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Some new Hermite-Hadamard type inequalities for h-convex functions via quantum integral on finite intervals

Limin Yang, Ruiyun Yang*

Basic Department, Henan Vocational College of Water Conservancy and Environment, Zhengzhou 450008, Henan, P. R. China.

Abstract

In this paper, we establish some new Hermite-Hadamard type inequalities for h-convex functions via quantum integral on finite intervals. The results presented here would provide extensions and corrections of those given in earlier works.

Keywords: Hermite-Hadamard type inequalities, h-convex functions, integral inequalities, quantum calculus.

2010 MSC: 26D10, 34A08, 26D15.

1. Introduction

The function $f : [a, b] \to \mathbb{R}$ is said to be convex if the following inequality holds

$$f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha)f(y)$$

for all $x, y \in [a, b]$ and $\alpha \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

Let $f : I \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

is known in the literature as Hermite-Hadamard's inequality for convex mapping, see [7, 14]. Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard's inequality (1.1) may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Since Hermite-Hadamard's inequality for convex functions has been considered the most useful inequality in mathematical analysis, it has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found, see, for example, [1, 2, 4–6, 9, 11, 19, 21] and the references cited therein. The author [22] obtained some Hermite-Hadamard type inequalities for (p_1, h_1)-(p_2, h_2)-convex function on the co-ordinates.

In [8], Fejér gave a weighted generalization of the inequalities (1.1) as the following.

*Corresponding author
Email addresses: liminyang777@126.com (Limin Yang), zfysoft@126.com (Ruiyun Yang)

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Theorem 1.1. *Let* $f : I \to \mathbb{R}$ *be a convex mapping and* $a, b \in I$ *with* $a < b$. *Then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leqslant \int_a^b f(x)w(x)dx \leqslant \frac{f(a)+f(b)}{2}\int_a^b w(x)dx,$$

where $w : [a, b] \to \mathbb{R}$ *is nonnegative, integrable, and symmetric about* $x = (a+b)/2$ (*i.e.*, $w(x) = w(a+b-x)$).

In [20], Varosanec introduced the class of h-convex functions defined as follows.

Definition 1.2. Let $h : [0, 1] \to \mathbb{R}$ be a positive function. We say that $f : I \to \mathbb{R}$ is h-convex function, or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $\alpha \in [0, 1]$ we have

$$f(\alpha x + (1-\alpha)y) \leqslant h(\alpha)f(x) + h(1-\alpha)f(y). \tag{1.2}$$

If inequality (1.2) is reversed, then f is said to be h-concave, i.e., $f \in SV(h, I)$.

In [15], Sarikaya et al. obtained the following inequality Hermite-Hadamard type for h-convex function.

Theorem 1.3. *Let* $f \in SX(h, I)$, $a, b \in I$ *with* $a < b$ *and* $f \in L_1([a, b])$. *Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant [f(a)+f(b)]\int_0^1 h(\alpha)d\alpha.$$

Recently, Tariboon and Ntouyas introduced the quantum calculus on finite intervals in the paper [17]. In [13], Noor et al. applied quantum analogue of classical integral identity to establish some quantum estimates for Hermite-Hadamard inequalities for q-differentiable convex functions and q-differentiable quasi convex functions. Chen and Yang [3] and Liu and Yang [12] obtained some new Chebyshev and Grüss type inequalities via quantum integral on finite intervals, respectively. In [23], some inequalities of Fejér type for twice differentiable mappings were established via quantum calculus on finite intervals. In [18], Tariboon and Ntouyas extended the Hölder, Hermite-Hadamard, trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss, and Grüss-Čebyšev integral inequalities to quantum calculus on finite intervals. In [16], Sudsutad et al. obtained some new Hermite-Hadamard type quantum integral inequalities for convex functions. However, it is easy to see that the Hermite-Hadamard type quantum integral inequality given by Tariboon and Ntouyas [18] is incorrect. Relaying on Hermite-Hadamard type quantum integral inequality given by Tariboon and Ntouyas [18], Sudsutad et al. gave the inequality (ii) of Theorem 4.3 and Theorem 4.4 in [16]. So the inequality (ii) of Theorem 4.3 and Theorem 4.4 in [16] are also mistaken. We will correct them in Remark 3.4 of Section 3. Motivated by the results mentioned above, the main aim of this paper is to establish some new Hermite-Hadamard type inequalities for h-convex functions via quantum integral on finite intervals. The results presented here would provide extensions of those given in earlier works when $q^- \to 1$.

2. Preliminaries

Let $I = [a, b] \subset \mathbb{R}$, $I^0 = (a, b)$ and $0 < q < 1$ be a constant. We give the definition q-derivative of a function $f : I \to \mathbb{R}$ at a point $x \in I$ as follows.

Definition 2.1 ([17]). Assume $f : I \to \mathbb{R}$ is a continuous function and let $x \in I$. Then the expression

$${}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a, \quad {}_aD_q f(a) = \lim_{x \to a} {}_aD_q f(x), \tag{2.1}$$

is called the q-derivative on I of function f at x.

