STARLIKENESS AND SHARPNESS RESULTS OF SPECIAL SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract

In the present paper, we investigate starlikeness of certain operators which are defined here by means of convolution. Also, by using the technique of finite Blaschke product (see [3,8]), we prove the sharpness of those results which were obtained earlier by authors in [1].

Keywords and phrases: Starlike function, sharpness, Blaschke product.

1. Introduction

Let \( H = H(\mathbb{D}) \) be the class of all analytic functions in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C}: |z| < 1 \} \). For \( n \in \mathbb{N} \) let \( \mathcal{A}_n \) denote the subclass of \( H \) containing the functions \( f(z) \) of the form

\[
  f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots (z \in \mathbb{D})
\]

with \( \mathcal{A}_1 = \mathcal{A} \). A function \( f \in \mathcal{A} \) is said to be starlike if it is univalent and \( f(\mathbb{D}) \) is starlike domain with respect to the origin. The class of starlike functions is denoted by \( S^* \). A special subclass of \( S^* \) is the class of starlike functions of order \( \gamma \) with \( 0 \leq \gamma < 1 \), given by

\[
  S^*(\gamma) = \left\{ f \in \mathcal{A}: \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \gamma, z \in \mathbb{D} \right\}.
\]

It is well known that \( S^*(0) = S^*(\text{, see [2]}). \)

For functions \( f, g \in H \) given by

\[
  f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k
\]

the Hadamard product (or convolution) of \( f, g \) in \( \mathbb{D} \) is defined by
\[(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z).\]

For \(n \in \mathbb{N}, 0 < \alpha \leq 1, 0 \leq \mu < \alpha n\) and \(\lambda > 0\) let \(U_n(\alpha, \mu, \lambda)\) be defined as follows
\[
U_n(\alpha, \mu, \lambda) = \left\{ f \in A_n : \left| (1 - \alpha) \left( \frac{Z}{f(z)} \right)^\mu + \alpha \left( \frac{Z}{f(z)} \right)^{\mu+1} f'(z) - 1 \right| < \lambda, z \in \mathbb{D} \right\}
\]
with \(U_f(\alpha, \mu, \lambda) = U(\alpha, \mu, \lambda)\). The special case of this class has been studied in [5].

For \(f \in U_n(\alpha, \mu, \lambda)\) we define the operator \(G(z)\) by
\[
G(z) = z \left( \frac{I}{\left( \frac{Z}{f(z)} \right)^\mu * \Phi(a; c; z) \right)^{\frac{1}{\mu}}} (1)
\]
where \(a, c \in \mathbb{C}, c \neq 0, -1, -2, \ldots, (\frac{Z}{f(z)})^\mu * \Phi(a; c; z) \neq 0\) and
\[
\Phi(a; c; z) = \sum_{k=0}^{\infty} (\frac{a}{c})_k z^k, (z \in \mathbb{D})
\]
with \((a)_k = a(a + 1) (a + 2) \cdots (a + k - 1)\) and \((a)_0 = 1\). Also, let
\[
H(z) = z \left( \frac{I}{\left( \frac{Z}{f(z)} \right)^\mu * \Phi(m, \gamma, z) \right)^{\frac{1}{\mu}}} (f \in U_n(\alpha, \mu, \lambda))
\]
where \(m < 1, \gamma \neq 0, \text{Re} \gamma \geq 0, (\frac{Z}{f(z)})^\mu * \Phi(m, \gamma, z) \neq 0\) and
\[
\Phi(m, \gamma, z) = 1 + (1 - m) \sum_{k=1}^{\infty} \frac{z^k}{k^\mu + 1}, (z \in \mathbb{D}).
\]

In [1] certain sufficient conditions in terms of \(\alpha, \mu, \lambda, \gamma\) and \(n\) were obtained, so that functions in \(U_n(\alpha, \mu, \lambda)\) belong to \(S^*(\gamma)\). Similarly, other conditions for these parameters were obtained such that the analytic functions \(G(z)\) and \(H(z)\) be in \(S^*\). In all these cases, the sharpness part was not proved. In this paper, by using the same techniques as in [8] we prove the sharpness part. Also, we give another proof for the starlikeness of \(G(z)\) and \(H(z)\). In order to prove our results we need the following lemmas.

**Lemma 1.1.** ([3]) Let \(\varphi, \psi \in \mathbb{R}\). There exists a sequence \(\{b_n\}\) of finite Blaschke products such that \(b_n(0) = 0, b_n(1) = e^{i\psi}\) and \(b_n(z) \to e^{i\psi} z\) uniformly on compact subsets of \(\mathbb{D}\).

