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# Solutions Exact to Fredholm Fuzzy Integral Equations with Optimal Homotopy Asymptotic Method

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### *Abstract*

A analytic approximate technique for addressing nonlinear problems, namely the Optimal Homotopy Asymptotic Method (OHAM), is proposed. This approach does not depend upon any small/large parameters. This method provides us with a convenient way to control the convergence of approximation series and adjust convergence regions when necessary. The series solution is developed and the recurrence relations are given explicitly. The results reveal that the proposed method is effective and easy to use. This method is illustrated by solving an examples.

**Keywords**: fredholm fuzzy integral equations, optimal homotopy asymptotic method (OHAM).

## 1. Introduction

The concept of integration of fuzzy functions was first introduced by Dubois and Prade [1]. The topics of numerical methods for solving fuzzy integral equations have been rapidly growing in recent years and have been studies by authors of [2]. The numerical methods for fuzzy differential equations have been studied by S. Abbasbandy, T. Allahviranloo, [3, 4, 5] and others. Alternative approaches were later suggested by Goetschel and Vaxman [6], Kaleva [10] and others.In this paper, the application of optimal homotopy asymptotic method (OHAM) will be extended to obtain approximate solutions fredholm fuzzy integral equations. Once an efficient method for integrating fuzzy functions is available, it may be useful for solving fuzzy integral equations as well. It should be noted that usually it is easier to treat fuzzy integral equations with kernels than fuzzy differential equations [7-15]. Indeed, the arithmetic operation of difference performed in the process of calculating a derivative of a fuzzy function may yield results out of the convex cone E l of fuzzy numbers [16, 17]. This complicates the process of solving a fuzzy differential equation even if it is replaced by an equivalent integral equation. However, in the case of integral equations with kernels, the formulation of the basic procedure for solving these equations is quite simple and straightforward forarbitrary kernels. This method is straight forward, reliable and it does not need to look for h curves like HAM. This method provides a convenient way to control the convergence of the series solution and allows the adjustment of convergence region where ever it is needed. Some plots and tables are presented to show the reliability and simplicity of the method.

## 2. Preliminaries

In this section the most basic notations used in fuzzy calculus are introduced. We start with defining a fuzzy number.

**Definition 2.1:** A fuzzy number is a fuzzy set $u: R^1 \rightarrow [0,1]$ which satisfies

i) $u$ is upper semicontinuous.
ii) $u(x) = 0$ outside some interval $[c, d]$
iii) There are real numbers $a, b : c \leq a \leq b \leq d$

1. $u(x)$ is monotonic increasing on $[c, a]$
2. $u(x)$ is monotonic increasing on $[b, d]$
3. $u(x) = 1,\ \ a \leq x \leq b.$

The set of all fuzzy numbers (as given by Definition 1) is denoted by $E^1$ . An alternative definition or parametric form of a fuzzy number which yields the same $E^1$ is given by Kaleva [8].

**Definition 2.2:** A fuzzy number $u$ is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r);\ \ 0 \leq x \leq 1$ which satisfying the following requirements:

i) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
ii) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,
iii) $\underline{u}(r) \leq \overline{u}(r)\, ; 0 \leq r \leq 1$

For arbitrary $u = \left(\underline{u}, \overline{u}\right),\ v = \left(\underline{v}, \overline{v}\right)$ and $k > 0$ we define addition $u + v$ and multiplication by $k$ as

$$\underline{(u+v)}(r) = \underline{u}(r) + \underline{v}(r), \qquad \overline{(u+v)}(r) = \overline{u}(r) + \overline{v}(r)$$
$$\underline{ku}(r) = k\underline{u}(r), \qquad \overline{\mathrm{ku}}\,(\mathrm{r}) = \mathrm{k}\overline{\mathrm{u}}(\mathrm{r}) \tag{1}$$

The collection of all the fuzzy numbers with addition and multiplication as defined by Eq. (1) is denoted by $E^1$ and is a convex cone. It can be shown that Eq. (1) are equivalent to the addition and the extension principles [8]. We will next define the fuzzy function notation and a metric D in $E^1$ [13].

