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Introducing a Novel Bivariate Generalized Skew-Symmetric Normal Distribution

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Abstract

We introduce a generalization of the bivariate generalized skew-symmetric normal distribution [5]. We denote this distribution by $SGN_n(\lambda_1,\lambda_2)$. We obtain some properties of $SGN_n(\lambda_1,\lambda_2)$ and derive the moment generating function.

Keywords: Generalized-skew-normal distribution, $SGN_n(\lambda_1,\lambda_2)$, Conditional distribution.

1. Introduction

The skew-normal distribution introduced by Azzalini [2]. This density has been studied and generalized by some researchers. For example, Azzalini and Dalla Valle [4], Azzalini and Capitanio [3], Arellano-Valle [1], Jamalizadeh [7], Sharafi and Behbodian [8], Hasanalipour and sharafi [6] and Yadegari [9].
Fathi and Hasanalipour [5] considered a generalization of $SGN_2(\lambda_1,\lambda_2)$ distribution and they called it the bivariate generalized skew-symmetric normal distribution.
Its probability density function is given by

$$f(x,y;\lambda_1,\lambda_2)=2\phi(x)\phi(y)\Phi\left(\frac{\lambda_1 xy}{1+\lambda_2(xy)^2}\right),\qquad x,y\in R,\quad \lambda_1\in R,\qquad \lambda_2\geq 0 \qquad (1)$$

This distribution denoted by $(X,Y)\sim SGN_2(\lambda_1,\lambda_2)$. In this paper, we introduce a new family of skew-normal distribution which generalizes (1) while preserving most of its properties.
In section 2, we present the definition and some properties of $SGN_n(\lambda_1,\lambda_2)$ class and section 3, gives some important theorems about conditional distributions of $SGN_n(\lambda_1,\lambda_2)$.

2. A novel bivariate generalized skew-symmetric normal distribution

In this section, we define the $SGN_n(\lambda_1,\lambda_2)$ class and obtain some its properties.

2.1. $SGN_n(\lambda_1,\lambda_2)$

Definition 1. Vector (X,Y) has $SGN_n(\lambda_1,\lambda_2)$ distribution if and only if for every $n \geq 1$ it has the following density

$$f_n(x,y;\lambda_1,\lambda_2)=c_n(\lambda_1,\lambda_2)\phi(x)\phi(y)\Phi^n(\frac{\lambda_1 x\,y}{\sqrt{1+\lambda_2(xy)^2}}), \qquad x,y\in R, \tag{2}$$

where $\lambda_1 \in R$ and $\lambda_2 \geq 0$. The coefficient $c_n(\lambda_1,\lambda_2)$, which is a function of n and the parameters λ_1,λ_2 is given by

$$c_n(\lambda_1,\lambda_2)=\frac{1}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x)\phi(y)\Phi^n(\frac{\lambda_1 x\,y}{\sqrt{1+\lambda_2(xy)^2}})dx\,dy}, \tag{3}$$

with this properties:

1. $\lim_{\lambda_1\to\infty} c_n(\lambda_1,\lambda_2)=4$ for all $\lambda_2 \geq 0$

2. $c_n(-\lambda_1,\lambda_2)=c_n(\lambda_1,\lambda_2)$

We denote this by $(X,Y)\sim SGN_n(\lambda_1,\lambda_2)$.

2.2. Some simple properties of $SGN_n(\lambda_1,\lambda_2)$

We now present some properties of this novel distribution.

1. $SGN_n(0,\lambda_2)=\phi(x)\phi(y)$, for all $\lambda_2 \geq 0$.

2. $SGN_1(\lambda_1,0)=2\phi(x)\phi(y)\Phi(\lambda_1 xy)$.

3. $X\mid\{Y=y\}=c_n(\lambda_1 y)\phi(x)\Phi^n(\frac{\lambda_1 x\,y}{\sqrt{1+\lambda_2(xy)^2}})\sim GBSN_n(\lambda_1 y)$ [6].

4. $Y\mid\{X=x\}=c_n(\lambda_1 x)\phi(y)\Phi^n(\frac{\lambda_1 x\,y}{\sqrt{1+\lambda_2(xy)^2}})\sim GBSN_n(\lambda_1 x)$ [6].

5. If $(X,Y)\sim SGN_n(\lambda_1,\lambda_2)$, then $(-X,Y)\sim SGN_n(-\lambda_1,\lambda_2)$, $(X,-Y)\sim SGN_n(-\lambda_1,\lambda_2)$ and

$(-X,-Y)\sim SGN_n(\lambda_1,\lambda_2)$ [1].

