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Common Fixed Point Theorem for Uniformly R-subweakly Commuting Mappings in Hausdorff Locally Convex Space

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Abstract

We show common fixed point theorem for generalized non-expansive and uniformly R −subweakly commuting mappings satisfying a more uniformly asymptotically regular condition in Hausdorff locally convex space.

Keywords : Uniformly R − subweakly commuting mapping , non-expansive ,uniformly asymptotically regular .

1. Introduction and Preliminaries

With introduction of a class R-weakly commuting mappings , Pant obtained common fixed point results. We introduce a new class of commuting mappings namely R −subweakly commuting mappings and R −subcommuting mappings. Simultaneously, Hussain and Berinde [6] proved common fixed point results for generalized non-expansive R −subweakly commuting mappings on non-starshaped domain. More recently , Ismat Beg et al[2] extended Cho's result to asymptotically I non-expansive mappings introducing a new class of non-commuting mappings as "uniformly R −subweakly commuting mappings". In this paper we present definition of uniformly R −subweakly commuting mappings on Hausdorff locally convex space that it is extension the definition of uniformly R −subweakly commuting on normed space.

The purpose of this paper is to generalize Beg's result for generalized asymptotically I non-expansive and uniformly R −subweakly commuting mappings . Also we present a common fixed point lemma and apply it to find new common fixed point result. The existence of common fixed point is established for two mappings T and I that are uniformly R −subweakly commuting on a Hausdorff locally convex space.

Let (X ,τ) be a Hausdorff locally convex topological vector space and M be a τ-bounded subset of X.

Definition 1.1: Let (X ,τ) be a Hausdorff locally convex topological vector space. A family $\{P_\alpha : \alpha \in I\}$ of semi-norms defined on X is said to be an associated family of semi-norms for τ if the family $\{\Upsilon u : \Upsilon > 0\}$, where

$$u = \bigcap_{i=1}^{n} u_{\alpha_i}.$$

$$u_{\alpha_i} = \{x : P_{\alpha_i}(x) < 1\}.$$

forms a base of neighborhoods of zero for τ. the associated family of semi norms for τ be denoted as A(τ)

Definition 1.2: A family $\{P_\alpha : \alpha \in I\}$ of semi-norms defined on X is called an augmented associated family for τ if $\{P_\alpha : \alpha \in I\}$ is an associated family with the property that the semi-norms,

$$\max\{P_\alpha, P_\beta\} \in \{P_\alpha : \alpha \in I\}.$$

for any α , β ∈ I. the augmented associated family of semi norms for τ be denoted by $A^*(\tau)$.

Definition 1.3: A subset M of X is τ -bounded iff each P_α is bounded on M.

Remark 1.4: If M is a τ -bounded subset of X, then a number $\lambda_\alpha > 0$ for each α ∈ I is selected such that $M \subset \lambda_\alpha u_\alpha$ where $u_\alpha = \{x : P_\alpha(x) \le 1\}$. Clearly B= $\bigcap_\alpha \lambda_\alpha u_\alpha$,τ _closed and τ _bounded, absolutely convex and contains M.

Definition 1.5: let T and I be self-maps on M. The map T is called

(1) $A^*(\tau)$-nonexpansive if for all $x, y \in M$

$$P_\alpha(Tx - Ty) \le P_\alpha(x - y).$$

For each $P_\alpha \in A^*(\tau)$.

(2) $A^*(\tau)$-I-nonexpansive if for all $x, y \in M$

$$P_\alpha(Tx - Ty) \le P_\alpha(Ix - Iy) .$$

For each $P_\alpha \in A^*(\tau)$.

For simplicity, we shall call $A^*(\tau)$-nonexpansive ($A^*(\tau)$-I-nonexpansive) maps to non-expansive (I −nonexpansive).

Definition 1.6: A self mapping T of M is said to be uniformly asymptotically regular on M if for each ε> 0, there exists $N(\varepsilon) = N$ such that $P_\alpha(T^n x - T^{n+1} x) < \varepsilon$, for all $n \ge N$, and $x \in M$.

Definition 1.7. A self mapping T of a nonempty subset M of a locally convex space X is said to be;

(1) *compact*, if $\{x_n\}$ is a bounded sequence in M, then $\{T x_n\}$ has a convergent subsequence $\{T x_m\}$ in M.