**Definition 2.3:** For arbitrary fuzzy numbers $u = \left(\underline{u}, \overline{u}\right)$ and $v = \left(\underline{v}, \overline{v}\right)$the quantity

$$D(u, v) = max\left\{sup_{0\leq r\leq 1}\left|\underline{u}(r) - \underline{v}(r)\right|, sup_{0\leq r\leq 1}|\overline{u}(r) - \overline{v}(r)|\,\right\}$$

is the distance between u and v.

This metric is equivalent to the one used by Puri and Ralescu [13] and Kaleva [8]. It is shown [14] that $(E^1, D)$ is a complete metric space. We now follow Goetschel and Voxman [6] and define the integral of a fuzzy function using the Riemann integral concept.

Let $f: [a, b] \rightarrow E^1$. For each partition $P = \{t_0, \cdots, t_n\}$ of [a,b] with $h = max|t_1 \rightarrow t_1 - 1|$ and for arbitrary $\varepsilon_i: t_{i-1} \leq \varepsilon_i \leq t_{i,}1 \leq i \leq n$ let

$$R_p = \sum_{i=1}^{n} f(\varepsilon_i)(t_i - \ t_{i-1}). \tag{2}$$

The definite integral of $f(t)$ over [a, b] is

$$\int_a^b f(t)dt = \lim_{h\to 0} R_p \qquad (3)$$

provided that this limit exists in the metric D. If the fuzzy function f(t) is continuous in the metric D, its definite integral exists [6]. Furthermore,

$$\underline{\left(\int_a^b f(t,r)dt\right)} = \int_a^b \underline{f}(t,r)dt,$$

$$\overline{\left(\int_a^b f(t,r)dt\right)} = \int_a^b \overline{f}(t,r)dt$$

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [8]. However, if f (t) is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral using Eqs. (2) and (3) is more convenient for numerical calculations. More detailsabout the properties of the fuzzy integral are given in [6, 8].

**Definition:** [6] For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity

$$D(u,v) = [\int_0^1 \left(\underline{u}(r) - \underline{v}(r)\right)^2 dr + \int_0^1 (\overline{u}(r) - \overline{v}(r))^2\, dr]^{\frac{1}{2}}$$

is the distance between and v.

## 3. Fuzzy Fredholm Integral Equation

In this section, the fuzzy integral equations of the second kind are introduced. The Fredholm integral equation of the second kind is [19]

$$f(x) = y(x) + \lambda \int_a^b k(x,t)f(t)dt$$

Where $\lambda > 0, k(x,t)$ is an arbitrary kernel function over the square a and b is fix. f and y are fuzzy functions on $[a,b]$. If $f(t)$ is a crisp function then the solutions of fuzzy integral equations is crisp .However, if y is a fuzzy function then this equation may only possess fuzzy solutions. Sufficient conditions for the existence of a unique solution to the fuzzy Fredholm integral equations of the second kind, i.e. fuzzy integral equations where $y(t)$ is fuzzy function, have been given in [19].

## 4. Analysis of the method

In this section, we recall the basic idea of the Optimal Homotopy Asymptotic Method (OHAM), consider the following differential equation:

$$A\big(u(\tau)\big) + f(\tau) = 0 \qquad (4)$$

subject to a boundary condition :

$$B(\tau) = 0 \qquad (5)$$

Where $A$ is a general differential operator, $B$ is a boundary operator, and $f(\tau)$ is a known function.

The operator can, generally speaking, be divided into two parts: $L$ a linear part and $N$ a nonlinear part. By means of OHAM one first constructs a family of equations and the boundary condition is:

$$(1-p)[L(\varphi(\tau,p)) + f(\tau)] = H(p)\big[L\big(\varphi(\tau,p)\big) + f(\tau) + N\big(\varphi(\tau,p)\big)\big] \qquad (6)$$

$$B\big(\varphi(\tau,p)\big) = 0 \qquad (7)$$

In Eq. (6), $\varphi(\tau,p)$ is an unknown function, $p \in [0,1]$ is an embedding parameter and $H(p)$ is a nonzero auxiliary function for $p \neq 0$ and $(0) = 0$ . Thus, when $p$ increases from 0 to 1 the solution $\varphi(\tau,p)$ changes between the initial guess $u_0(\tau)$ and the solution $u(\tau)$ . Obviously, when $p = 0$ and $p = 1$ it holds that

