equation

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Abstract

In this paper, we will show the existence of positive semidefinite solution of Furuta type operator equation

$$\sum_{j=0}^{n-1} A^j X A^{n-j-1} = Y,$$

where Y can be expressed by a comprehensive form.

Keywords: Furuta type operator equation, generalized Furuta inequality, positive definite operator and positive semidefinite operator.

2010 MSC: 47A62, 47A63.

1. Introduction and main result

A capital letter, such as T, stands for an operator on a Hilbert space $\mathscr{H}$.

In 2010, T. Furuta investigated operator equation $\sum_{j=0}^{n-1} A^j X A^{n-j-1} = Y$ and obtained the following result.

Theorem 1.1 ([2]). *Let m and n be natural numbers. If A and B are a positive definite operator and a positive semidefinite operator, respectively, then there exists positive semidefinite operator solution X satisfying the following operator equation:*

$$\sum_{j=0}^{n-1} A^j X A^{n-j-1} = A^{\frac{nr}{2(m+r)}} \left(\sum_{i=1}^{m} A^{\frac{n(m-i)}{m+r}} B A^{\frac{n(i-1)}{m+r}} \right) A^{\frac{nr}{2(m+r)}}$$

for r such that $\begin{cases} r \geqslant 0, & \text{if } n \geqslant m; \\ r \geqslant \frac{m-n}{n-1}, & \text{if } m \geqslant n \geqslant 2. \end{cases}$

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In 2014, we extends Furuta's result as follows.

Theorem 1.2 ([3]). *Let* m, n *and* k *be positive integers. If* A *and* B *are a positive definite operator and a positive semidefinite operator, respectively, then for each* $t \in [0,1]$, *there exists positive semidefinite operator solution* X *which satisfies the following operator equation:*

$$\sum_{j=0}^{n-1} A^j X A^{n-j-1} = A^{\frac{nr}{2[(m-t)k+r]}} \left(\sum_{i=1}^{k} \sum_{j=1}^{m} A^{\frac{n[2(m-t)(k-i)-t+2(m-j)]}{2[(m-t)k+r]}} B A^{\frac{n[2(j-1)-t+2(m-t)(i-1)]}{2[(m-t)k+r]}} \right) A^{\frac{nr}{2[(m-t)k+r]}}$$

for r *such that* $\begin{cases} r \geqslant t, & \text{if } (1-t)n \geqslant (m-t)k\,; \\ r \geqslant \max\{\frac{(m-t)k-(1-t)n}{n-1}, t\}, & \text{if } (m-t)k \geqslant (1-t)n \text{ with } n \geqslant 2\,. \end{cases}$

As a continuation, in this short note, we extend Theorem 1.2 as follows.

Theorem 1.3. *Let* $k_1, k_2, k_3, k_4, j, j_1, j_2, j_3, j_4$ *be nonnegative integers. If* A *and* B *are a positive definite operator and a positive semidefinite operator, respectively, then for* $t \in [0,1]$, *there exist a positive semidefinite solution* X *satisfying*

$$\sum_{j=0}^{n-1} A^j X A^{n-j-1} = A^{\frac{nr}{2\delta}} \left(\sum_{j_4=0}^{k_4-1} H^{j_4} \widetilde{H} H^{k_4-j_4-1} \right) A^{\frac{nr}{2\delta}},$$

where

$$H = A^{\frac{\{[(k_1-t)k_2+t]k_3-t\}n}{\delta}}, \qquad \widetilde{H} = A^{-\frac{nt}{2\delta}} \Big(\sum_{j_3=0}^{k_3-1} K^{j_3} \widetilde{K} K^{k_3-j_3-1} \Big) A^{-\frac{nt}{2\delta}},$$

$$K = A^{\frac{[(k_1-t)k_2+t]n}{\delta}}, \qquad \widetilde{K} = A^{\frac{nt}{2\delta}} \Big(\sum_{j_2=0}^{k_2-1} L^{j_2} \widetilde{L} L^{k_2-j_2-1} \Big) A^{\frac{nt}{2\delta}},$$

$$L = A^{\frac{(k_1-t)n}{\delta}}, \qquad \widetilde{L} = A^{-\frac{nt}{2\delta}} \Big(\sum_{j_1=0}^{k_1-1} A^{\frac{nj_1}{\delta}} B A^{\frac{n(k_1-j_1-1)}{\delta}} \Big) A^{-\frac{nt}{2\delta}},$$

$$\delta = \{[(k_1-t)k_2+t]k_3-t\}k_4 + r,$$

r *is a positive number such that*

$$\begin{cases} r \geqslant t, & \text{if } (1-t)n \geqslant \{[(k_1-t)k_2+t]k_3-t\}k_4\,; \\ r \geqslant \max\{\frac{\{[(k_1-t)k_2+t]k_3-t\}k_4-(1-t)n}{n-1}, t\}, & \text{if } \{[(k_1-t)k_2+t]k_3-t\}k_4 \geqslant (1-t)n \text{ with } n \geqslant 2\,. \end{cases}$$

In order to prove the main result above, we list a useful lemma first.