Here a finite Blaschke product is a function as the type
\[
b(z) = e^{i\psi} \prod_{k=1}^{n} \frac{Z - a_k}{1 - \alpha a_k}, (\{a_k\} \subseteq \mathbb{D}, \gamma \in \mathbb{R}).
\]

**Lemma 1.2.** ([7]) If \(f\) and \(g\) are analytic and \(F\) and \(G\) are convex (univalent) such that \(f < F\) and \(g < G\), then \(f * g < F * G\), where \(<\) denotes the usual subordination, (see [2]).

**Lemma 1.3.** ([6]) Let \(c \in \mathbb{C}\) with \(\text{Re} c < 1\) and \(F_c(z) = \sum_{n=1}^{\infty} \frac{z^n}{n-c} \in H\). Then
\[
\sup_{z \in \mathbb{D}} |f(z) * F_c(z)| \leq \sup_{z \in \mathbb{D}} |f(z)|, (f \in H).
\]
2. Main Results

We begin with the following lemma that will be used in the next theorems.

**Lemma 2.1.** For fixed real numbers \( n \in \mathbb{N}, 0 < \alpha \leq 1, 0 < \mu < \alpha n \) and \( \lambda > 0 \) let \( f \in U_n(\alpha, \mu, \lambda) \). There exists an analytic function \( w(z) \) in \( \mathbb{D} \) where \( |w(z)| < 1 \) and \( w(0) = w'(0) = \cdots = w^{(n-1)}(0) = 0 \), such that

\[
\frac{zf'(z)}{f(z)} = \frac{1}{\alpha} \left( \frac{1 + \lambda w(z)}{1 - \frac{\lambda \mu}{\alpha} \int_{0}^{1} \frac{w(tz)^\alpha}{t^\alpha} \, dt} + \alpha - 1 \right), \quad (z \in \mathbb{D}).
\]

**Proof.** For \( f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{A}_n \), we can write

\[
\left( \frac{z}{f(z)} \right)^{\mu + 1} = \frac{1}{1 + (\mu + 1)a_{n+1}z^n + \cdots}.
\]

So, we obtain

\[
(1 - \alpha) \left( \frac{z}{f(z)} \right)^\mu + \alpha \left( \frac{z}{f(z)} \right)^{\mu + 1} f'(z) = (1 - (\mu + 1)a_{n+1}z^n + \cdots)(1 + (\alpha n)a_{n+1}z^n + \cdots)
\]

\[
= 1 + (\alpha n - \mu)a_{n+1}z^n + \cdots = 1 + \alpha \left( \frac{\alpha n - \mu}{\lambda} \right)a_{n+1}z^n + \cdots.
\]

Therefore, there exists an analytic function \( w(z) \) in \( \mathbb{D} \) with \( |w(z)| < 1 \) and \( w(0) = w'(0) = \cdots = w^{(n-1)}(0) = 0 \), such that

\[
(1 - \alpha) \left( \frac{z}{f(z)} \right)^\mu + \alpha \left( \frac{z}{f(z)} \right)^{\mu + 1} f'(z) = 1 + \lambda w(z). \tag{5}
\]

Let \( p(z) = \left( \frac{z}{f(z)} \right)^\mu \). Then \( p(z) \) is analytic in \( \mathbb{D} \) and \( p(0) = 1 \). Differentiating \( p(z) \) we obtain

\[
p(z) - \frac{\alpha}{\mu} z p'(z) = 1 + \lambda w(z). \tag{6}
\]

Solving the first order differential equation (6) we conclude that

\[
p(z) = 1 - \frac{\lambda \mu}{\alpha} \int_{0}^{1} \frac{w(tz)}{t^{\alpha + 1}} \, dt
\]

or equally

\[
\left( \frac{f(z)}{z} \right)^\mu = \frac{1}{1 - \frac{\lambda \mu}{\alpha} \int_{0}^{1} \frac{w(tz)}{t^{\alpha + 1}} \, dt}. \tag{7}
\]

Using (5) and (7) we obtain the required result. This completes the proof. \( \square \)