$$\varphi(\tau,0) = u_0(\tau), \qquad \varphi(\tau,1) = u(\tau) \qquad (8)$$

We choose the auxiliary function $H(p)$ in the form

$$H(p) = pC_1 + p^2C_2 + p^3C_3 + \cdots, \qquad (9)$$

where $C_1, C_2, C_3, \cdots$ are constants which can be determined later. Expanding in a series with respect to one has

$$B(\tau,p,C_i) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau,C_i)p^m, \qquad i = 1,2,\cdots. \qquad (10)$$

Substituting Eq. (9) into Eq. (4), collecting the same powers of $p$ , and equating each coefficient of $p$ to zero, we obtain:

$$L\big(u_0(\tau)\big) + f(\tau) = 0, \qquad B(u_0) = 0$$

$$L\big(u_1(r)\big) = C_1N_0\big(u_0(\tau)\big), \qquad B(u_1) = 0 \qquad (11)$$

$$L\big(u_m(\tau)\big) + u_{\{m-1\}}(\tau) = C_mN_0\big(u_0(r)\big) + \textstyle\sum_{i=}^{m-1} C_i\, L\big(u_{m-1}(\tau)\big) +$$

$$N_{m-1}\big(u_0(\tau), u_1(\tau), \cdots, u_{m-1}(\tau)\big), \quad \mathrm{m} = 2,3,\cdots, \qquad \mathrm{B}(u_m) = 0$$

Where $N_i$, $i \geq 0$ , are the coefficients of $p_i$ in the nonlinear operator N:

$$N\big(u(\tau)\big) = N_0\big(u_0(\tau)\big) + pN_1\big(u_0(\tau), u_1(\tau)\big) + p^2N_2\big(u_0(\tau), u_1(\tau), u_2(\tau)\big) + \cdots \qquad (12)$$

It should be emphasized that the $u_k$ for $k \geq 0$ are governed by the linear Eq. (10) with the linear boundary conditions that come from original problem, which can be easily solved. The convergence of the series Eq. (9) depends upon the auxiliary constants $C_1, C_2, \cdots$ . If it is convergent at $P = 1$ , one has

$$u(\tau,C_i) = u_0(\tau) + \sum_{m=0}^{\infty} u_m(\tau,C_i) \qquad (13)$$

The result of the $k$ th order approximations are given:

$$\overline{u}^k = u_0(\tau) + \sum_{m=1}^{k} u_m(\tau, C_i) \tag{14}$$

Substituting Eq. (14) into Eq. (4), there results the following residual:

$$R(\tau, C_i) = L\left(\overline{u}^k(\tau, C_i)\right) + f(\tau) + N\left(\overline{u}^k(\tau, C_i)\right), \quad i = 1,2,\cdots \tag{15}$$

If $R\left(\overline{u}^k(\tau, C_i)\right) = 0$ then $\overline{u}^k(\tau, C_i)$ happens to be the exact solution. Generally such a case will not arise for nonlinear problems, but we can minimize the functional:

$$J(C_1, C_2, \cdots, C_n) = \int_a^b R^2\,(\tau, C_1, C_2, \cdots, C_n)\, d\tau \tag{16}$$

Where $a$ and $b$ are two values, depending on the given problem. With these constants known, the approximate solution (of order ) is well determined.

## 4. Optimal Homotopy Asymptotic Method for Fuzzy Integral Equations

Consider the following Fuzzy Integral Equations

$$y(s) = f(s) + \lambda \int_a^b k(x,t) y(t) dt \tag{24}$$

for Eqs. (24) reads

$$L\underline{\varphi}\,(x,t) = \underline{\varphi}(x,t)$$
$$L\overline{\varphi}\,(x,t) = \overline{\varphi}(x,t)$$

$$N\underline{\varphi}\,(x,t) = -f(x) - \lambda \int_a^b k(x,t)\underline{\varphi}\,(x,t) dt$$
$$N\overline{\varphi}\,(x,t) = -f(x) - \lambda \int_a^b k(x,t)\overline{\varphi}\,(x,t) dt$$

a linear part L and a nonlinear part N . By means of OHAM one first constructs a family of equations: And the boundary condition is:

$$(1-p)\left(L\underline{\varphi}\,(x,t) - f(x)\right) = H(p)\left(L\underline{\varphi}\,(x,t) + N\underline{\varphi}\,(x,t) - f(x)\right)$$
$$(1-p)\left(L\overline{\varphi}\,(x,t) - f(x)\right) = H(p)(L\overline{\varphi}\,(x,t) + N\overline{\varphi}\,(x,t) - f(x))$$

