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Hyers-Ulam Stability of Fibonacci Functional Equation in Modular Functional Spaces

Maryam Naderi Parizi[1], Madjid Eshaghi Gordji[2,*]

[1]Department of Mathematics, Payame noor University, Tehran, Iran

[2]Department of Mathematics, Semnan University ,P.O.BOX35195-363, Semnan. Iran

E-mail

mrym_ndr@yahoo.com[1], madjid.eshaghi@gmail.com[2]

*The corresponding author (M. E. Gordji)

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Abstract

In this paper, we prove the Hyers-Ulam stability of functional equation

$$f(x) = f(x-1) + f(x-2) \tag{1.1}$$

which called the Fibonacci functional equation in modular functional space.

Keywords: Hyers-Ulam stability, Fibonacci functional equation, modular functional space.

1. Introduction

In 1950 Nakano [16] introduced the theory of modular spaces in connection with the theory of ordered spaces, Musielak and Orlicz [15] in 1959, redefined and generalized it to obtain a generalization of the classical function spaces L^p.

Also the stability of functional equations originated from a question of Ulam [19] in 1940. In the next year, Hyers [9] proved the problem for the Cauchy functional equation. The stability problems of functional equations have been extensively investigated by several mathematician (see [3,4,5,6,7,8,10,11,12,14,17] and [18]).

Recently, Jung [13] investigated the Hyers-Ulam stability of Fibonacci functional equation f(x)=f(x-1)+f(x-2).Here we prove the Hyers-Ulam stability of this functional equation in modular functional space. By α and β we denote roots of the equation $x^2 - x - 1 = 0$. $\alpha + \beta = 1$ α is greater than one and β is negative root. We have $\alpha + \beta = 1$ and $\alpha\beta = -1$.

2. Preliminaries

We recall some basic notions and facts about Modular spaces.

Definition2.1: Let X be an arbitrary vector space over a complex or real field.

(a) A function $\rho : X \to [0,+\infty]$ is called a modular If

(i) $\rho(x) = 0 \Leftrightarrow x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for every scaler $|\alpha|$ with $|\alpha = 1|$,

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ $\alpha + \beta = 1, \alpha \geq 0, \beta \geq 0$

for all $x, y \in \mathbf{R}$.

(b) If (iii) is replaced by

$\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ *if* $\alpha + \beta = 1$

And $\alpha \geq 0, \beta \geq 0$ we say that ρ is convex modular.

(c) A modular ρ defines a corresponding modular space, i.e.the vector space X_ρ given by

$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \, as \, \lambda \to 0\}$

Example2.2: Let $(\mathrm{X}, \|.\|)$ be a norm space, then $\|.\|$ is a convex modular on X . But the converse is not true.

In general the modular ρ does not behave as a norm or a distance because it is not sub--additive. But one can associate to a modular the F-norm (see [9]).

Definition2.3: The modular space X_ρ can be equipped with the F-norm defined by

$$|x|_\rho = \inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq \alpha\}$$

Namely, if ρ be convex, then the functional

$$\|x\|_\rho = \inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq 1\}$$

is a norm called the Luxemburg norm in X_ρ which is equivalent to the $|.|_\rho$.

Definition2.4: Let X_ρ be a modular space.

(a) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ρ is said to be:

(i) ρ - convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$;

(ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$;

(b) X_ρ is ρ-complete if every ρ-Cauchy sequence is ρ - convergent.

(c) A subset $B \subseteq X_\rho$ is said to be ρ-closed if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset B$ with $x_n \to x$, then $x \in B$.

(d) A subset $B \subseteq X_\rho$ is called ρ-bounded if

$$\delta_\rho(B)=\sup\{\rho(x-y):x,y\in B\}<\infty$$

where $\delta_\rho(B)$ is called the ρ-diameter of B .

(e) We say that ρ has the Fatou property if

$$\rho(x-y)\le \underline{lim}\,\rho(x_n-y_n)$$

whenever $\rho(x_n-x)\to 0, \rho(y_n-y)\to 0$ as $n\to\infty$.

