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An eighth order frozen Jacobian iterative method for solving nonlinear IVPs and BVPs

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Abstract

A frozen Jacobian iterative method is proposed for solving systems of nonlinear equations. In particular, we are interested in solving the systems of nonlinear equations associated with initial value problems (IVPs) and boundary value problems (BVPs). In a single instance of the proposed iterative method DEDF, we evaluate two Jacobians, one inversion of the Jacobian and four function evaluations. The direct inversion of the Jacobian is computationally expensive, so, for a moderate size, LU factorization is a good direct method to solve the linear system. We employed the LU factorization of the Jacobian to avoid the direct inversion. The convergence order of the proposed iterative method is at least eight, and it is nine for some particular classes of problems. The discretization of IVPs and BVPs is employed by using Jacobi-Gauss-Lobatto collocation (J-GL-C) method. A comparison of J-GL-C methods is presented in order to choose best collocation method. The validity, accuracy and the efficiency of our DEDF are shown by solving eleven IVPs and BVPs problems.

Keywords: Frozen Jacobian iterative methods, systems of nonlinear equations, nonlinear initial-boundary value problems, Jacobi-Gauss-Lobatto quadrature, collocation method.
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1. Introduction

The closed form solutions of nonlinear problems are not always available. It means we need an iterative method to solve them. We are interested in iterative numerical methods for solving systems of nonlinear equations associated with nonlinear IVPs and BVPs. At the first place, we find some efficient and accurate discretization method to approximate the nonlinear differential equations. Once, we translated the continuous problem into a discrete nonlinear problem. We proceed by finding some efficient iterative solvers for the associated systems of nonlinear equations.

The pseudospectral collocation methods offer excellent accuracy. Doha et al. [7] employed J-GL-C method for the discretization of nonlinear 1+1 Schrödinger for spatial dimensions and got a system of

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ordinary differential equations. They used implicit Runge-Kutta method to solve the system of ordinary differential equations and obtained highly accurate numerical solutions, for four different kinds of nonlinear 1+1 Schrödinger equations. In another article Bhrawy et al. [5] solved the nonlinear reaction-diffusion equations by using J-GL-C method. In the solution of the complex generalized Zakharov system of equation [4], the J-GL-C method is also the method of discretization. The further application of pseudospectral collocation techniques for solving nonlinear IVPs and BVPs can be found in [6, 8, 9, 17] and references therein. The J-GL-C method is a parametric pseudospectral collocation method. By selecting different values of the parameters, we can get different pseudospectral collocation methods. The Legendre, Chebyshev, and Gegenbauer pseudospectral collocation methods are special cases of J-GL-C method [16].

The Jacobi polynomials are the eigenfunctions of a singular Strum-Liouville problem [16]

$$(1-x^2)\sigma''(x) + (\theta - \phi + (\theta + \phi + 2)x)\sigma'(x) + n(n + \theta + \phi + 1)\sigma(x) = 0\,.$$

The following recurrence relation produces the Jacobi polynomials

$$\begin{aligned}
J_{k+1}^{(\theta,\phi)}(x) &= \left(a_k^{(\theta,\phi)} - b_k^{(\theta,\phi)}\right) J_k^{(\theta,\phi)}(x) - c_k^{(\theta,\phi)} J_{k-1}^{(\theta,\phi)}(x), \quad k \geqslant 1\,,\\
J_0^{(\theta,\phi)}(x) &= 1, \quad J_1^{(\theta,\phi)}(x) = \frac{1}{2}(\theta + \phi + 2)x + \frac{1}{2}(\phi - \theta)\,,
\end{aligned}$$

where

$$\begin{aligned}
a_k^{(\theta,\phi)} &= \frac{(2k + \theta + \phi + 1)(2k + \theta + \phi + 2)}{2(k+1)(k + \theta + \phi + 1)}\,,\\
b_k^{(\theta,\phi)} &= \frac{(\theta^2 - \phi^2)(2k + \theta + \phi + 1)}{2(k+1)(k + \theta + \phi - 1)(2k + \theta + \phi)}\,,\\
c_k^{(\theta,\phi)} &= \frac{(k+\theta)(k+\phi)(2k + \theta + \phi + 2)}{(k+1)(k + \theta + \phi - 1)(2k + \theta + \phi)}\,.
\end{aligned}$$