We say that f is q-differentiable on I provided ${}_aD_q f(x)$ exists for all $x \in I$. Note that if $a = 0$ in (2.1), then ${}_0D_q f = D_q f$, where D_q is the well-known q-derivative of the function $f(x)$ defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

For more details, see [10].

Definition 2.2 ([17]). Assume $f : I \to \mathbb{R}$ is a continuous function. Then the q-integral on I is defined by

$$I_q^a f(x) = \int_a^x f(t)_a d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \tag{2.2}$$

for $x \in I$. Moreover, if $c \in (a, x)$ then the definite q-integral on I is defined by

$$\begin{aligned}\int_c^x f(t)_a d_q t &= \int_a^x f(t)_a d_q t - \int_a^c f(t)_a d_q t \\ &= (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - (1-q)(c-a) \sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a).\end{aligned}$$

Note that if $a = 0$, then (2.2) reduces to the classical q-integral of a function $f(x)$ defined by (see [10])

$$\int_0^x f(t)_0 d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x), \quad \forall x \in [0, \infty).$$

Lemma 2.3 ([3]). *Assume $f, g : I \to \mathbb{R}$ are two continuous functions and $f(t) \leqslant g(t)$ for all $t \in I$. Then*

$$\int_a^x f(t)_a d_q t \leqslant \int_a^x g(t)_a d_q t.$$

Theorem 2.4 ([17]). *Let $f : I \to \mathbb{R}$ be a continuous function. Then we have*

(a) ${}_aD_q \int_a^x f(t)_a d_q t = f(x)$;

(b) $\int_c^x {}_aD_q f(t)_a d_q t = f(x) - f(c)$, *for* $c \in (a, x)$.

Theorem 2.5 ([17]). *Let $f, g : I \to \mathbb{R}$ be two continuous functions and $\alpha \in \mathbb{R}$. Then, for $x \in I$, we have*

(a) $\int_a^x [f(t) + g(t)]_a d_q t = \int_a^x f(t)_a d_q t + \int_a^x g(t)_a d_q t$;

(b) $\int_a^x (\alpha f)(t)_a d_q t = \alpha \int_a^x f(t)_a d_q t$;

(c) $\int_c^x f(t)_a D_q g(t)_a d_q t = [(fg)(t)]_c^x - \int_c^x g(qt + (1-q)a)_a D_q f(t)_a d_q t$, *for* $c \in (a, x)$.

3. Hermite-Hadamard type inequalities via quantum integral on finite intervals

In this section, we will give some Hermite-Hadamard type inequalities via quantum integral on finite intervals.

Theorem 3.1. *Let $f \in SX(h, I)$ and $a, b \in I$ with $a < b$. Then the following inequalities hold:*

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_a^b w(x)_a d_q x &\leqslant \int_a^b [f(x) + f(a+b-x)] w(x)_a d_q x \\ &\leqslant [f(a) + f(b)] \int_a^b \left[h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right)\right] w(x)_a d_q x.\end{aligned} \tag{3.1}$$

Proof. The function $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric, then

$$\begin{aligned}\int_0^1 f((1-t)a+tb)w(ta+(1-t)b)_0 d_q t &= \int_0^1 f((1-t)a+tb)w((1-t)a+tb)_0 d_q t\\ &= \frac{1}{b-a}\left((1-q)(b-a)\sum_{n=0}^{\infty} q^n f((1-q^n)a+q^n b)w((1-q^n)a+q^n b)\right)\\ &= \frac{1}{b-a}\int_a^b f(x)w(x)_a d_q x,\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\int_0^1 f(ta+(1-t)b)w(ta+(1-t)b)_0 d_q t &= (1-q)\sum_{n=0}^{\infty} q^n f(q^n a+(1-q^n)b)w(q^n a+(1-q^n)b)\\ &= \frac{1}{b-a}\left((1-q)(b-a)\sum_{n=0}^{\infty} q^n f(a+b-((1-q^n)a+q^n b))w(a+b-((1-q^n)a+q^n b))\right)\\ &= \frac{1}{b-a}\int_a^b f(a+b-x)w(a+b-x)_a d_q x = \frac{1}{b-a}\int_a^b f(a+b-x)w(x)_a d_q x.\end{aligned} \tag{3.3}$$

According to (1.2) with $x = ta+(1-t)b$, $y = (1-t)a+tb$ and $\alpha = 1/2$, we find that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant h\left(\frac{1}{2}\right)f(ta+(1-t)b) + h\left(\frac{1}{2}\right)f((1-t)a+tb)\\ &= h\left(\frac{1}{2}\right)\big(f(ta+(1-t)b)+f((1-t)a+tb)\big).\end{aligned} \tag{3.4}$$