That

$$\varphi(x,t) = \sum_{i=0}^{m} \varphi_i(x,p)(x), \quad H(p) = \sum_{i=1}^{m} c_i p^i,$$

Rearranging based on powers of p-terms.

## 5. Numerical results

To give a clear overview of the method and show the ability of the method, we present the following an example.

**Example.** Consider the fuzzy integral equation where [5]

$$\underline{f}(x;\alpha) = x^3(\alpha + \alpha^2),$$
$$\overline{f}(x;r) = x^3(4 - \alpha - \alpha^3),$$

and kernel

$$k(x,t) = x + 1, \qquad -1 \leq x,t \leq 1.$$

and a = −1, b = 1. The exact solution in this case is given

$$\underline{y}(x,r) = x^3(\alpha^2 + \alpha)$$
$$\underline{y}(x,r) = s^3(4 - \alpha^3 - \alpha) \ .$$

To apply the OHAM, first we rewrite Fuzzy Integral Equations in the following form

$$L\underline{\varphi}\ (x,t) + N\underline{\varphi}\ (x,t) - \underline{\underline{f}}(x;\alpha) = 0$$
$$L\overline{\varphi}\ (x,t) + N\overline{\varphi}\ (x,t) - \overline{f}(x;\alpha) = 0$$

Following the OHAM formulation presented in Section 3 we start with:

$$L\underline{\varphi}\ (x,t) = \underline{y}(x,\alpha), \qquad L\overline{\varphi}\ (x,t) = \overline{y}(x,\alpha)$$
$$N\underline{\varphi}\ (x,t) = -\lambda \int_a^b k(x,t)\underline{\varphi}\ (x,t)dt, \qquad N\overline{\varphi}\ (x,t) = -\lambda \int_a^b k(x,t)\overline{\varphi}\ (x,t)dt$$

that

$$\varphi(x,t) = \sum_{i=0}^{m} \varphi_i(x,p)(x)\,, \qquad H(p) = \sum_{i=1}^{m} c_i p^i$$

This generates a series of problems the first of these is the zeroth order problem. We have using Mathematics software.

$$-\underline{\alpha}x^3 - \underline{\alpha}^2 s x^3 + \underline{\mathrm{y}_0}(\mathrm{x},\underline{\alpha}) = 0$$
$$-4x^3 + \overline{\alpha}x^3 + \overline{\alpha}^3 x^3 + \overline{\mathrm{y}_0}(\mathrm{x},\overline{\alpha}) = 0$$

we obtain:

$$\underline{\mathrm{y}_0}(\mathrm{x},\underline{\alpha}) = \underline{\alpha}x^3 + \underline{\alpha}^2 x^3,$$
$$\overline{\mathrm{y}_0}(\mathrm{x},\overline{\alpha}) = -4x^3 + \overline{\alpha}x^3 + \overline{\alpha}^3 x^3,$$

The first-order problem is defined as:

$$\underline{\alpha}x^3+\underline{\alpha}^2x^3+\underline{\alpha}\,x^3\underline{c}_1+\underline{\alpha}^2x^3\underline{c}_1+\left(\int_{-1}^{1}(1+t)\left(\underline{y_0}(t,\underline{\alpha})\right)dt\right)\underline{c}_1-\underline{y_0}(x,\underline{\alpha})-\underline{c}_1\underline{y_0}(x,\underline{\alpha})$$
$$+\underline{y_1}(x,\underline{\alpha})=0,$$
$$4x^3-\overline{\alpha}x^3-\overline{\alpha}^3x^3+4x^3\overline{c_1}-\overline{\alpha}x^3\overline{c_1}-\overline{\alpha}^3x^3\overline{c_1}+\left(\int_{-1}^{1}(1+t)\left(\overline{y_0}(t,\overline{\alpha})\right)ds\right)\overline{c_1}-\overline{y_0}(x,\overline{\alpha})$$
$$-c_1\overline{y_0}(x,\overline{\alpha})+\overline{y_1}(x,\overline{\alpha})=0,$$