Now, we restate the sharp version of Theorem 2.1 in [1] and prove the sharpness part.
Theorem 2.2 [1. Theorem 2.1.] Let \( n \in \mathbb{N}, n \geq 2, \frac{n+1}{2n} < \alpha \leq l \) and \( n(l-\alpha) < \mu < \alpha n \). If \( f \in U_n(\alpha, \mu, \lambda) \), then \( f \in \mathcal{S}^*(\gamma) \) for \( 0 < \lambda \leq \lambda(\alpha, \mu, n, \gamma) \), where

\[
\lambda(\alpha, \mu, n, \gamma) = \begin{cases} 
\frac{(an-\mu)\sqrt{2\alpha(1-\gamma) - l}}{(an-\mu)^2 + \mu^2(2\alpha(1-\gamma) - l)}; & 0 \leq \gamma < \frac{\mu - n(l-\alpha)}{\mu(l+n)} < \gamma < l, \\
\frac{(an-\mu)(1-\gamma)}{n + \mu(y - l)}; & \frac{\mu - n(l-\alpha)}{\mu(l+n)} < \gamma < l, 
\end{cases}
\]

also, all bounds for \( \lambda \) are the best possible.

Proof. Suppose that \( f(z) = z + a_n z^{n+1} + \cdots \in U_n(\alpha, \mu, \lambda) \). By Lemma 2.1 there exists an analytic function \( w(z) \) in \( \mathbb{D} \) with \( |w(z)| < 1 \) and \( w(0) = w'(0) = \cdots = w^{(n-1)}(0) = 0 \) such that

\[
\frac{zf'(z)}{f(z)} = \frac{l}{\alpha} \left( \frac{1 + \lambda w(z)}{l - \frac{\alpha}{\mu} \int_0^1 \frac{w(tx)}{t^{n+1}} dt} + \alpha - l \right),
\]

and therefore

\[
\frac{l}{l - \gamma} \left( \frac{zf'(z)}{f(z)} - \gamma \right) = \frac{(\alpha - l) - \alpha \gamma}{\alpha (l - \gamma)} \left( \alpha - \lambda \mu \int_0^1 \frac{w(tx)}{t^{n+1}} dt \right) + \alpha (l + \lambda w(z))
\]

Now, we have to show that \( \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \gamma \). To do this, according to a well-known result in [6] and the last equation, it is sufficient to show that

\[
\frac{(\alpha - l) - \alpha \gamma}{\alpha (l - \gamma)} \left( \alpha - \lambda \mu \int_0^1 \frac{w(tx)}{t^{n+1}} dt \right) \neq -iT, (T \in \mathbb{R}),
\]

which is equivalent to

\[
\lambda \left( \frac{w(z) + \mu \left( \frac{\alpha y + l - \alpha}{\alpha} - T(l - \gamma) i \right) \int_0^1 \frac{w(tx)}{t^{n+1}} dt}{\alpha (l - \gamma)(l + Ti)} \right) \neq -l, (T \in \mathbb{R}).
\]

Let

\[
M = \sup_{z \in \mathbb{D}, w \in \mathcal{B}_n, T \in \mathbb{R}} \left| w(z) + \mu \left( \frac{\alpha y + l - \alpha}{\alpha} - T(l - \gamma) i \right) \int_0^1 \frac{w(tx)}{t^{n+1}} dt \right| \alpha (l - \gamma)(l + Ti),
\]

with

\[
B_n = \{ w \in H(\mathbb{D}) : |w(z)| < 1 \text{ and } w^{(k)}(0) = 0, k = 0, 1, 2, \ldots, n - 1 \}.
\]
Then \( f \in S^*(\gamma) \) if \( \lambda M \leq 1 \). This shows that it is sufficient to find \( M \). By the general Schwarz lemma we have \( |w(z)| \leq |z|^\alpha \), so we see that

\[
M \leq \sup_{T \in \mathbb{R}} \left\{ \frac{f'_k(\frac{1}{2\pi})}{\sqrt{\frac{\alpha^2 + 1}{2\pi} + T^2}} \right\}.
\]  

(8)