Multiplying both sides of (3.4) by $w(ta+(1-t)b)$, and integrating the resulting inequality obtained with respect to t from 0 to 1, by (3.2) and (3.3), we have

$$f\left(\frac{a+b}{2}\right)\left(\frac{1}{b-a}\int_a^b w(x)_a d_q x\right) \leqslant h\left(\frac{1}{2}\right)\left(\frac{1}{b-a}\int_a^b [f(x)+f(a+b-x)]w(x)_a d_q x\right), \tag{3.5}$$

and the first inequality is proved. The proof of the second inequality follows by using (1.2) with $x = a$ and $y = b$. Then we have

$$f(\alpha a+(1-\alpha)b) \leqslant h(\alpha)f(a)+h(1-\alpha)f(b). \tag{3.6}$$

Multiplying both sides of (3.6) by $w(\alpha a+(1-\alpha)b)$ and $w((1-\alpha)a+\alpha b)$, integrating with respect to α over $[0, 1]$, respectively, and adding the obtained results, then we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b [f(x)+f(a+b-x)]w(x)_a d_q x &\leqslant [f(a)+f(b)]\\ &\quad\times\left(\frac{1}{b-a}\int_a^b \left[h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right]w(x)_a d_q x\right).\end{aligned} \tag{3.7}$$

We obtain inequality (3.1) from (3.5) and (3.7). The proof is complete. □

It is easy to see that the following two corollaries hold.

Corollary 3.2. *Let $f \in SX(h, I)$, $a, b \in I$ with $a < b$. Then following inequality holds*

$$\frac{1}{h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b [f(x)+f(a+b-x)]_a d_q x \leqslant [f(a)+f(b)]\int_0^1 [h(x)+h(1-x)]_0 d_q x.$$

Corollary 3.3. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a convex mapping and* $a, b \in I$ *with* $a < b$. *Then following inequality holds*

$$2f\left(\frac{a+b}{2}\right)\int_a^b w(x)_a d_q x \leqslant \int_a^b [f(x)+f(a+b-x)]w(x)_a d_q x \leqslant [f(a)+f(b)]\int_a^b w(x)_a d_q x.$$

Specially,

$$2f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b [f(x)+f(a+b-x)]_a d_q x \leqslant [f(a)+f(b)].$$

Remark 3.4. We should note that in the proof of [18, Theorem 3.2], the authors used the following equalities

$$\int_0^1 f((1-t)a+tb)_0 d_q t = \frac{1}{b-a}\int_a^b f(x)_a d_q x \quad \text{and} \quad \int_0^1 f(ta+(1-t)b)_0 d_q t = \frac{1}{b-a}\int_a^b f(x)_a d_q x.$$

In fact, the second equality does not hold always. From (3.3), when $w(x) = 1$, we have

$$\int_0^1 f(ta+(1-t)b)_0 d_q t = \frac{1}{b-a}\int_a^b f(a+b-x)_a d_q x.$$

For example, $f(x) = x$, then we get

$$\int_0^1 f((1-t)a+tb)_0 d_q t = \frac{b+aq}{1+q} \quad \text{and} \quad \int_0^1 f(ta+(1-t)b)_0 d_q t = \frac{a+bq}{1+q}.$$

In the proof of inequality (ii) of Theorem 4.3 in [16], the authors used the following equality

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)_0 d_q t = \frac{1}{b-a}\int_a^b f(x)g(x)_a d_q x.$$

The above equality does not hold obviously. In the proof of Theorem 4.4 in [16], the authors used [18, Theorem 3.2]. So [16, Theorem 4.4] is also not right. Next, we will give the correct results in Theorems 3.6 and 3.9.

Remark 3.5. If $q \to 1$, then inequality (3.1) reduces to the Hermite-Hadamard integral inequality for h-convex functions

$$\begin{aligned}\frac{1}{h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx &\leqslant \int_a^b (f(x)+f(a+b-x))w(x)dx \\ &\leqslant (f(a)+f(b))\int_a^b \left[h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right]w(x)dx.\end{aligned} \tag{3.8}$$

In fact, by letting $q \to 1$ and changing variables, we obtain

$$\lim_{q\to1}\int_a^b f(x)_a d_q x = \int_a^b f(x)dx \quad \text{and} \quad \lim_{q\to1}\int_a^b f(a+b-x)_a d_q x = \int_a^b f(a+b-x)dx = \int_a^b f(x)dx.$$

In (3.8), if we take $w(x) = 1/(b-a)$, then inequality (3.8) reduces to the result in [15].

Theorem 3.6. *Let* $f \in SX(h_1, I)$ *and* $g \in SX(h_2, I)$, $a, b \in I$ *with* $a < b$. *Then following inequalities hold*

$$\begin{aligned}\frac{1}{b-a}\int_a^b [f(x)g(x)+f(a+b-x)g(a+b-x)]_a d_q x \leqslant{}& M(a,b)\int_0^1 [h_1(x)h_2(x)+h_1(1-x)h_2(1-x)]_0 d_q x \\ &+N(a,b)\int_0^1 [h_1(x)h_2(1-x)+h_1(1-x)h_2(x)]_0 d_q x,\end{aligned} \tag{3.9}$$

and

$$\frac{f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} - \frac{1}{b-a}\int_a^b [f(x)g(x)+f(a+b-x)g(a+b-x)]_a d_q x$$
$$\leqslant M(a,b)\int_0^1 [h_1(x)h_2(1-x)+h_1(1-x)h_2(x)]_0 d_q x \tag{3.10}$$
$$+ N(a,b)\int_0^1 [h_1(x)h_2(x)+h_1(1-x)h_2(1-x)]_0 d_q x,$$

where $M(a,b) = f(a)g(a)+f(b)g(b)$ *and* $N(a,b) = f(a)g(b)+f(b)g(a)$.