(2) *demiclosed* at 0, if for every net $\{x_n\}$ in M converging weakly to x and $\{T x_n\}$ converging strongly to 0, T x =0.

Definition 1.8: Two self mappings T and I defined on M is said to be $R-$ *commuting* if

$$P_\alpha(TIx - ITx) \le R P_\alpha(Ix - Tx).$$

For all $x \in M$ and $P_\alpha \in A^*(\tau)$ and real number $R > 0$ .if R=1 the self mappings T and I are said to be *commuting*.

Definition 1.9: Let M be nonempty , T and I –invariant and q −starshaped with $q \in F(I)$. Two self mappings T and I defined on M are said to be $R - subcommuting$ if for all $x \in M$ and $P_\alpha \in A^*(\tau)$,there exists $R > 0$ such that $P_\alpha(TIx - ITx) \leq \left(\frac{R}{K}\right) P_\alpha(((1-K)q + KTx) - Ix)$ for all $K \in (0,1)$. Moreover if $R = 1$ The self mappings T and I are said to be $subcommuting$.

Definition 1.10: Let M be T and I-invariant and $q - starshaped$ with q∈ $F(I)$. Two self mappings T and I are said to be $R - subweakly\ commuting$ if for all $x \in M$ and $P_\alpha \in A^*(\tau)$ There exists Real number $R > 0$ such that $P_\alpha(TIx - ITx) \leq R d_{P_\alpha}(Ix, [q, Tx])$. Such that

$$d_{P_\alpha}(Ix, [q, Tx]) = \inf\{P_\alpha(Ix - y): y \in [q, Tx]\}$$

for all $0 \leq K \leq 1$, where, $[q, Tx] = (1-K)q + KTx$.

Remark 1.11: It is clear from the definition that $commuting$ mappings are $R - subcommuting$ and $R - subweakly\ commuting$ mappings are $R - weakly\ commuting$ mappings.

Example 1.12: Let $X = \mathbb{R}$ with norm $\|x\| = |x|$ and $= [1, \infty)$. Let two self mappings I and T of M be defined as:

$$Tx = x^2$$

$$Ix = 2x - 1$$

Then T and I are $R - weakly\ commuting$ mappings to R $= 2$. But T and I are not $R - subcommuting$.

Example 1.13: Let $X = [0,1]$ with Euclidean metric and define

$$Tx = \frac{x}{x+4}$$

$$Ix = \frac{x}{2}$$

For $\in X$. But for any $x \neq 0$

$$TIx = \frac{x}{x+8} > \frac{x}{2x+8} = ITx$$

Example 1.14: Let $X = \mathbb{R}$ with norm $\|x\| = |x|$ and $= [1, \infty]$. Let two self mappings T and I of M be defined as

$$Tx = 4x - 3$$

$$Ix = 2x^2 - 1$$

Then M is T and $I - invariant$ and $P - starshaped$ with $P = 1 \in F(I)$ and

$$|TIx - ITx| = 24(x-1)^2$$

Moreover for all $x \in M$ and R= 12 , $P = 1 = F(I)$ We have

$$|TIx - ITx| \leq (\frac{R}{K})|KTx + (1-K)P - Ix|$$

Therefore two self mappings T and I are $R - subweakly\ commuting$ mappings , but are not $commuting$.

Example 1.15: Let $X = \mathbb{R}^2$ with norm $\|(x, y)\| = \max\{|x|, |y|\}$. And define T and I by

$$\mathrm{T(x, y) = (2x - 3, y^3)}$$

$$\mathrm{I(x, y) = (x^2, y^2)}$$

Then two self mappings T and I are $R - subweakly\ commuting$ on $M = \{(x, y): x \geq 1 , y \geq 1\}$

, but are not $commuting$.

2. main result

The following is the definition of uniformly $R-$ $subweakly\ commuting$ mappings in the subset of normed space.