The p-th derivative of k-degree Jacobi polynomial $J_k^{(\theta,\phi)}$ can be computed as

$$D^{(p)} J_k^{(\theta,\phi)}(x) = \frac{\Gamma(j + \theta + \phi + p + 1)}{2^p \Gamma(j + \theta + \phi + 1)} J_{k-p}^{(\theta+p,\phi+p)}(x)\,.$$

The Jacobi polynomials are orthogonal over the domain $[-1, 1]$ with respect to weight function $(1+x)^{\theta}(1-x)^{\phi}$. The J-GL-C method is attractive because the numerical differentiation matrix is easy to construct for approximating differential operators of different orders. The construction of such differentiation matrices can be found in [14]. Assume Q is the Jacobi-Gauss-Lobatto differentiation matrix of the first order derivative operator over the domain $[-1, 1]$ .Then a derivative of order p can be approximated over the interval $[a, b]$ as follows

$$\frac{d^p}{dx^p} \approx \left(\frac{2}{b-a} Q\right)^p\,.$$

The discretization of IVPs and BVPs gives a system of nonlinear algebraic equations $\mathbf{F}(\mathbf{y}) = \mathbf{0}$, where $\mathbf{y} = [y_1, y_2, y_3, \cdots, y_n]^T$. The classical iterative method for solving system of nonlinear equations is the Newton method [11, 18], which can be written as

$$\mathrm{NR} = \begin{cases} \mathbf{y}_0 = \text{initial guess}\,,\\ \mathbf{y}_{n+1} = \mathbf{y}_n - \mathbf{F}'(\mathbf{y}_n)^{-1}\mathbf{F}(\mathbf{y}_n)\,, \end{cases}$$

where $\det\big(\mathbf{F}'(\mathbf{y}_n)\big) \neq 0$. Many researchers [1–3, 10, 12, 13, 15, 19, 20] have proposed the frozen Jacobian multi-step iterative method for solving the system of nonlinear equations. The frozen Jacobian iterative methods are computationally efficient, because inversion of the Jacobian is too expensive. For moderate size system of nonlinear equations, it is good idea to use the LU factorization for solving the system of linear equations. The benefit of using the LU factorization of frozen Jacobian is apparent because in DEDF method we use eight systems of linear equations with fixed augmented matrix, which is a frozen Jacobian. In fact, we solve eight lower, and eight upper triangular systems and the solution of the triangular system is computationally economical. In the DEDF method, we get an increment in the convergence order of one per solving a system of linear equations. It is a good idea to design higher order frozen Jacobian iterative methods, which provide us rapid convergence in the vicinity of the root.