Proof. Since $f \in SX(h_1, I)$ and $g \in SX(h_2, I)$, we have

$$f(tb+(1-t)a) \leqslant h_1(t)f(b)+h_1(1-t)f(a), \tag{3.11}$$

and

$$g(tb+(1-t)a) \leqslant h_2(t)g(b)+h_2(1-t)g(a). \tag{3.12}$$

Multiplying (3.11) with (3.12), we get

$$\begin{aligned} f(tb+(1-t)a)g(tb+(1-t)a) \leqslant & h_1(t)h_2(t)f(b)g(b)+h_1(1-t)h_2(1-t)f(a)g(a) \\ & + h_1(t)h_2(1-t)f(b)g(a)+h_1(1-t)h_2(t)f(a)g(b). \end{aligned} \tag{3.13}$$

Integrating (3.13) with respect to t from 0 to 1, we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)g(x)_a d_q x \leqslant & f(b)g(b)\int_0^1 h_1(x)h_2(x)_0 d_q x + f(a)g(a)\int_0^1 h_1(1-x)h_2(1-x)_0 d_q x \\ & + f(b)g(a)\int_0^1 h_1(x)h_2(1-x)_0 d_q x + f(a)g(b)\int_0^1 h_1(1-x)h_2(x)_0 d_q x. \end{aligned} \tag{3.14}$$

Similarly, we can obtain the following inequality

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(a+b-x)g(a+b-x)_a d_q x \leqslant & f(a)g(a)\int_0^1 h_1(x)h_2(x)_0 d_q x \\ & + f(b)g(b)\int_0^1 h_1(1-x)h_2(1-x)_0 d_q x \\ & + f(a)g(b)\int_0^1 h_1(x)h_2(1-x)_0 d_q x \\ & + f(b)g(a)\int_0^1 h_1(1-x)h_2(x)_0 d_q x. \end{aligned} \tag{3.15}$$

We obtain inequality (3.9) from (3.14) and (3.15).

Due to the fact that $(a+b)/2 = (ta+(1-t)b)/2 + ((1-t)a+tb)/2$ and (3.11) and (3.12), we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)g\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \\ &\leqslant h_1\left(\frac{1}{2}\right)[f(ta+(1-t)b)+f((1-t)a+tb)]\cdot h_2\left(\frac{1}{2}\right)[g(ta+(1-t)b)+g((1-t)a+tb)] \\ &= h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big(f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g(ta+(1-t)b) \end{aligned}$$

$$
\begin{aligned}
&+f(ta+(1-t)b)g((1-t)a+tb)+f((1-t)a+tb)g((1-t)a+tb)\Big)\\
&\leqslant h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big(f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\\
&+[h_1(t)f(a)+h_1(1-t)f(b)][h_2(1-t)g(a)+h_2(t)g(b)]\\
&+[h_1(1-t)f(a)+h_1(t)f(b)][h_2(t)g(a)+h_2(1-t)g(b)]\Big).
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leqslant{}& h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[f(ta+(1-t)b)g(ta+(1-t)b)\\
&+f((1-t)a+tb)g((1-t)a+tb)]\\
&+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big([h_1(t)h_2(1-t)+h_1(1-t)h_2(t)]M(a,b)\\
&+[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]N(a,b)\Big).
\end{aligned}
\tag{3.16}
$$

Multiplying both sides of (3.16) by $1/\big(h_1(\frac{1}{2})h_2(\frac{1}{2})\big)$ and integrating the obtained result with respect to t from 0 to 1, we have

$$
\begin{aligned}
\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leqslant{}& \frac{1}{b-a}\int_a^b [f(x)g(x)+f(a+b-x)g(a+b-x)]_a d_q x\\
&+M(a,b)\int_0^1 [h_1(x)h_2(1-x)+h_1(1-x)h_2(x)]_0 d_q x\\
&+N(a,b)\int_0^1 [h_1(x)h_2(x)+h_1(1-x)h_2(1-x)]_0 d_q x,
\end{aligned}
$$

which implies (3.10). This completes the proof. □

Corollary 3.7. *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$ be convex mappings and $a, b \in I$ with $a < b$. Then following inequalities hold*

$$
\frac{1}{b-a}\int_a^b [f(x)g(x)+f(a+b-x)g(a+b-x)]_a d_q x \leqslant \frac{(1+2q+q^3)M(a,b)+2q^2N(a,b)}{(1+q)(1+q+q^2)},
$$

and

$$
\begin{aligned}
4f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b [f(x)g(x)+f(a+b-x)g(a+b-x)]_a d_q x&\\
\leqslant \frac{2q^2M(a,b)+(1+2q+q^3)N(a,b)}{(1+q)(1+q+q^2)}&,
\end{aligned}
$$

where $M(a,b)$ and $N(a,b)$ are defined in Theorem 3.6.