Definition 2.1: If M is $q-$ $starshaped$ subset of normed space with q ∈ F(T), then T and I are said to be $uniformly\ R-$ $subweakly\ commuting$ on $M \setminus \{q\}$ if there exists a real number $R > 0$ such that $\|T^n Ix - IT^n x\| \leq Rdist(Ix, [T^n x, q])$ for all $x \in M$ and $R > 0$ where

$dist(Ix, [\,T^n x, q]) = inf\ \{\|Ix - y_n\|: y_n \in [\,T^n x, q] =$
$\alpha_n T^n\, x + (1 - \alpha_n\,)q\ , where\ \{\alpha_n\,\}\ is\ a\ sequence\ of\ real\ numbers\ such\ that\ 0 < \alpha_n <$
$1\ and\ \lim_{n\to\infty} \alpha_n = 1\ \}$.

Now present definition of uniformly $R - subweakly\ commuting$ mappings in Hausdorff locally convex space.

Definition 2.2: Let M is T and $I - invariant$ and $q - starshaped$ with q∈ $F(I)$. Two self mappings T and I are said to be uniformly $R - subweakly\ commuting$ on $M \setminus \{q\}$ if there exists a real number $R > 0$ such that for all $P_\alpha \in A^*(\tau)$ and all $x \in M$ we have;

$$P_\alpha(T^n Ix - IT^n x) \leq Rd_{\,P_\alpha}(Ix, [T^n x], q)$$

where,

$d_{\,P_\alpha}(Ix, [T^n x, q]) = \inf\{\, P_\alpha((I(x - y_n): y_n \in [T^n x, q] =$
$\alpha_n T_n x + (1 - \alpha_n)q\ , and\ \{\alpha_n\} is\ a\ sequence\ of\ real\ numbers\ such\ that\ 0 < \alpha_n <$
$1\ \ and\ \lim_{n\to\infty} \alpha_n = 1\}$.

Lemma 2.3: Let M be a nonempty , $\tau-bounded$ and $\tau-closed$ of a Hausdorff locally convex space (X, τ). Let $T\ , I: M \to M$ be self mappings and $T(M - \{q\}) \subseteq I(M) - \{q\}$ and, suppose there exists $k \in (0,1)$ such that :

$$P_\alpha(Tx-Ty) \le kmax\left\{ P_\alpha(Ix-Iy), P_\alpha(Ix-Tx), P_\alpha(Iy-Ty), \frac{1}{2}[\,P_\alpha(Ix-Ty)+\ P_\alpha(Iy-Tx)]\right\}$$

For all $x, y \in M$ and $P_\alpha \in A^*(\tau)$. Further if T is continuous and $cl[T(M-\{q\})]$ is τ–sequentially completely . T and I are R-weakly commuting on $M-\{q\}$. Then F(I)⋂F(T) is singleton.

Proof: Let $x_0 \in M-q$. Since $(M-\{q\}) \subseteq I(M)-\{q\}$, we define a sequence $\{x_{\rm n}\}$ in $M-\{q\}$ as $Ix_n = Tx_{n-1}$ for each $n \ge 1$. Then

$$\begin{aligned}
\mathrm{P}_\alpha(Ix_{n+1}-Ix_n) &= \mathrm{P}_\alpha(\mathrm{T}x_n-\mathrm{T}x_{n-1})\\
&\le \mathrm{kmax}\left\{\mathrm{P}_\alpha(Ix_n-Ix_{n-1}), \mathrm{P}_\alpha(Ix_n-\mathrm{T}x_n), \mathrm{P}_\alpha(Ix_{n-1}-\mathrm{T}x_{n-1}), \frac{1}{2}[\mathrm{P}_\alpha(Ix_n-\mathrm{T}x_{n-1})\right.\\
&\left.+\ \mathrm{P}_\alpha(\mathrm{T}x_n-Ix_{n-1})]\right\}\\
&= \mathrm{kmax}\left\{\mathrm{P}_\alpha(Ix_n-Ix_{n-1}), \mathrm{P}_\alpha(Ix_n-Ix_{n+1}), \mathrm{P}_\alpha(Ix_{n-1}-Ix_n), \frac{1}{2}[\mathrm{P}_\alpha(Ix_n-Ix_n)\right.\\
&\left.+\ \mathrm{P}_\alpha(Ix_{n+1}-Ix_{n-1})]\right\}\\
&= \mathrm{kmax}\{\mathrm{P}_\alpha(Ix_n-Ix_{n-1}), \mathrm{P}_\alpha(Ix_n-Ix_{n+1}), \frac{1}{2}[\mathrm{P}_\alpha(Ix_{n+1}-Ix_n)\\
&= \mathrm{kmax}\{\mathrm{P}_\alpha(Ix_n-Ix_{n-1}), \mathrm{P}_\alpha(Ix_n-Ix_{n+1}), \frac{1}{2}[\mathrm{P}_\alpha(Ix_{n+1}-Ix_n)+\mathrm{P}_\alpha(Ix_n-Ix_{n-1})]\}\\
&\le \mathrm{kP}_\alpha(Ix_n-Ix_{n-1})
\end{aligned}$$