we obtain:

$$\underline{y_1}(x,\underline{\alpha})=-\frac{2}{5}\left(\underline{\alpha}\underline{c}_1+\underline{\alpha}^2\underline{c}_1\right)$$
$$\overline{y_1}(x,\overline{\alpha})=\frac{2}{5}(4c_1+\overline{\alpha}c_1+\overline{\alpha}^3c_1);,$$

The second-order problem is defined as:

$$\underline{\alpha}x^3\underline{c}_2+\underline{\alpha}^2x^3\underline{c}_2+\left(\int_{-1}^{1}(1+t)(\underline{y_1}(t,\underline{\alpha}))\,dt\right)\underline{c}_2-\underline{c}_2\underline{y_0}(x,\underline{\alpha})-\underline{y_1}(x,\underline{\alpha})-\underline{c}_1\underline{y_1}(x,\underline{\alpha})$$
$$+\underline{y_2}(x,\underline{\alpha})=0$$
$$4x^3\overline{c_2}-\overline{\alpha}x^3\overline{c_2}-\overline{\alpha}^3x^3\overline{c_2}+\left(\int_{-1}^{1}(1+t)(\overline{y_1}(t,\overline{\alpha}))\,dt\right)\overline{c_2}-\overline{c_2y_0}(x,\overline{\alpha})-\overline{y_1}(x,\overline{\alpha})-\overline{c_1y_1}(x,\overline{\alpha})$$
$$+\overline{y_2}(x,\overline{\alpha})=0$$

we obtain:

$$\underline{y_2}(x,\underline{\alpha})=-\frac{2}{5}\left(\underline{\alpha}\underline{c}_1+\underline{\alpha}^2\underline{c}_1+\underline{\alpha}\underline{c}_1{}^2+\underline{\alpha}^2\underline{c}_1{}^2-2\underline{\alpha}\underline{c}_1\underline{c}_2-2\underline{\alpha}^2\underline{c}_1\underline{c}_2\right),$$
$$\overline{y_2}(x,\overline{\alpha})=\frac{2}{5}\left(4\overline{c_1}+\overline{\alpha c_1}+\overline{\alpha}^3\overline{c_1}+4\overline{c_1}^2+\overline{\alpha c_1}^2+\overline{\alpha}^3\overline{c_1}^2-8\overline{c_1c_2}-2\overline{\alpha c_1c_2}-2\overline{\alpha}^3\overline{c_1c_2}\right),$$

The third-order problem is defined as:

$$\underline{\alpha}x^3\underline{c}_3+\underline{\alpha}^2x^3\underline{c}_3+\left(\int_{-1}^{1}(1+t)(\underline{y_2}(t,\underline{\alpha})))\,dt\right)\underline{c}_3-\underline{c}_3\underline{y_0}(x,\underline{\alpha})-\underline{c}_2\underline{y_1}(x,\underline{\alpha})-\underline{y_2}(x,\underline{\alpha})$$
$$-\underline{c}_2\underline{y_2}(x,\underline{\alpha})+\underline{y_3}(x,\underline{\alpha})=0$$
$$4x^3\overline{c_3}-\overline{\alpha}x^3\overline{c_3}-\overline{\alpha}^3x^3\overline{c_3}+\left(\int_{-1}^{1}(1+t)(\overline{y_2}(t,\overline{\alpha})))\,dt\right)\overline{c_3}-\overline{c_3y_0}(x,\overline{\alpha})-\overline{c_2y_1}(x,\overline{\alpha})-\overline{y_2}(x,\overline{\alpha})$$
$$-\overline{c_1y_2}(x,\overline{\alpha})+\overline{y_3}(x,\overline{\alpha})=0$$

we obtain:

$$\underline{y_3}(x,\underline{\alpha})=-\frac{2}{5}\Big(\underline{\alpha}\underline{c}_1+\underline{\alpha}^2\underline{c}_1+2\underline{\alpha}\underline{c}_1^2+2\underline{\alpha}^2\underline{c}_1{}^2+\underline{\alpha}\underline{c}_1{}^3+\underline{\alpha}^2\underline{c}_1{}^3-\underline{\alpha}\underline{c}_1\underline{c}_2-\underline{\alpha}^2\underline{c}_1\underline{c}_2-2\underline{\alpha}\underline{c}_1{}^2\,\underline{c}_2$$
$$-2\underline{\alpha}^2\underline{c}_1{}^2\,\underline{c}_2-2\underline{\alpha}\underline{c}_1\underline{c}_3-2\underline{\alpha}^2\underline{c}_1\underline{c}_3-2\underline{\alpha}\underline{c}_1{}^2\,\underline{c}_3-2\underline{\alpha}^2\underline{c}_1{}^2\,\underline{c}_3+4\underline{\alpha}\underline{c}_1\underline{c}_2\underline{c}_3$$
$$+4\underline{\alpha}^2\underline{c}_1\underline{c}_2\underline{c}_3\Big),$$

$$\overline{y_3}(\mathrm{x},\overline{\alpha}) = \frac{2}{5}\left(4\overline{c_1} + \overline{\alpha c_1} + \overline{\alpha}^3\overline{c_1} + 8\overline{c}_1^2 + 2\overline{\alpha c}_1^2 + 2\overline{\alpha}^3\overline{c}_1^2 + 4\overline{c}_1^3 + \overline{\alpha c}_1^3 + \overline{\alpha}^3\overline{c}_1^3 - 4\overline{c_1c_2} - \overline{\alpha c_1c_2} - \overline{\alpha}^3\overline{c_1c_2} - 8\overline{c}_1^2\overline{c_2} - 2\overline{\alpha c}_1^2\overline{c_2} - 2\overline{\alpha}^3\overline{c}_1^2\overline{c_2} - 8\overline{c_1c_3} - 2\overline{\alpha c_1c_3} - 2\overline{\alpha}^3\overline{c_1c_3} - 8\overline{c}_1^2\overline{c_3} - 2\overline{\alpha c}_1^2\overline{c_3} - 2\overline{\alpha}^3\overline{c}_1^2\overline{c_3} + 16\overline{c_1c_2c_3} + 4\overline{\alpha c_1c_2c_3} + 4\overline{\alpha}^3\overline{c_1c_2c_3}\right),$$

By OHAM for p = 1 is:

$$\underline{y}(x,\underline{\alpha}) = \underline{\mathrm{y}_0}(\mathrm{x},\underline{\alpha}) + \underline{\mathrm{y}_1}(\mathrm{x},\underline{\alpha}) + \underline{\mathrm{y}_2}(\mathrm{x},\underline{\alpha}) + \underline{\mathrm{y}_3}(\mathrm{x},\underline{\alpha}),$$
$$\overline{y}(x,\overline{\alpha}) = \overline{\mathrm{y}_0}(\mathrm{x},\overline{\alpha}) + \overline{\mathrm{x}_1}(\mathrm{x},\overline{\alpha}) + \overline{\mathrm{y}_2}(\mathrm{x},\overline{\alpha}) + \overline{\mathrm{y}_3}(\mathrm{x},\overline{\alpha}),$$

Following the procedure described in Section 3 on the domain between a = -1 and b = 1, using the residual,

$$\underline{R} = \underline{y}(x,\underline{\alpha}) - \int_{-1}^{1}(t+1)(\underline{y}(t,\underline{\alpha}))\,\mathrm{d}t - \underline{f}(x;\underline{\alpha}),$$
$$\overline{R} = \overline{y}(x,\overline{\alpha}) - \int_{-1}^{1}(t+1)(\overline{y}(x,\overline{\alpha}))\,\mathrm{d}t - \overline{f}(x;\overline{\alpha}),$$

Put $\underline{\alpha}$ = 0.1 and $\overline{\alpha}$ = 1.

the following values of $c_i$'s are obtained :

$$\underline{c_1} = -57.973, \quad \underline{c_2} = -38.4271, \quad \underline{c_3} = -28.9278,$$
$$\overline{c_1} = -3, \quad \overline{c_2} = -1, \quad \overline{c_3} = -0.75,$$