6. $\lim_{\lambda_1 \to \infty} f_n(x, y; \lambda_1, \lambda_2) = 4\phi(x)\phi(y)\, I_{\{x>0, y>0\}}$.

7. $\lim_{\lambda_1 \to -\infty} f_n(x, y; \lambda_1, \lambda_2) = 4\phi(x)\phi(y)\, I_{\{x<0, y<0\}}$.

8. $\lim_{\lambda_1 \to \infty} \left\{ f_{X|Y}(x, y; \lambda_1, \lambda_2) + f_{X|Y}(x, y; -\lambda_1, \lambda_2) \right\} = 2\phi(x)$.

9. $\lim_{\lambda_1 \to \infty} \left\{ f_{Y|X}(x, y; \lambda_1, \lambda_2) + f_{Y|X}(x, y; -\lambda_1, \lambda_2) \right\} = 2\phi(y)$.

2. Some theorems about conditional distributions of $SGN_n(\lambda_1, \lambda_2)$

Theorem 1. If $X, Y, Z_1, \ldots, Z_n$ are i.i.d. $N(0,1)$ distribution then we have:

$$(X, Y) \Big| \left\{ Z_{(n)} \le \frac{\lambda_1 XY}{\sqrt{1+\lambda_2 (XY)^2}} \right\} \sim SGN_n(\lambda_1, \lambda_2) \qquad (4)$$

Where $Z_{(n)} = \max\{Z_1, \ldots, Z_n\}$.

Proof: Suppose $A = (Z_{(n)} \le \frac{\lambda_1 XY}{\sqrt{1+\lambda_2 (XY)^2}})$. Then, we write

$$f_{(X,Y)|A}(x, y | A) = \frac{P(A | X = x, Y = y) f(x, y)}{P(A)}$$

$$= \frac{P\left(Z_{(n)} \le \frac{\lambda_1 XY}{\sqrt{1+\lambda_2 (XY)^2}} \,\Big|\, X = x, Y = y\right)\phi(x)\phi(y)}{P\left(Z_{(n)} \le \frac{\lambda_1 XY}{\sqrt{1+\lambda_2 (XY)^2}}\right)}$$

$$= \frac{P\left(Z_1 \le \frac{\lambda_1 xy}{\sqrt{1+\lambda_2 (xy)^2}}, \ldots, Z_n \le \frac{\lambda_1 xy}{\sqrt{1+\lambda_2 (xy)^2}}\right)\phi(x)\phi(y)}{P\left(Z_1 \le \frac{\lambda_1 XY}{\sqrt{1+\lambda_2 (XY)^2}}, \ldots, Z_n \le \frac{\lambda_1 XY}{\sqrt{1+\lambda_2 (XY)^2}}\right)}$$

$$= c_n(\lambda_1, \lambda_2)\phi(x)\phi(y)\Phi^n\left(\frac{\lambda_1 x\, y}{\sqrt{1+\lambda_2 (xy)^2}}\right)$$

For random number generation, it is more efficient to use single variant of this result, namely to put

$$Z=(Z_1,Z_2)=\begin{cases}(X,Y) & Z_{(n)}\le \dfrac{\lambda_1 XY}{\sqrt{1+\lambda_2(XY)^2}}\\ (-X,-Y) & Z_{(n)}> \dfrac{\lambda_1 XY}{\sqrt{1+\lambda_2(XY)^2}}\end{cases} \tag{5}$$

This make an important point for $SGN_n(\lambda_1,\lambda_2)$ distribution, comparing with acceptance-rejection method simulation of independent normal distribution.

Theorem 2. If $(X,Y)\sim SGN_n(\lambda_1,\lambda_2)$, then $(X^2|Y)\xrightarrow{L}\chi^2_{(1)}$ as $\lambda_1\to\infty$, where $\chi^2_{(1)}$ shows chi-square random variable with one degree of freedom.