Remark 3.8. If $q^- \to 1$, then inequalities (3.9) and (3.10) reduce to the following integral inequalities for h-convex functions

$$
\frac{1}{b-a}\int_a^b f(x)g(x)dx \leqslant M(a,b)\int_0^1 h_1(x)h_2(x)dx + N(a,b)\int_0^1 h_1(x)h_2(1-x)dx,
$$

and

$$
\frac{f(\frac{a+b}{2})g(\frac{a+b}{2})}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}-\frac{1}{b-a}\int_a^b f(x)g(x)dx \leqslant M(a,b)\int_0^1 h_1(x)h_2(1-x)dx + N(a,b)\int_0^1 h_1(x)h_2(x)dx,
$$

where $M(a,b)$ and $N(a,b)$ are defined in Theorem 3.6. See [15].

Theorem 3.9. *Let* $f \in SX(h_1, I)$ *and* $g \in SX(h_2, I)$, $a, b \in I$ *with* $a < b$. *Then following inequalities hold*

$$\begin{aligned}
&\int_a^b \int_a^b \int_0^1 [f(ty+(1-t)x) + f(a+b-(ty+(1-t)x))][g(ty+(1-t)x)\\
&\quad + g(a+b-(ty+(1-t)x))]_0 d_q t_a d_q x_a d_q y\\
&\qquad \leqslant (b-a) \int_0^1 [h_1(t)h_2(t) + h_1(1-t)h_2(1-t)]_0 d_q t\\
&\qquad\quad \times \int_a^b [f(x) + f(a+b-x)][g(x) + g(a+b-x)]_a d_q x\\
&\qquad\quad + (b-a)^2 \int_0^1 [h_1(t)h_2(1-t) + h_1(1-t)h_2(t)]_0 d_q t\\
&\qquad\quad \times \int_0^1 [h_1(t) + h_1(1-t)]_0 d_q t \int_0^1 [h_2(t) + h_2(1-t)]_0 d_q t [M(a,b) + N(a,b)],
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&\int_a^b \int_0^1 \left[f\left(ty + (1-t)\left(\frac{a+b}{2}\right)\right) g\left(ty + (1-t)\left(\frac{a+b}{2}\right)\right)\right.\\
&\quad \left. + f\left(a+b-\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\right) g\left(a+b-\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\right)\right]_0 d_q t_a d_q y\\
&\quad \leqslant \int_0^1 h_1(t)h_2(t)_0 d_q t \int_a^b [f(y)g(y) + f(a+b-y)g(a+b-y)]_a d_q y\\
&\quad + (b-a) \int_0^1 [h_1(t) + h_1(1-t)]_0 d_q t\\
&\quad \times \int_0^1 [h_2(t) + h_2(1-t)]_0 d_q t \left(2h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \int_0^1 h_1(1-t)h_2(1-t)_0 d_q t\right.\\
&\quad + h_2\left(\frac{1}{2}\right) \int_0^1 h_1(t)h_2(1-t)_0 d_q t\\
&\quad \left. + h_1\left(\frac{1}{2}\right) \int_0^1 h_1(1-t)h_2(t)_0 d_q t\right) [M(a,b) + N(a,b)],
\end{aligned} \tag{3.18}$$

where $M(a,b)$ *and* $N(a,b)$ *are defined in Theorem 3.6.*

Proof. Since $f \in SX(h_1, I)$ and $g \in SX(h_2, I)$, for all $t \in [0,1]$, $x, y \in I$, we have

$$f(ty + (1-t)x) \leqslant h_1(t)f(y) + h_1(1-t)f(x), \tag{3.19}$$

and

$$g(ty + (1-t)x) \leqslant h_2(t)g(y) + h_2(1-t)g(x). \tag{3.20}$$

Multiplying (3.19) with (3.20), we get

$$\begin{aligned}
f(ty+(1-t)x)g(ty+(1-t)x) \leqslant\; & h_1(t)h_2(t)f(y)g(y) + h_1(1-t)h_2(1-t)f(x)g(x)\\
& + h_1(t)h_2(1-t)f(y)g(x) + h_1(1-t)h_2(t)f(x)g(y).
\end{aligned} \tag{3.21}$$

Integrating (3.21) with respect to t from 0 to 1, we have

$$\begin{aligned}
\int_0^1 f(ty+(1-t)x)g(ty+(1-t)x)_0 d_q t \leqslant\; & f(y)g(y) \int_0^1 h_1(t)h_2(t)_0 d_q t\\
& + f(x)g(x) \int_0^1 h_1(1-t)h_2(1-t)_0 d_q t\\
& + f(y)g(x) \int_0^1 h_1(t)h_2(1-t)_0 d_q t\\
& + f(x)g(y) \int_0^1 h_1(1-t)h_2(t)_0 d_q t.
\end{aligned} \tag{3.22}$$