For all n . This implies that $\{Ifx_{\rm n}\}$ is Cauchy sequence in $M-\{q\}$. So $\{\mathrm{T}x_{\rm n}\}$ is a Cauchy sequence in $M-\{q\}$ and since $Cl[T(M)-q]$is τ –complete, $Tx_n \to\ y \in M$ and consequently $x_n \to y$. Since T, I are $R-weakly\ commuting$ on $M-\{q\}$, $\mathrm{P}_\alpha(\mathrm{T}Ix_n-\mathrm{IT}x_n) \le \mathrm{RP}_\alpha(\mathrm{T}x_n-\mathrm{I}x_n)$

Which implies $ITx_n \to Ty$ as $n \to \infty$.

Since

$$\mathrm{P}_\alpha(\mathrm{T}x_n-T\mathrm{T}x_n) \le \mathrm{k.max}\{\mathrm{P}_\alpha(\mathrm{I}x_n-\mathrm{IT}x_n), \mathrm{P}_\alpha(\mathrm{I}x_n-\mathrm{T}x_n), \mathrm{P}_\alpha(\mathrm{IT}x_n-\mathrm{TT}x_n)\,,$$

$$\tfrac{1}{2}[\mathrm{P}_\alpha(\mathrm{I}x_n-\mathrm{TT}x_n)+\mathrm{P}_\alpha(\mathrm{IT}x_n-\mathrm{T}x_n)\,]\}$$

Taking the limited as $n \to \infty$, We get

$$\mathrm{P}_\alpha(\mathrm{y}-\mathrm{Ty}) \le \mathrm{k.max}\{\mathrm{P}_\alpha(\mathrm{y}-\mathrm{Ty}), \mathrm{P}_\alpha(\mathrm{y}-\mathrm{y}), \mathrm{P}_\alpha(\mathrm{Ty}-\mathrm{Ty}),$$

$$\tfrac{1}{2}[\mathrm{P}_\alpha(\mathrm{y}-\mathrm{Ty})+\mathrm{P}_\alpha(\mathrm{Ty}-\mathrm{y})\,]\}$$

Thus $y = Ty$

Suppose $y = q$. Since $T\ and\ I$ are R-weakly commuting on $M-\{q\}$, it follow that

$$0 = \mathrm{P}_\alpha(\mathrm{ITq}-\mathrm{TIq}) \le \mathrm{RP}_\alpha(\mathrm{Tq}-\mathrm{Iq}) = 0$$

A contradiction . Since $y = Ty \in T(M - q)$ and $T(M - \{q\}) \subset I(M) - \{q\}$, there exists $z \in M - \{q\}$ such that $y = Ifz$. Now we shall show that $Iz = Tz$, and

$$P_\alpha(TTx_n - Tz) \leq k.\max\{P_\alpha(ITx_n - Iz), P_\alpha(ITx_n - TTx_n), P_\alpha(Iz - Tz),$$

$$\frac{1}{2}[P_\alpha(ITx_n - Tz) + P_\alpha(TTx_n - Iz)]\}$$

Now, as $n \to \infty$ we have,

$$P_\alpha(Ty - Tz) \leq k.\max\{P_\alpha(Ty - Iz), P_\alpha(Ty - Tz), P_\alpha(Iz - Tz), \quad \frac{1}{2}[P_\alpha(Ty - Tz) + P_\alpha(Iz - Ty)]\}.$$

Therefore,

$$P_\alpha(y - Tz) \leq k.\max\{P_\alpha(y - Ty), P_\alpha(y - Tz), P_\alpha(y - Tz),$$

$$\frac{1}{2}[P_\alpha(y - Tz) + P_\alpha(y - Tz)]\} .$$

So,

$$P_\alpha(y - Tz) < P_\alpha(y - Tz).$$

Hence $y = Tz = Iz$ .Since

$$P_\alpha(TIz - ITz) \leq RP_\alpha(Iz - Tz)$$

We have ,$TIz = ITz$.