2. Frozen Jacobian iterative method

A new iterative method (DEDF) can be described as

$$
\text{DEDF} = \begin{cases} \text{Convergence order} & \geqslant 8 \\ \text{Function evaluations} & = 4 \\ \text{Jacobian evaluations} & = 2 \\ \text{LU-factorization} & = 1 \\ \text{Matrix-vector} & \\ \quad \text{multiplications} & = 4 \\ \text{Vector-vector} & \\ \quad \text{multiplications} & = 8 \\ \text{Number of lower and} & \\ \text{upper triangular systems} & = 8 \end{cases}
\begin{cases} \mathbf{y}_0 = \text{initial guess}, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{y}_0), \\ \mathbf{y}_1 = \mathbf{y}_0 - \boldsymbol{\phi}_1, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_2 = \mathbf{F}(\mathbf{y}_1), \\ \mathbf{y}_2 = \mathbf{y}_1 - \boldsymbol{\phi}_2, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_3 = \mathbf{F}(\mathbf{y}_2), \\ \mathbf{y}_3 = \mathbf{y}_2 - \alpha_1\,\boldsymbol{\phi}_3, \\ \mathbf{y}_{31} = \mathbf{y}_2 - \alpha_2\,\boldsymbol{\phi}_3, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_4 = \mathbf{F}(\mathbf{y}_3), \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_5 = \mathbf{F}'(\mathbf{y}_{31})\,\boldsymbol{\phi}_4, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_6 = \mathbf{F}'(\mathbf{y}_{31})\,\boldsymbol{\phi}_5, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_7 = \mathbf{F}'(\mathbf{y}_{31})\,\boldsymbol{\phi}_6, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_8 = \mathbf{F}'(\mathbf{y}_{31})\,\boldsymbol{\phi}_7, \\ \mathbf{y}_4 = \mathbf{y}_2 - \beta_1\,\boldsymbol{\phi}_3 - \beta_2\,\boldsymbol{\phi}_4 - \beta_3\,\boldsymbol{\phi}_5 - \beta_4\,\boldsymbol{\phi}_6 - \beta_5\,\boldsymbol{\phi}_7 - \beta_6\,\boldsymbol{\phi}_8, \\ \mathbf{y}_0 = \mathbf{y}_4\,, \end{cases}
$$

where

$$
\begin{aligned}
\alpha_2 &= -\frac{\sqrt[3]{1724 + 68\sqrt{9757}}}{204} + \frac{29}{17\sqrt[3]{1724 + 68\sqrt{9757}}} + \frac{95}{102}, \\
\alpha_1 &= 4\alpha_2 - 3, \\
\beta_1 &= -\frac{6\,\alpha_2 - 5}{(4\,\alpha_2 - 3)\,(2\,\alpha_2 - 3)}, \\
\beta_2 &= -1/32\,\frac{960\,\alpha_2{}^3 - 2560\,\alpha_2{}^2 + 2260\,\alpha_2 - 659}{(2\,\alpha_2{}^3 - 7\,\alpha_2{}^2 + 8\,\alpha_2 - 3)\,(4\,\alpha_2 - 3)}, \\
\beta_3 &= 1/8\,\frac{160\,\alpha_2{}^2 - 305\,\alpha_2 + 146}{2\,\alpha_2{}^3 - 7\,\alpha_2{}^2 + 8\,\alpha_2 - 3}, \\
\beta_4 &= -3/16\,\frac{120\,\alpha_2{}^2 - 226\,\alpha_2 + 107}{2\,\alpha_2{}^3 - 7\,\alpha_2{}^2 + 8\,\alpha_2 - 3}, \\
\beta_5 &= 1/8\,\frac{96\,\alpha_2{}^2 - 179\,\alpha_2 + 84}{2\,\alpha_2{}^3 - 7\,\alpha_2{}^2 + 8\,\alpha_2 - 3}, \\
\beta_6 &= -1/32\,\frac{80\,\alpha_2{}^2 - 148\,\alpha_2 + 69}{2\,\alpha_2{}^3 - 7\,\alpha_2{}^2 + 8\,\alpha_2 - 3}\,.
\end{aligned}
$$

The DEDF is an efficient iterative method, since it requires only one inversion of the Jacobian regarding LU factorization. In summary, the DEDF employs four functions, two Jacobian evaluations, and eight matrix-vector multiplications.

3. Convergence analysis

We used the symbolic algebra of Maple software to deal with symbolic computations. The Fréchet differentiability condition on the system of nonlinear equations is the essential because it ensure the linearization of the nonlinear function.