Lemma 1.4 ([1, Generalized Furuta inequality]). *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, $p_1, p_2, p_3, p_4 \geqslant 1$, *then*

$$A^{1-t+r} \geqslant \left\{ A^{\frac{r}{2}} \left[A^{-\frac{t}{2}} \{ A^{\frac{t}{2}} (A^{-\frac{t}{2}} B^{p_1} A^{-\frac{t}{2}})^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{-\frac{t}{2}} \right]^{p_4} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{\{[(p_1-t)p_2+t]p_3-t\}p_4+r}}$$

holds for $t \in [0,1]$ *and* $r \geqslant t$.

2. Proof of the main result

In this section, we prove Theorem 1.3, which is the main result. We use the same method as in [2] and [3].

Proof of Theorem 1.3. For $A + xB \geqslant A > 0$, $x > 0$, $A^{-1} \geqslant (A+xB)^{-1} > 0$.

Replacing A by A^{-1}, B by $(A+xB)^{-1}$ in generalized Furuta inequality, then

$$A^{-(1-t+r)} \geqslant \left\{ A^{-\frac{r}{2}} \left[A^{\frac{t}{2}} \{ A^{-\frac{t}{2}} (A^{\frac{t}{2}} (A+xB)^{-p_1} A^{\frac{t}{2}})^{p_2} A^{-\frac{t}{2}} \}^{p_3} A^{\frac{t}{2}} \right]^{p_4} A^{-\frac{r}{2}} \right\}^{\frac{1-t+r}{\{[(p_1-t)p_2+t]p_3-t\}p_4+r}}. \tag{2.1}$$

Let $p_1 = k_1$, $p_2 = k_2$, $p_3 = k_3$, $p_4 = k_4$ in (2.1), take reverse and apply Löwner-Heinz inequality for $\alpha \in [0,1]$, we have

$$\left\{A^{\frac{r}{2}}\left[A^{-\frac{t}{2}}\{A^{\frac{t}{2}}(A^{-\frac{t}{2}}(A+xB)^{k_1}A^{-\frac{t}{2}})^{k_2}A^{\frac{t}{2}}\}^{k_3}A^{-\frac{t}{2}}\right]^{k_4}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{\delta}\alpha} \geqslant A^{(1-t+r)\alpha}, \tag{2.2}$$

where $\delta = \{[(k_1-t)k_2+t]k_3-t\}k_4+r$.

Let $\frac{\delta}{(1-t+r)\alpha}$ be some a positive integer n, i.e., $\frac{\delta}{(1-t+r)\alpha} = n$. Because $\alpha = \frac{\delta}{(1-t+r)n} \in [0,1]$, then $r \geqslant \frac{\{[(k_1-t)k_2+t]k_3-t\}k_4-(1-t)n}{n-1}$ if $\{[(k_1-t)k_2+t]k_3-t\}k_4 \geqslant (1-t)n$.

Put $F(x) = \left\{A^{\frac{r}{2}}\left[A^{-\frac{t}{2}}\{A^{\frac{t}{2}}(A^{-\frac{t}{2}}(A+xB)^{k_1}A^{-\frac{t}{2}})^{k_2}A^{\frac{t}{2}}\}^{k_3}A^{-\frac{t}{2}}\right]^{k_4}A^{\frac{r}{2}}\right\}^{\frac{1}{n}}$. Together with (2.2) we can obtain that

$$F(x) \geqslant F(0) = A^{(1-t+r)\alpha} = A^{\frac{\delta}{n}}$$

holds for any $x \geqslant 0$. Thus $F'(x)\Big|_{x=0} \geqslant 0$.