In fact, in the sequel, we prove that equality holds in the above relation, hence the sharpness is established. By Lemma 1.1 given \( \psi, \phi \in \mathbb{R} \) there exists a sequence of finite Blaschke products \( \{w_k(z)\} \) such that \( w_k(I) = e^{i\psi} \) and \( w_k(z) \to e^{i\phi} z^n \) uniformly on compact subsets of \( \mathbb{D} \). Therefore, we have the following relation for each \( T \in \mathbb{R} \):

\[
\sup_{z \in \mathbb{D}, w \in \mathbb{R}_n} \left| \frac{w(z) + \mu \left( \frac{ay + 1 - a}{a} - T(I - \gamma)i \right) \int_0^{1/T} w(tz) \frac{dt}{t+1}}{\alpha(I - \gamma)(1 + T^2)} \right| \leq \sup_{\psi, \phi \in \mathbb{R}} \left| e^{i\psi} + \frac{\mu a}{\alpha - \mu} \left( \frac{ay + 1 - a}{a} \right)^2 + (I - \gamma)^2 T^2 e^{i(\phi + \theta)} \right| \left/ \alpha(I - \gamma) \sqrt{1 + T^2} \right.,
\]

where \( \theta = Arg \left( \frac{ay + 1 - a}{a} - T(I - \gamma)i \right) \). Fixing \( \phi \) and choosing \( \psi = \phi + \theta \), we get the required equality in (8). Thus the bound for \( M \) is sharp as a function of \( T \).  

By taking \( \gamma = 0 \) in Theorem 2.2 we obtain the following sharp result.

**Corollary 2.3.** Let \( n \in \mathbb{N}, n \geq 2, \frac{n+1}{2n} < \alpha \leq 1 \) and \( n(I - \alpha) < \mu < \alpha n \). If \( f \in U_n(\alpha, \mu, \lambda) \), then \( f \in S^* \) for \( 0 < \lambda \leq \frac{(an - \mu)\sqrt{2a - 1}}{\sqrt{(an - \mu)^2 + \mu^2(2a - 1)}} \), and the bound for \( \lambda \) is sharp.

**Theorem 2.4.** Let \( n \in \mathbb{N}, n \geq 2, \frac{n+1}{2n} < \alpha \leq 1 \) and \( n(I - \alpha) < \mu < \alpha n \). Also, let \( \varphi(z) = 1 + b_1z + b_2z^2 + \cdots \) with \( b_n \neq 0 \) be convex (univalent) in \( \mathbb{D} \). If \( f(z) = z + a_n + z^{n+1} + \cdots \in U_n(\alpha, \mu, \lambda) \) and \( \Phi(a; c; z) \) defined by (2) satisfies the conditions

\[
\left( \frac{Z}{f(z)} \right)^\mu \ast \Phi(a; c; z) \neq \emptyset, \Phi(a; c; z) < \varphi(z), (z \in \mathbb{D})
\]

then the function \( G(z) \) defined by (1) has the following properties:

- \( G \in U_n(\alpha, \mu, \lambda|b_n|) \),
- \( G \in S^* \) for \( 0 < \lambda \leq \frac{(an - \mu)\sqrt{2a - 1}}{|b_n|\sqrt{(an - \mu)^2 + \mu^2(2a - 1)}} \)

In the case 2 the bound for \( \lambda \) is sharp.

**Proof.** The definition of \( G \) shows that

\[
\left( \frac{Z}{G(z)} \right)^\mu = \left( \frac{Z}{f(z)} \right)^\mu \ast \Phi(a; c; z).
\]

Also, a simple calculation gives

\[
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\]
\[
\frac{Z}{\mu} \left( \frac{Z}{G(z)} \right)^{\mu'} = \left( \frac{Z}{G(z)} \right)^{\mu} - \left( \frac{Z}{G(z)} \right)^{\mu+1} G'(z).
\]

Therefore, we obtain
\[
(1 - \alpha) \left( \frac{Z}{G(z)} \right)^{\mu} + \alpha \left( \frac{Z}{G(z)} \right)^{\mu+1} G'(z) = \left( \frac{Z}{G(z)} \right)^{\mu} - \frac{\alpha z}{\mu} \left( \frac{Z}{G(z)} \right)^{\mu'},
\]
\[
= \left( \frac{Z}{f(z)} \right)^{\mu} \cdot \Phi(a; c; z) - \alpha \left( \frac{Z}{f(z)} \right)^{\mu} \cdot \Phi(a; c; z) + \alpha \left( \frac{Z}{f(z)} \right)^{\mu+1} f'(z) \cdot \Phi(a; c; z)
\]
\[
= \left( (1 - \alpha) \left( \frac{Z}{f(z)} \right)^{\mu} + \alpha \left( \frac{Z}{f(z)} \right)^{\mu+1} f'(z) \right) \cdot \Phi(a; c; z).
\]

Since \( I + \lambda z^n \) and \( \varphi(z) \) are convex in \( \mathbb{D} \) and by the assumption (also, see relation (5))
\[
(1 - \alpha) \left( \frac{Z}{f(z)} \right)^{\mu} + \alpha \left( \frac{Z}{f(z)} \right)^{\mu+1} f'(z) < (1 + \lambda z^n) \cdot \Phi(a; c; z) \leq \varphi(z)
\]
so, by Lemma 1.2, we deduce that
\[
(1 - \alpha) \left( \frac{Z}{G(z)} \right)^{\mu} + \alpha \left( \frac{Z}{G(z)} \right)^{\mu+1} G'(z) < (1 + \lambda b_n z^n).
\]

Case 1 now follows from the last subordination, while 2 is a simple consequence of Corollary 2.3.