By considering these values our approximate solution becomes,

$$\underline{y}(t,\underline{\alpha}) = 1.14575 \times 10^{-13} + 2t^3,$$
$$\overline{y}(t,\overline{\alpha}) = 2.69296 \times 10^{-12} + 0.11\,t^3,$$

We earn suitable $\underline{c_i}$ and $\overline{c_i}$ in tables 1 and 2:

**Table 1. For $\underline{\mathrm{y}}(\mathrm{x},\underline{\alpha})$ (x = 0. 1)**

| $\underline{\alpha}$ | $\underline{c_1}$ | $\underline{c_2}$ | $\underline{c_3}$ | Exact | OHAM |
|---|---|---|---|---|---|
| 0.1 | -57.973 | -38.4271 | -38.4271 | 1.1(-4) | 1.1 (-4) |
| 0.2 | -73.2968 | -38.0852 | -40.4353 | 2.4(-4) | 2.4(-4) |
| 0.3 | -34.909 | -9.11796 | -16.1955 | 3.9(-4) | 3.90001(-4) |
| 0.4 | -12.351 | -6.42951 | -6.42951 | 5.6(-4) | 5.6(-4) |
| 0.5 | -40.5846 | 211.16 | -19.522 | 7.5(-4) | 7.5(-4) |
| 0.6 | -4.08846 | -0.55633 | -1.15651 | 9.6(-4) | 9.6(-4) |
| 0.7 | -3.00636 | 0.98691 | -0.704309 | 1.19(-3) | 1.19(-3) |
| 0.8 | -3.06961 | -0.943058 | -0.689965 | 1.44(-3) | 1.44(-3) |
| 0.9 | -0.654751 | 0.37351 | -1.03676 | 1.77(-3) | 1.77(-3) |

**Table 1. For $\overline{y}(x, \overline{\alpha})$ (x = 0. 1)**

| $\overline{\alpha}$ | $\overline{c_1}$ | $\overline{c_2}$ | $\overline{c_3}$ | **Exact** | **OHAM** |
|---|---|---|---|---|---|
| 0.1 | -7.97336 | -4.04438 | -4.35138 | 3.184(-3) | 3.184(-3) |
| 0.2 | -6.14848 | - 2.51466 | 4.28123 | 3.792(-3) | 3.792(-3) |
| 0.3 | -6.14848 | -2.51466 | 4.28123 | 3.673(-3) | 3.673(-3) |
| 0.4 | -51.8861 | -25.2093 | 0.947523 | 3.536(-3) | 3.536(-3) |
| 0.5 | -1432126 | -5.17019 | -5.38035 | 3.375(-3) | 3.375(-3) |
| 0.6 | -14.6667 | 10.2697 | -6.49807 | 3.184(-3) | 3.184(-3) |
| 0.7 | -2.99685 | -1.00644 | -0.6993 | 2.957(-3) | 2.957(-3) |
| 0.8 | -1.57645 | 0.39709 | -0.297878 | 2.688(-3) | 2.957(-3) |
| 0.9 | -44.1662 | -23.8564 | -23.5967 | 2.371(-3) | 2.37171(-3) |
| 1 | -3 | -1 | -0.75 | 2(-3) | 2(-3) |