(f) ρ is said to satisfies the Δ_2-condition if

$\rho(2x_n)\to 0$ *as* $n\to\infty$ whenever $\rho(x_n)\to 0$ as $n\to\infty$.

It is easy to check that for every modular ρ and $x,y\in X_\rho$;

(1) $\rho(\alpha x)\le\rho(\beta x)$ for each positive numbers with $\alpha\le\beta$.

(2) $\rho(x+y)\le\rho(2x)+\rho(2y)$.

3. Hyers-Ulam Stability of Fibonacci equation in modular space

As already stated, α denotes the positive root of the equation $x^2-x-1=0$ and β is its negetive root. We can prove the Hyers-Ulam stability of the Fibonacci functional equation (1.1) as we see in the following theorem.

Theorem3.1: Let (X,ρ) be a Banach modular space. If $f:\mathbf{R}\to X$ satisfies the inequality

$$\rho(f(x)-f(x-1)-f(x-2))\le\varepsilon \qquad (3.1)$$

for all $x\in\mathbf{R}$ and for some $\varepsilon>0$ then there exists a Fibonacci function $G:\mathbf{R}\to X$ such that

$$\rho(f(x)-G(x))\le(1+\frac{2}{\sqrt{5}})\varepsilon \qquad (3.2)$$

for all $x\in\mathbf{R}$.

Proof .We get from (3.1):

$$\rho(f(x)-\alpha f(x-1)-\beta[f(x-1)-\alpha f(x-2)])\le\varepsilon$$

If we replace x by $x-k$ in the last inequality, then we have,

$$\rho(f(x-k)-\alpha f(x-k-1)-\beta[f(x-k-1)-\alpha f(x-k-2)])\le\varepsilon$$

and furthermore,

$$\rho(\beta^k(f(x-k)-\alpha f(x-k-1)-\beta^{k+1}(f(x-k-1)-\alpha f(x-k-2))$$

$$\le|\beta^k|\,p(f(x-k)-\alpha f(x-k-1)-\beta[f(x-k-1)-\alpha f(x-k-2)])$$

$$\le|\beta^k|\,\varepsilon \qquad (3.3)$$

for all $x\in\mathbf{R}$ and natural number k. By (3.3), we obviously have,

$$\rho(f(x)-\alpha f(x-1)-\beta^n[f(x-n)-\alpha f(x-n-1)])$$

$$=\rho(\sum_{k=0}^{n-1}\beta^k(f(x-k)-\alpha(f(x-k-1)-\beta^{k+1}(f(x-k-1)-\alpha f(x-k-2))$$

$$\leq \sum_{k=0}^{n-1} |\beta|^k \ \rho((f\ (x-k)-\alpha f\ (x-k-1)-\beta(f\ (x-k-1)-\alpha f\ (x-k-2)))$$

$$\leq \sum_{k=0}^{n-1} |\beta|^k \ \text{ò} \leq \frac{\text{ò}}{1-|\beta|} \qquad (3.4)$$

for all $x \in \mathbf{R}$ and natural number n.

For any $x \in \mathbf{R}$ (3.3) implies that $\{\beta^n (f\ (x-n)-\alpha f\ (x-n-1))\}$ is a cauchy sequence .

Therefore, we can define a function by

$$G_1(x) = \lim_{n\to\infty} \beta^n [f\ (x-n)-\alpha f\ (x-n-1)]$$

since X is complete. In view of the above definition of G_1, we obtain:

$$G_1(x-1)+G_1(x-2) = \beta^{-1} \lim_{n\to\infty} \beta^{n+1}[f\ (x-(n+1))-\alpha f\ (x-(n+1)-1)]$$
$$+\beta^{-2} \lim_{n\to\infty} \beta^{n+2}[f\ (x-(n+2)-\alpha f\ (x-(n+2)-1)]$$
$$= \beta^{-1}G_1(x)+\beta^{-2}G_1(x) = G_1(x)$$

for all $x \in \mathbf{R}$. Hence, G_1 is a Fibonacci function. If n goes to infinity, then (3.3) yields:

$$\rho(f\ (x)-\alpha f\ (x-1)-G_1) \leq \frac{3+\sqrt{5}}{2}\varepsilon \qquad (3.5)\ .$$

for all $x \in \mathbf{R}$.on the other hand, it also follows from (3.1) that

$$\rho(f\ (x)-\beta f\ (x-1)-\alpha[f\ (x-1)-\beta f\ (x-2)]) \leq \varepsilon$$

Analogous to (3.3), replacing x by $x+k$ in the above

Inequality, then we have:

$$\rho(f\ (x+k)-\beta[f\ (x+k-1)-\alpha f\ (x+k-1)-\beta f\ (x+k-2)]) \leq \varepsilon$$

and

$$\rho(\alpha^{-k}[f\ (x+k)-\beta f\ (x+k-1)]-\alpha^{-k+1}[f\ (x+k-1)-\beta f\ (x+k-2)]) \leq |\alpha|^{-k}\ \varepsilon \quad (3.6)$$

for all $x \in \mathbf{R}$. By using (3.6), we further obtain:

$$\rho(\alpha^{-n}[f\ (x+n)-\beta f\ (x+n-1)]-[f\ (x)-\beta f\ (x-1)])$$

$$\leq \sum_{k=1}^{n} \rho(\alpha^{-k}[f\ (x+k)-\beta f\ (x+k-1)]-\alpha^{-k+1}[f\ (x+k-1)-\beta f\ (x+k-2)])$$

$$\leq \sum_{k=1}^{n} \alpha^{-k} \rho([f\ (x+k)-\beta f\ (x+k-1)]-\alpha[f\ (x+k-1)-\beta f\ (x+k-2)])$$

$$\leq \sum_{k=1}^{n} \alpha^{-k} \varepsilon \quad (3.7)$$

for all $x \in \mathbf{R}$ and $n\in\BN$ .Thus $\{\alpha^{-n}[f\ (x+n)-\beta f\ (x+n-1)]\}$ is a cauchy sequence,

for any fixed $x \in \mathbf{R}$. Hence, we can define a function by

$$G_2(x) = \lim_{n\to\infty} \alpha^{-n}[f\ (x+n)-\beta f\ (x+n-1)]$$

Using the above definition of G_2, we get:

$$G_2(x-1)+G_2(x-2)=\alpha^{-1}\lim_{n\to\infty}\alpha^{-(n-1)}[f(x+n-1)-\beta f(x+(n-1)-1)]$$
$$+\alpha^{-2}\lim_{n\to\infty}\alpha^{-(n-2)}[f(x+n-2)-\beta f(x+(n-2)-1)]$$
$$=\alpha^{-1}G_2(x)+\alpha^{-2}G_2(x)=G_2(x)$$

For any $x\in\mathbf{R}$. So, G_2 is also a Fibonacci function. If we

Let n goes to infinity, then it follows from (3.7) that

$$\rho(G_2(x)-f(x)-\beta f(x-1))\le\frac{\sqrt{5}+1}{2}\varepsilon \quad (3.8)$$

for $x\in\mathbf{R}$. By (3.5), (3.8) we have,

$$\rho(f(x)-[\frac{\beta}{\beta-\alpha}G_1-\frac{\alpha}{\beta-\alpha}G_2])$$
$$\le\frac{\alpha}{\alpha-\beta}\rho(f(x)-\beta f(x-1)-G_2)-\frac{\beta}{\alpha-\beta}\rho(f(x)-\alpha f(x-1)-G_1)$$
$$=\rho(\frac{\alpha}{\alpha-\beta}(f(x)-\beta f(x-1)-G_2)-\frac{\beta}{\alpha-\beta}(f(x)-\alpha f(x-1)-G_1))\le(1+\frac{2}{\sqrt{5}})\epsilon$$

for all $x\in\mathbf{R}$. We now set:

$$G(x)=\frac{\beta}{\beta-\alpha}G_1-\frac{\alpha}{\beta-\alpha}G_2$$

It is easy to show that G is a Fibonacci function satisfying (3.2).

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