Next, taking double q-integral to both sides of (3.22) with respect to x, y on (a, b), we obtain

$$\begin{aligned}
&\int_a^b\int_a^b\int_0^1 f(ty+(1-t)x)g(ty+(1-t)x)\,{}_0d_qt\,{}_ad_qx\,{}_ad_qy\\
&\quad\leqslant (b-a)\int_0^1 h_1(t)h_2(t)\,{}_0d_qt\int_a^b f(y)g(y)\,{}_ad_qy\\
&\qquad+(b-a)\int_0^1 h_1(1-t)h_2(1-t)\,{}_0d_qt\int_a^b f(x)g(x)\,{}_ad_qx\\
&\qquad+\int_0^1 h_1(t)h_2(1-t)\,{}_0d_qt\int_a^b f(y)\,{}_ad_qy\int_a^b g(x)\,{}_ad_qx\\
&\qquad+\int_0^1 h_1(1-t)h_2(t)\,{}_0d_qt\int_a^b f(x)\,{}_ad_qx\int_a^b g(y)\,{}_ad_qy\\
&\quad=(b-a)\int_0^1[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]\,{}_0d_qt\\
&\qquad\times\int_a^b f(x)g(x)\,{}_ad_qx+\int_0^1[h_1(t)h_2(1-t)\\
&\qquad+h_1(1-t)h_2(t)]\,{}_0d_qt\int_a^b f(x)\,{}_ad_qx\int_a^b g(x)\,{}_ad_qx.
\end{aligned}\tag{3.23}$$

Due to the fact that $a+b-(ty+(1-t)x)=t(a+b-y)+(1-t)(a+b-x)$ and (3.21), similarly, we can have the following inequalities

$$\begin{aligned}
&\int_a^b\int_a^b\int_0^1 f(a+b-(ty+(1-t)x))g(ty+(1-t)x)\,{}_0d_qt\,{}_ad_qx\,{}_ad_qy\\
&\quad\leqslant(b-a)\int_0^1[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]\,{}_0d_qt\int_a^b f(a+b-x)g(x)\,{}_ad_qx\\
&\qquad+\int_0^1[h_1(t)h_2(1-t)+h_1(1-t)h_2(t)]\,{}_0d_qt\int_a^b f(a+b-x)\,{}_ad_qx\int_a^b g(x)\,{}_ad_qx,
\end{aligned}\tag{3.24}$$

$$\begin{aligned}
&\int_a^b\int_a^b\int_0^1 f(a+b-(ty+(1-t)x))g(ty+(1-t)x)\,{}_0d_qt\,{}_ad_qx\,{}_ad_qy\\
&\quad\leqslant(b-a)\int_0^1[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]\,{}_0d_qt\int_a^b f(x)g(a+b-x)\,{}_ad_qx\\
&\qquad+\int_0^1[h_1(t)h_2(1-t)+h_1(1-t)h_2(t)]\,{}_0d_qt\int_a^b f(x)\,{}_ad_qx\int_a^b g(a+b-x)\,{}_ad_qx,
\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}
&\int_a^b\int_a^b\int_0^1 f(a+b-(ty+(1-t)x))g(a+b-(ty+(1-t)x))\,{}_0d_qt\,{}_ad_qx\,{}_ad_qy\\
&\quad\leqslant(b-a)\int_0^1[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]\,{}_0d_qt\int_a^b f(a+b-x)g(a+b-x)\,{}_ad_qx\\
&\qquad+\int_0^1[h_1(t)h_2(1-t)+h_1(1-t)h_2(t)]\,{}_0d_qt\int_a^b f(a+b-x)\,{}_ad_qx\int_a^b g(a+b-x)\,{}_ad_qx.
\end{aligned}\tag{3.26}$$

Adding (3.23), (3.24), (3.25), (3.26), we can get

$$\begin{aligned}&\int_a^b\int_a^b\int_0^1[f(ty+(1-t)x)+f(a+b-(ty+(1-t)x))][g(ty+(1-t)x)\\&\quad+g(a+b-(ty+(1-t)x))]_0d_qt_ad_qx_ad_qy\\&\qquad\leqslant(b-a)\int_0^1[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]_0d_qt\\&\qquad\quad\times\int_a^b[f(x)+f(a+b-x)][g(x)+g(a+b-x)]_ad_qx\\&\qquad\quad+\int_0^1[h_1(t)h_2(1-t)+h_1(1-t)h_2(t)]_0d_qt\\&\qquad\quad\times\int_a^b[f(x)+f(a+b-x)]_ad_qx\int_a^b[g(x)+g(a+b-x)]_ad_qx.\end{aligned}\tag{3.27}$$

From Corollary 3.2 and the h-convexity of f and g, then we have

$$\int_a^b[f(x)+f(a+b-x)]_ad_qx\leqslant(b-a)[f(a)+f(b)]\int_0^1[h_1(t)+h_1(1-t)]_0d_qt,\tag{3.28}$$

and

$$\int_a^b[g(x)+g(a+b-x)]_ad_qx\leqslant(b-a)[g(a)+g(b)]\int_0^1[h_2(t)+h_2(1-t)]_0d_qt.\tag{3.29}$$

We obtain inequality (3.17) from (3.27), (3.28), and (3.29).