It is well-known that if \( \alpha > 0 \) and \( c \geq \max\{2, \alpha\} \), then \( \Phi(a; c, z) \) defined by (2) is convex in \( \mathbb{D} \), (see [4]). So, if we take \( \varphi(z) = \Phi(a; c; z) \) in Theorem 2.4, we obtain the following sharp result.

**Corollary 2.5.** Let \( n \in \mathbb{N}, n \geq 2, \alpha > 0 \) and \( c \geq \max\{2, \alpha\} \). Also, let \( \frac{n+1}{2n} < \alpha \leq 1 \) and \( n(1 - \alpha) < \mu < an \). If \( f \in U_n(\alpha, \mu, \lambda) \) and \( \Phi(a; c; z) \) defined by (2) satisfy the condition \( \left( \frac{Z}{f(z)} \right)^{\mu} \cdot \Phi(a; c; z) \neq 0 \) for all \( z \in \mathbb{D} \), then the function \( G(z) \) defined by (1) has the following properties:

- \( G \in U_n \left( \alpha, \mu, \frac{2\|a\|}{\|c\|} \right) \).
- \( G \in S^* \) where \( 0 < \lambda \leq \frac{\|a\|\|\alpha - \mu\|^{2\mu - 1}}{\|c\|^{2\lambda - 1}} \).

Also, the bound for \( \lambda \) is sharp.

**Theorem 2.6.** For \( n \in \mathbb{N}, n \geq 2, \frac{n+1}{2n} < \alpha \leq 1 \) and \( n(1 - \alpha) < \mu < an \) let \( f \in U_n(\alpha, \mu, \lambda) \). If \( m < 1, \Re \varphi > 0 \) and \( \Psi(m, \gamma, z) \) defined by (4) satisfy the condition \( \left( \frac{Z}{f(z)} \right)^{\mu} \cdot \Psi(m, \gamma, z) \neq 0 \) for all \( z \in \mathbb{D} \), then the function \( H(z) \) given by (3) has the following properties:

- \( H \in U_n \left( \alpha, \mu, \lambda(1 - m) \right) \).
- \( H \in S^* \) where \( 0 < \lambda \leq \frac{(\alpha - \mu)^{2\mu - 1}}{(1 - m)\|\alpha - \mu\|^{2\mu - 1}} \).
In the case 2 the bound for $\lambda$ is best possible.

**Proof.** Using the same steps as in the proof of Theorem 2.4 we obtain

$$\left(1 - \alpha\right)\left(\frac{z}{H(z)}\right)^\mu + \alpha \left(\frac{z}{H(z)}\right)^{\mu+1} H'(z) - I =$$

$$= \left(1 - \alpha\right)\left(\frac{z}{f(z)}\right)^\mu + \alpha \left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) * \Psi(m, \gamma, z) - I$$

$$= \left(1 - \alpha\right)\left(\frac{z}{f(z)}\right)^\mu + \alpha \left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - I * \left(1 + \left(1 - m\right) \sum_{k=1}^{\infty} \frac{\gamma^k}{k} \right)$$

$$= \left(1 - \alpha\right)\left(\frac{z}{f(z)}\right)^\mu + \alpha \left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - I * \left(1 - \frac{m}{(1 - 1/\gamma)} \right)$$

Now, by using Lemma 1.3 with $c = 1 - \frac{i}{\gamma}$, we conclude that

$$\left|\left(1 - \alpha\right)\left(\frac{z}{H(z)}\right)^\mu + \alpha \left(\frac{z}{H(z)}\right)^{\mu+1} H'(z) - I\right|$$

$$\leq \left(1 - m\right) \sup_{z \in D} \left|\left(1 - \alpha\right)\left(\frac{z}{f(z)}\right)^\mu + \alpha \left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - I\right|$$

$$\leq \left(1 - m\right) \lambda.$$ 

This proves the case 1. Case 2 follows simply from Corollary 2.3.■

**References**