Differentiate $F^n(x) = A^{\frac{r}{2}}\left[A^{-\frac{t}{2}}\{A^{\frac{t}{2}}(A^{-\frac{t}{2}}(A+xB)^{k_1}A^{-\frac{t}{2}})^{k_2}A^{\frac{t}{2}}\}^{k_3}A^{-\frac{t}{2}}\right]^{k_4}A^{\frac{r}{2}}$, and take $x=0$, we have

$$\begin{aligned}\frac{d}{dx}[F^n(x)]\Big|_{x=0} &= \frac{d}{dx}\left\{A^{\frac{r}{2}}\left[A^{-\frac{t}{2}}\{A^{\frac{t}{2}}(A^{-\frac{t}{2}}(A+xB)^{k_1}A^{-\frac{t}{2}})^{k_2}A^{\frac{t}{2}}\}^{k_3}A^{-\frac{t}{2}}\right]^{k_4}A^{\frac{r}{2}}\right\}\Big|_{x=0}\\ &= A^{\frac{r}{2}}\left\{\frac{d}{dx}\left[A^{-\frac{t}{2}}\{A^{\frac{t}{2}}(A^{-\frac{t}{2}}(A+xB)^{k_1}A^{-\frac{t}{2}})^{k_2}A^{\frac{t}{2}}\}^{k_3}A^{-\frac{t}{2}}\right]^{k_4}\Big|_{x=0}\right\}A^{\frac{r}{2}}\\ &= A^{\frac{r}{2}}\left\{\sum_{j_4=0}^{k_4-1}H^{j_4}(x)H'(x)H^{k_4-j_4-1}(x)\Big|_{x=0}\right\}A^{\frac{r}{2}},\end{aligned} \tag{2.3}$$

where

$$H(x) = A^{-\frac{t}{2}}\{A^{\frac{t}{2}}(A^{-\frac{t}{2}}(A+xB)^{k_1}A^{-\frac{t}{2}})^{k_2}A^{\frac{t}{2}}\}^{k_3}A^{-\frac{t}{2}}. \tag{2.4}$$

It is easy to obtain that

$$H(0) = A^{[(k_1-t)k_2+t]k_3-t} \tag{2.5}$$

and

$$\begin{aligned}H'(0) &= \frac{d}{dx}[H(x)]\Big|_{x=0}\\ &= A^{-\frac{t}{2}}\left\{\sum_{j_3=0}^{k_3-1}K^{j_3}(x)K'(x)K^{k_3-j_3-1}(x)\Big|_{x=0}\right\}A^{-\frac{t}{2}},\end{aligned} \tag{2.6}$$

where

$$K(x) = A^{\frac{t}{2}}(A^{-\frac{t}{2}}(A+xB)^{k_1}A^{-\frac{t}{2}})^{k_2}A^{\frac{t}{2}}. \tag{2.7}$$

It is easy to show that

$$K(0) = A^{(k_1-t)k_2+t} \tag{2.8}$$

and

$$\begin{aligned}K'(0) &= \frac{d}{dx}[K(x)]\Big|_{x=0}\\ &= A^{\frac{t}{2}}\left\{\sum_{j_2=0}^{k_2-1}L^{j_2}(x)L'(x)L^{k_2-j_2-1}(x)\Big|_{x=0}\right\}A^{\frac{t}{2}},\end{aligned} \tag{2.9}$$

where

$$L(x) = A^{-\frac{t}{2}}(A + xB)^{k_1}A^{-\frac{t}{2}}. \tag{2.10}$$

Similarly,

$$L(0) = A^{k_1 - t} \tag{2.11}$$

and

$$\begin{aligned} L'(0) =& \frac{d}{dx}[L(x)]\Big|_{x=0} \\ =& A^{-\frac{t}{2}}\Big\{\sum_{j_1=0}^{k_1-1}(A+xB)^{j_1}(A+xB)'(A+xB)^{k_1-j_1-1}\Big|_{x=0}\Big\}A^{-\frac{t}{2}} \\ =& A^{-\frac{t}{2}}\Big\{\sum_{j_1=0}^{k_1-1}A^{j_1}BA^{k_1-j_1-1}\Big\}A^{-\frac{t}{2}}. \end{aligned} \tag{2.12}$$

Notice that

$$\frac{d}{dx}[F^n(x)]\Big|_{x=0} = \sum_{j=0}^{n-1}F^j(x)F'(x)F^{n-j-1}(x)\Big|_{x=0} = \sum_{j=0}^{n-1}F^j(0)F'(0)F^{n-j-1}(0) \tag{2.13}$$

and $F(0) = A^{\frac{\delta}{n}}$.

Let $X = F'(0)$, therefore,

$$\sum_{j=0}^{n-1}A^{\frac{\delta j}{n}}XA^{\frac{\delta(n-j-1)}{n}} = \frac{d}{dx}[F^n(x)]\Big|_{x=0}. \tag{2.14}$$

Replacing A by $A^{\frac{n}{\delta}}$ in (2.3)-(2.14), and letting $H = H(0)$, $\widetilde{H} = H'(0)$, $K = K(0)$, $\widetilde{K} = K'(0)$, $L = L(0)$, $\widetilde{L} = L'(0)$, then we finish the proof.

□

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