Therefore, $y = Ty = Iy$.

Theorem 2.4: Let T and I be continuous self mappings of a nonempty, τ–bounded, τ–closed, $\tau\text{–}sequentialluy$ complete and $q - starshaped$ subset M of a Hausdorff locally convex space (X, τ) . Suppose I is $A^*(\tau)$-nonexpansive and affine with respect to $q \in F(I)$ and $I(M) = M$ and $T(M \setminus \{q\}) \subseteq I(M) - \{q\}$ if T, I are uniformly $R - subweakly\ commuting$ and T is $I - nonexpansive$ and there exists a sequence $\{K_n\}$ of real numbers with $K_n \geq 0$ and $\lim_{n\to\infty} K_n = 1$ such that T is uniformly asymptotically regular satisfying

$$\begin{aligned} &P_\alpha(T^n x - T^n y) \\ &\quad \leq K_n \max\left\{P_\alpha(Ix - Iy), dist(Ix, [T^n x, q]), dist(Iy, [T^n y, q]), \frac{1}{2}[dist(Ix, [T^n y, q]) \right. \\ &\quad \left. + dist(Iy, [T^n x, q])]\right\} \end{aligned}$$

For all $x, y \in M\ and\ P_\alpha \in A^*(\tau)$ and $n \in \mathbb{N}$.

Then T and I have a common fixed point provided any one of the following condition hold:

1) $cl[T(M \setminus \{q\})]$ is τ–sequentially compact ;

2) M is weakly compact and $(I - T^m)$ is $demiclosed$ in 0.

Proof: For each $n \geq 1$, define T_n on M by $T_n = \mu_n T^n x + (1-\mu_n)q$, for all $x \in M$. Where $\mu_n = \frac{\lambda_n}{K_n}$

and $\{K_n\}$ is a sequence of real numbers with $0 < \lambda_n < 1$ such that $\lim_{n\to\infty} \lambda_n = 1$ and$\{K_n\}$ is defined as above. Let T_n is a self mappings on M such that $T(M \setminus \{q\}) \subseteq I(M) - \{q\}$, from uniformly $R-$ $sub\ weakly\ commuting$ of T and I on $M \setminus \{q\}$, since I is affine with respect to q , it follow that;

$$P_\alpha(T_n Ix - IT_n x) = \mu_n P_\alpha(T^n Ix - IT^n x) \leq R\mu_n d_{P_\alpha}(Ix, [T^n x, q] \leq R\mu_n d_{P_\alpha}(Ix - T_n x).$$

For all $x \in M \setminus \{q\}$, which it implies T_n and f are μ_n $R - subweakly\ commuting$ .Hence by lemma 2.3 ,there exists $\{x_n\}$ such that $T_n x_n = Ix_n = x_n$. Hence $\mu_n T^n x + (1-\mu_n)q = Ix_n = x_n$. Also ,

$$P_\alpha(x_n - T^n x_n) = P_\alpha(T_n x_n - T^n x_n)$$
$$= P_\alpha(\mu_n T^n x_n + (1-\mu_n)q - T^n x_n)$$
$$= (1-\mu_n)P_\alpha(q - T^n x_n)$$

Since $T(M \setminus \{q\})$ is τ–bounded as $\mu_n \to \infty$ and $P_\alpha(x_n - T^n x_n) \to 0$ as $n \to \infty$. Therefore