Next, we prove that (3.18) holds. Since $f\in SX(h_1,I)$ and $g\in SX(h_2,I)$, for all $t\in[0,1]$, $y\in I$, we have

$$f\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\leqslant h_1(t)f(y)+h_1(1-t)f\left(\frac{a+b}{2}\right),\tag{3.30}$$

and

$$g\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\leqslant h_2(t)g(y)+h_2(1-t)g\left(\frac{a+b}{2}\right).\tag{3.31}$$

Multiplying (3.30) with (3.31), we get

$$\begin{aligned}&f\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\\&\leqslant h_1(t)h_2(t)f(y)g(y)+h_1(1-t)h_2(1-t)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\quad+h_1(t)h_2(1-t)f(y)g\left(\frac{a+b}{2}\right)+h_1(1-t)h_2(t)f\left(\frac{a+b}{2}\right)g(y).\end{aligned}\tag{3.32}$$

Integrating the obtained result (3.32) with respect to t from 0 to 1, we have

$$\begin{aligned}&\int_0^1 f\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)_0d_qt\\&\leqslant\int_0^1 h_1(t)h_2(t)_0d_qtf(y)g(y)+\int_0^1 h_1(1-t)h_2(1-t)_0d_qtf\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\quad+\int_0^1 h_1(t)h_2(1-t)_0d_qtf(y)g\left(\frac{a+b}{2}\right)+\int_0^1 h_1(1-t)h_2(t)_0d_qtf\left(\frac{a+b}{2}\right)g(y).\end{aligned}\tag{3.33}$$

Similarly, we can get the following inequality

$$\begin{aligned}&\int_0^1 f\left(a+b-\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\right)g\left(a+b-\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\right)_0d_qt\\&\leqslant\int_0^1 h_1(t)h_2(t)_0d_qtf(a+b-y)g(a+b-y)+\int_0^1 h_1(1-t)h_2(1-t)_0d_qtf\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\quad+\int_0^1 h_1(t)h_2(1-t)_0d_qtf(a+b-y)g\left(\frac{a+b}{2}\right)+\int_0^1 h_1(1-t)h_2(t)_0d_qtf\left(\frac{a+b}{2}\right)g(a+b-y).\end{aligned}\tag{3.34}$$

Adding (3.33) with (3.34) and integrating the obtained result with to y from a to b, we have

$$\begin{aligned}
&\int_a^b \int_0^1 \Bigg[f\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right) \\
&\quad + f\left(a+b-\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\right)g\left(a+b-\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\right)\Bigg]{}_0d_qt\,{}_ad_qy \\
&\qquad \leqslant \int_0^1 h_1(t)h_2(t)\,{}_0d_qt \int_a^b [f(y)g(y)+f(a+b-y)g(a+b-y)]\,{}_ad_qy \\
&\qquad + 2(b-a)\int_0^1 h_1(1-t)h_2(1-t)\,{}_0d_qt \\
&\qquad \times f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) + \int_0^1 h_1(t)h_2(1-t)\,{}_0d_qt \int_a^b [f(y)+f(a+b-y)]\,{}_ad_qt g\left(\frac{a+b}{2}\right) \\
&\qquad + \int_0^1 h_1(1-t)h_2(t)\,{}_0d_qt f\left(\frac{a+b}{2}\right)\int_a^b [g(y)+g(a+b-y)]\,{}_ad_qt.
\end{aligned} \tag{3.35}$$

By applying Corollary 3.4 to (3.35), we have

$$\begin{aligned}
&\int_a^b \int_0^1 \Bigg[f\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right) \\
&\quad + f\left(a+b-\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\right)g\left(a+b-\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\right)\Bigg]{}_0d_qt\,{}_ad_qy \\
&\qquad \leqslant \int_0^1 h_1(t)h_2(t)\,{}_0d_qt \int_a^b [f(y)g(y)+f(a+b-y)g(a+b-y)]\,{}_ad_qy \\
&\qquad + 2(b-a)\int_0^1 h_1(1-t)h_2(1-t)\,{}_0d_qt \\
&\qquad \times h_1\left(\frac{1}{2}\right)[f(a)+f(b)]\int_0^1 [h_1(t)+h_1(1-t)]\,{}_0d_qt[g(a)+g(b)]\int_0^1 [h_2(t)+h_2(1-t)]\,{}_0d_qt \\
&\qquad + \int_0^1 h_1(t)h_2(1-t)\,{}_0d_qt(b-a)[f(a)+f(b)]\int_0^1 [h_1(t)+h_1(1-t)]\,{}_0d_qx \\
&\qquad \times h_2\left(\frac{1}{2}\right)[g(a)+g(b)]\int_0^1 [h_2(t)+h_2(1-t)]\,{}_0d_qt + \int_0^1 h_1(1-t)h_2(t)\,{}_0d_qt \\
&\qquad \times h_1\left(\frac{1}{2}\right)[f(a)+f(b)]\int_0^1 [h_1(t)+h_1(1-t)]\,{}_0d_qt \times (b-a)[g(a)+g(b)]\int_0^1 [h_2(t)+h_2(1-t)]\,{}_0d_qt,
\end{aligned}$$