Proof: Let $(X^2|Y)=Z$. The density of Z is

$$f_Z(z,y;\lambda_1,\lambda_2)=\frac{1}{\sqrt{z}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z}{2}}c_n(\lambda_1 y)\left[\frac{\Phi^n(\frac{\lambda_1\sqrt{z}\,y}{\sqrt{1+\lambda_2 zy^2}})+\Phi^n(\frac{-\lambda_1\sqrt{z}\,y}{\sqrt{1+\lambda_2 zy^2}})}{2}\right]$$

$$=f_{\chi^2_{(1)}}(z)\left[a_n(z,y;\lambda_1,\lambda_2)\right]; \qquad z>0$$

with

$$a_n(z,y;\lambda_1,\lambda_2)=c_n(\lambda_1 y)\left[\frac{\Phi^n(\frac{\lambda_1\sqrt{z}\,y}{\sqrt{1+\lambda_2 zy^2}})+\Phi^n(\frac{-\lambda_1\sqrt{z}\,y}{\sqrt{1+\lambda_2 zy^2}})}{2}\right]$$

Since $c_n(\lambda_1 y)\to 2$ *as* $\lambda_1\to\infty$, we conclude that $a_n(z,y;\lambda_1,\lambda_2)\to 1$, *as* $\lambda_1\to\infty$. Therefore, the density $f_Z(z,y;\lambda_1,\lambda_2)$ converges to the distribution of $\chi^2_{(1)}$, as $\lambda_1\to\infty$. Hence, the distribution of Z converges to the distribution of $\chi^2_{(1)}$, i.e. $Z=(X^2|Y)\xrightarrow{L}\chi^2_{(1)}$.

Theorem 3. If $(X,Y)\sim SGN_n(\lambda_1,\lambda_2)$ and $Z\sim N(0,1)$, then $\frac{|X|}{Y}$ and $|Z|$ are identically distributed, i.e, $\lim\limits_{\lambda_1\to\infty}\frac{|X|}{Y}\xrightarrow{D}|Z|\sim HN(0,1)$, where $HN(0,1)$ denotes the standard half-normal distribution.

Proof: We know that $|Z|$ has density $2\phi(z)\,I_{\{z>0\}}$. The density $W=\frac{|X|}{Y}$ is

$$\begin{aligned} f_W(w) &= f_{X|Y|}(w) + f_{X|Y|}(-w) \\ &= c_n(\lambda_1 y)\phi(w)\Phi^n\left(\frac{\lambda_1 w\, y}{\sqrt{1+\lambda_2 (wy)^2}}\right) + c_n(\lambda_1 y)\phi(-w)\Phi^n\left(\frac{-\lambda_1 w\, y}{\sqrt{1+\lambda_2 (wy)^2}}\right) \\ &= c_n(\lambda_1 y)\phi(w)\left[\Phi^n\left(\frac{\lambda_1 w\, y}{\sqrt{1+\lambda_2 (wy)^2}}\right) + \Phi^n\left(\frac{-\lambda_1 w\, y}{\sqrt{1+\lambda_2 (wy)^2}}\right)\right] \\ &= \phi(w)\left[b_n(w, y; \lambda_1, \lambda_2)\right] \end{aligned}$$

Now, we can show that $b_n(w, y; \lambda_1, \lambda_2) \to 2$ as $\lambda_1 \to \infty$, then $\lim_{\lambda_1 \to \infty} W = 2\phi(w)$ for $w > 0$ and we have

$\lim_{\lambda_1 \to \infty} \frac{|X|}{Y} \xrightarrow{D} |Z|$.

Theorem 4. The moment generating function $(X, Y) \sim SGN_n(\lambda_1, \lambda_2)$ is

$$M_{X,Y}(t_1, t_2) = c_n(\lambda_1, \lambda_2) e^{\frac{t_1^2 + t_2^2}{2}} E\left\{E\left\{\Phi^n\left(\frac{\lambda_1 WK}{\sqrt{1+\lambda_2 (WK)^2}}\right)\right\}\right\}$$

where $W \sim N(t_1, 1),\ K \sim N(t_2, 1)$.

Proof:

$$\begin{aligned} M_{X,Y}(t_1, t_2) &= E(e^{t_1 X + t_2 Y}) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_n(\lambda_1, \lambda_2) e^{t_1 x + t_2 y} \phi(x)\phi(y)\Phi^n\left(\frac{\lambda_1 x\, y}{\sqrt{1+\lambda_2 (xy)^2}}\right) dx\, dy \\ &= c_n(\lambda_1, \lambda_2) e^{\frac{t_1^2 + t_2^2}{2}} E\left\{E\left\{\Phi^n\left(\frac{\lambda_1 WK}{\sqrt{1+\lambda_2 (WK)^2}}\right)\right\}\right\}. \end{aligned}$$

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