$$\begin{aligned}
\mathrm{P}_\alpha(x_n - \mathrm{T}x_n) &\leq \mathrm{P}_\alpha(x_n - T^{\mathrm{n}}x_n) + \mathrm{P}_\alpha(T^{\mathrm{n}}x_n - T^{\mathrm{n+1}}x_n) + \mathrm{P}_\alpha(T^{\mathrm{n+1}}x_n - \mathrm{T}x_n) \\
&\leq \mathrm{P}_\alpha(x_n - T^{\mathrm{n}}x_n) + \mathrm{P}_\alpha(T^{\mathrm{n}}x_n - T^{\mathrm{n+1}}x_n) + \mathrm{P}_\alpha(T(T^{\mathrm{n}}x_n) - \mathrm{T}x_n) \\
&= \mathrm{P}_\alpha(x_n - T^{\mathrm{n}}x_n) + \mathrm{P}_\alpha(T^{\mathrm{n}}x_n - T^{\mathrm{n+1}}x_n) \\
&\quad + k_1.max\{\mathrm{P}_\alpha(IT^{\mathrm{n}}x_n - \mathrm{I}x_n), \mathrm{P}_\alpha(Ix_n - \mathrm{T}x_n), \mathrm{P}_\alpha(IT^{\mathrm{n}}x_n - TT^{\mathrm{n}}x_n), \\
&\quad \tfrac{1}{2}[\mathrm{P}_\alpha(Ix_n - T^{\mathrm{n+1}}x_n) + \mathrm{P}_\alpha(IT^{\mathrm{n}}x_n - \mathrm{T}x_n)]\} \\
&\leq \mathrm{P}_\alpha(x_n - T^{\mathrm{n}}x_n) + \mathrm{P}_\alpha(T^{\mathrm{n}}x_n - T^{\mathrm{n+1}}x_n) + k_1\mathrm{P}_\alpha(IT^{\mathrm{n}}x_n - Ix_n) \\
&= \mathrm{P}_\alpha(x_n - T^{\mathrm{n}}x_n) + \mathrm{P}_\alpha(T^{\mathrm{n}}x_n - T^{\mathrm{n+1}}x_n) + K_1\mathrm{P}_\alpha(\mathrm{I}(\mathrm{T^n x_n} - \mathrm{x_n}).
\end{aligned}$$

Since I is continuous and affine with respect to , and T is uniformly asymptotic regular , it follow that $\mathrm{P}_\alpha(x_n - T^{\mathrm{n}}x_n) \to 0$ as $n \to \infty$.

Since $ClT(M - \{q\})$ is τ- compact, there exist a subsequence $\{x_{\mathrm{m}}\}$ of $\{x_{\mathrm{n}}\}$ such that $x_{\mathrm{m}} \to \mathrm{y} \in \mathrm{M}$ (1) as $\to \infty$. T is continuous , it follow that

$$Tx_m \to Ty = y$$

Moreover , $T(M - \{q\}) \subseteq I(M) - \{q\}$ implies $y = Ty = Iz$ for some z$\in M$.

$$\mathrm{P}_\alpha(Tx_m - \mathrm{Tz}) \leq k_1.max\{\mathrm{P}_\alpha(Ix_m - \mathrm{Iz}), \mathrm{P}_\alpha(Ix_m - \mathrm{T}x_{\mathrm{m}}), \mathrm{P}_\alpha(Iz - \mathrm{Tz}), \tfrac{1}{2}[\mathrm{P}_\alpha(Ix_m - \mathrm{Tz}) + \mathrm{P}_\alpha(\mathrm{T}x_m - \mathrm{Iz})\}$$

$\leq k_1\mathrm{P}_\alpha(Ix_m - \mathrm{Iz}) = k_1\mathrm{P}_\alpha(x_m - \mathrm{y})$, therefore

$Tx_{\mathrm{m}} \rightarrow Tz$, As m→ ∞. Hence , $y = Tz = Ty = Iz$ .Now

$$\mathrm{P}_{\alpha}(Iy - \mathrm{Ty}) = \mathrm{P}_{\alpha}(ITz - \mathrm{TIz}) \leq R\mathrm{P}_{\alpha}(Tz - \mathrm{Iz}) = 0$$

Which implies Ty= $Iy = y$.

(2) Since M is weakly compact, there exist a subsequence $\{x_{\mathrm{m}}\}$ of $\{x_{\mathrm{n}}\}$ converging weakly to some y∈ M .But I is affine and continuous in weakly topology in Hausdorff Spaces, it follow that $Iy = y$.