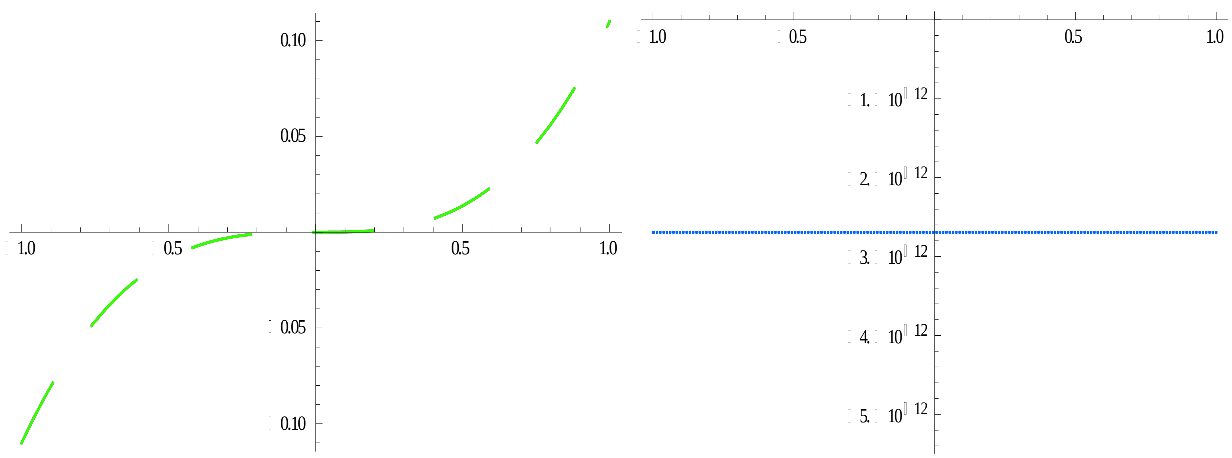


figure 1.a. exact solution and solution by OHAM ($\underline{\alpha} = 0.1$ , $t = 0.1$), b. error in OHAM

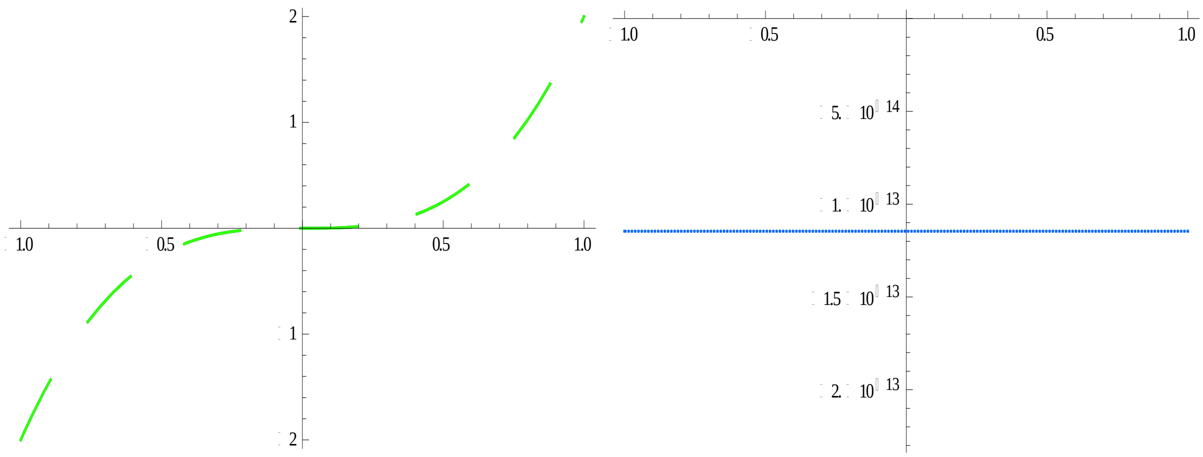


figure 2.a. exact solution and solution by OHAM ($\overline{\alpha} = 1, t = 0.1$), b. error in OHAM

## 5. Conclusion

In this paper, the Optimal Homotopy Asymptotic Method (OHAM) has been successfully applied to finding solutions of fuzzy integral equations. The solution obtained by the Optimal Homotopy Asymptotic Method (OHAM) is an infinite power series for which, with appropriate initial condition, can be expressed in a closed form, the exact solution. The results presented in this contribution show that the Optimal Homotopy Asymptotic Method (OHAM) is a powerful mathematical tool to solving fuzzy differential equation. For fuzzy differential equation, its exact solution can be obtained by Optimal Homotopy Asymptotic Method (OHAM) due to the fact that the Lagrange multiplier can be exactly identified. They converted the fuzzy differential equation to a system of linear fuzzy differential equations which is contain n, fuzzy integral equations of order one. But we directly applied Optimal Homotopy Asymptotic Method (OHAM) for this type of equations.Convergency of Optimal Homotopy Asymptotic Method (OHAM) has been discussed for fuzzy differential equation. It is also a promising method to solve other linear equations.

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