which implies (3.18). This completes the proof. □

Corollary 3.10. *Let* $f, g : I \to \mathbb{R}$ *be convex mappings and* $a, b \in I$ *with* $a < b$. *Then following inequality holds*

$$\begin{aligned}
&\int_a^b \int_a^b \int_0^1 [f(ty+(1-t)x)+f(a+b-(ty+(1-t)x))][g(ty+(1-t)x) \\
&\quad + g(a+b-(ty+(1-t)x))]\,{}_0d_qt\,{}_ad_qx\,{}_ad_qy \leqslant \frac{(1+2q+q^3)(b-a)}{(1+q)(1+q+q^2)}\int_a^b [f(x)+f(a+b-x)][g(x) \\
&\quad + g(a+b-x)]\,{}_ad_qx + \frac{2q^2(b-a)^2}{(1+q)(1+q+q^2)}[M(a,b)+N(a,b)],
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \int_0^1 \Bigg[f\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right) \\
&\quad + f\left(a+b-\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\right)g\left(a+b-\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)\right)\Bigg]{}_0d_qt\,{}_ad_qy
\end{aligned}$$

$$\leqslant \frac{1}{1+q+q^2}\int_a^b [f(y)g(y)+f(a+b-y)g(a+b-y)]_a d_q y$$
$$+\frac{q(1+q)(b-a)}{2(1+q+q^2)}[M(a,b)+N(a,b)],$$

where $M(a,b)$ *and* $N(a,b)$ *are defined in Theorem* 3.6.

Remark 3.11. If $q^- \to 1$, then inequalities (3.17) and (3.18) reduce to the following inequalities

$$\begin{aligned}&\int_a^b\int_a^b\int_0^1 [f(ty+(1-t)x)+f(a+b-(ty+(1-t)x))]g(ty+(1-t)x)dtdxdy\\ &\quad\leqslant (b-a)\int_0^1 h_1(t)h_2(t)dt\int_a^b [f(x)+f(a+b-x)]g(x)dx\\ &\qquad+4(b-a)^2\int_0^1 h_1(t)h_2(1-t)dt\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt[M(a,b)+N(a,b)],\end{aligned}$$

and

$$\begin{aligned}&\int_a^b\int_0^1 f\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(ty+(1-t)\left(\frac{a+b}{2}\right)\right)dtdy\\ &\leqslant \int_0^1 h_1(t)h_2(t)dt\int_a^b f(y)g(y)dy\\ &\quad+2(b-a)\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt\left(2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\int_0^1 h_1(1-t)h_2(1-t)dt\right.\\ &\quad\left.+h_2\left(\frac{1}{2}\right)\int_0^1 h_1(t)h_2(1-t)dt+h_1\left(\frac{1}{2}\right)\int_0^1 h_1(1-t)h_2(t)dt\right)[M(a,b)+N(a,b)],\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are defined in Theorem 3.6.

Remark 3.12. Let $f(x)=x^2\in[a,b]$. It is easy to see that $f(t)$ is h-convex function when $h(t)=t\in[0,1]$. From Corollary 3.2, we have

$$2\left(\frac{a+b}{2}\right)^2 \leqslant \frac{1}{b-a}\int_a^b [x^2+(a+b-x)^2]_a d_q x \leqslant a^2+b^2,$$

that is,

$$2\left(\frac{a+b}{2}\right)^2 \leqslant \frac{(1+2q+q^3)(a^2+b^2)+4q^2ab}{(1+q)(1+q+q^2)} \leqslant a^2+b^2.$$

Let $f(x)=g(x)=x^2\in[a,b]$. From Corollary 3.7, we have the following inequalities

$$\begin{aligned}&\frac{2(b-a)^4}{1+q+q^2+q^3+q^4}-\frac{4(b-a)^4}{1+q+q^2+q^3}+\frac{6(b-a)^2(a^2+b^2)}{1+q+q^2}\\ &\qquad-\frac{4(b-a)^2(a^2+ab+b^2)}{1+q}+a^4+b^4 \leqslant \frac{(1+2q+q^3)(a^4+b^4)+4q^2a^2b^2}{(1+q)(1+q+q^2)},\end{aligned}$$

and

$$\begin{aligned}&4\left(\frac{a+b}{2}\right)^4-\frac{2(b-a)^4}{1+q+q^2+q^3+q^4}+\frac{4(b-a)^4}{1+q+q^2+q^3}-\frac{6(b-a)^2(a^2+b^2)}{1+q+q^2}\\ &\qquad+\frac{4(b-a)^2(a^2+ab+b^2)}{1+q}-(a^4+b^4) \leqslant \frac{2q^2(a^4+b^4)+2(1+2q+q^3)a^2b^2}{(1+q)(1+q+q^2)}.\end{aligned}$$

4. Conclusions

In this paper, we obtained some new Hermite-Hadamard type inequalities for h-convex functions via quantum integral on finite intervals. Some mistakes are pointed out and corrections are provided as well. Some existing results are obtained as special cases of our theorems.

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