Now the $demiclosed$ of $(I - T^m)$ at 0 guarantees that $(I - T^m)$y=0 . Hence $T^m y = y$. Now we shall show that Ty=y. Since

$$\mathrm{P}_{\alpha}(Ty - T^m\mathrm{y}) = \mathrm{P}_{\alpha}(Ty - \mathrm{T}(T^{m-1})\mathrm{y})$$

$$\leq k_1.\max\{\mathrm{P}_{\alpha}(I\mathrm{y} - \mathrm{I}T^{m-1}\mathrm{y}), \mathrm{d}_{\mathrm{P}_{\alpha}}(\mathrm{Iy}, [\mathrm{Ty}, \mathrm{q}]), \mathrm{d}_{\mathrm{P}_{\alpha}}(\mathrm{I}T^{m-1}\mathrm{y}, [\mathrm{T}T^{m-1}\mathrm{y}, \mathrm{q}]),$$

$$\frac{1}{2}[\mathrm{d}_{\mathrm{P}_{\alpha}}(\mathrm{Iy}, [\mathrm{T}T^{m-1}\mathrm{y}, \mathrm{q}]) + \mathrm{d}_{\mathrm{P}_{\alpha}}(\mathrm{I}\,T^{m-1}\mathrm{y}, [\mathrm{Ty}, \mathrm{q}])\,]\}$$

Let $m \rightarrow \infty$ we have,

$$\mathrm{P}_{\alpha}(Ty - \mathrm{y}) \leq k_1 \max\{\mathrm{P}_{\alpha}(\mathrm{y} - \mathrm{y}), \mathrm{d}_{\mathrm{P}_{\alpha}}(\mathrm{Iy} - \mathrm{T}_1\mathrm{y}), \mathrm{P}_{\alpha}(\mathrm{Iy} - \mathrm{T}_{\mathrm{m}}\mathrm{y}), \frac{1}{2}[\mathrm{P}_{\alpha}(\mathrm{y} - \mathrm{T}_{\mathrm{m}}\mathrm{y}) + \mathrm{P}_{\alpha}(\mathrm{y} - \mathrm{Ty})\,]\} \quad .$$

A contradiction, hence Ty=y which implies $Iy = Ty = y$.

REFERENCES

[1] F. Akbar and A. R. Khan, Common fixed point and approximation results for noncommuting maps on locally convex spaces, Fixed point theory and applications, 2009article ID 207503, pp.14 pages**(2009).**

[2] I. Beg, D.R. Sahu and S.D. Diwan, Approximation of fixed points of uniformly *R*_subweakly commuting mappings, J.Math,Anal.appl,

[3] Y.J. Cho, D.R. Sahu, J.S. Jung, Approximation of fixed points of asymptotically pseudocontractiv mappings in Banach Spaces, Southwest.J.Pure.Appl.Math,2, 49-59**(2003).**

[4] K. Goebel ,W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc.Amer.Math.Soc.35, 171-174**(1972).**

[5] N. Hussain, G. Jungck, Common fixed points and invariant approximation results for noncommuting generalized (*f*, *g*) nonxpansive maps, J.Math,Anal.appl,**in press**.

[6] N. Hussain, V. Berinde, Common fixed point theorem for C_q-commuting mappings in certain metrizable topological vector spaces, Fixed point Theorey and applications, 2006 article ID 23582,1-13. 552. **(2006),**

[7] G. Kothe, Topological Vector Spaces, Springer verlag, New York, **(1969).**

[8]H. Shojaei, R. Mortezaei common fixed point for affine self-maps invariant approximation in p-normed space, J. Math. Computer Sci., 6, pp- 201**(2013).**

[9] H. Shojaei, K. Banaei, N. Shojaei, , J. Math. Computer Sci., 6, pp- 118-128**(2013).**

[10] H. Shojaei, R. Mortezaei, j. Math. . Computer Sci., 6, pp201-209 **(2013).**

[11] J. Schu, Iterative approximation of fixed points of nonexpansive mappings with starshaped domain, Comment.Math Univ.Carolin. 31(2) 277-282**(1990).**

[12] J. Schu, Approximation of fixed points of asymptotically nonexpanisve mappings , Proc. Amer. Math. Soc. 112(1) 143-151**(1991).**

[12] N. Shahzad, Invariant approximations, generalized *I*-contractions, and *R*-subweakly commuting maps, Fixed Point Theory and Applications.179-86**(2005).**