The existence of the following limits

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{||\mathbf{F}(\mathbf{y}+\mathbf{h})-\mathbf{F}(\mathbf{y})-\mathbf{A}\mathbf{h}||}{||\mathbf{h}||}=0\,,$$

ensures the Fréchet differentiability. The linear operator $\mathbf{A}$ is called the first order Fréchet derivative and we denote it by $\mathbf{F}'(\mathbf{y})$. The higher order Fréchet derivatives can be computed recursively

$$\begin{aligned}&\mathbf{F}'(\mathbf{y})=\text{Jacobian}\left(\mathbf{F}(\mathbf{y})\right),\\&\mathbf{F}^{j}(\mathbf{y})\mathbf{u}^{j-1}=\text{Jacobian}\left(\mathbf{F}^{j-1}(\mathbf{y})\mathbf{v}^{j-1}\right),\quad j\geqslant 2,\end{aligned}$$

where $\mathbf{u}$ is a vector independent from $\mathbf{y}$.

Theorem 3.1. *Let $\mathbf{F}:\Gamma\subseteq\mathbb{R}^n\to\mathbb{R}^n$ be a sufficiently Fréchet differentiable function on an open convex neighborhood Γ of $\mathbf{y}^*\in\mathbb{R}^n$ with $\mathbf{F}(\mathbf{y}^*)=0$ and $\det(\mathbf{F}'(\mathbf{y}^*))\neq 0$, where $\mathbf{F}'(\mathbf{y})$ denotes the Fréchet derivative of $\mathbf{F}(\mathbf{y})$. Let $\mathbf{A}_1=\mathbf{F}'(\mathbf{y}^*)$ and $\mathbf{A}_j=\dfrac{1}{j!}\mathbf{F}'(\mathbf{y}^*)^{-1}\mathbf{F}^{(j)}(\mathbf{y}^*)$, for $j\geqslant 2$, where $\mathbf{F}^{(j)}(\mathbf{y})$ denotes j-order Fréchet derivative of $\mathbf{F}(\mathbf{y})$. Then, with an initial guess in the neighborhood of $\mathbf{y}^*$, the sequence $\{\mathbf{y}_m\}$ generated by DEDF converges to $\mathbf{y}^*$ with local order of convergence at least eight and error*

$$\mathbf{e}_4=\mathbf{L}{\mathbf{e}_0}^8+O\left({\mathbf{e}_0}^9\right)\,,$$

where $\mathbf{e}_0=\mathbf{y}_0-\mathbf{y}^*$, ${\mathbf{e}_0}^p=\overbrace{(\mathbf{e}_0,\mathbf{e}_0,\ldots,\mathbf{e}_0)}^{p\ times}$, and

$$\begin{aligned}\mathbf{L}=\Big(&-16\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2^2\alpha_2^4+48\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^4\alpha_2^4+16\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^3\alpha_2^4-48\mathbf{A}_3\mathbf{A}_2^5\alpha_2^4-816\mathbf{A}_2^7\alpha_2^3+56\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2^2\alpha_2^3\\&-216\alpha_2^3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^4-56\alpha_2^3\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^3+216\alpha_2^3\mathbf{A}_3\mathbf{A}_2^5+2280\alpha_2^2\mathbf{A}_2^7-72\alpha_2^2\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2^2+360\alpha_2^2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^4\\&+72\alpha_2^2\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^3-360\alpha_2^2\mathbf{A}_3\mathbf{A}_2^5-2144\alpha_2\mathbf{A}_2^7+40\alpha_2\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2^2-264\alpha_2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^4-40\alpha_2\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^3\\&+264\alpha_2\mathbf{A}_3\mathbf{A}_2^5+678\mathbf{A}_2^7-8\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2^2+72\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^4+8\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^3-72\mathbf{A}_3\mathbf{A}_2^5\Big)/\Big((2\alpha_2-3)(\alpha_2^2-2\alpha_2+1)\Big),\end{aligned}$$

is a 8-linear function, i.e., $\mathbf{L}\in\mathbb{L}\overbrace{(\mathbb{R}^n,\mathbb{R}^n,\mathbb{R}^n,\cdots,\mathbb{R}^n)}^{8\ times}$ with $\mathbf{L}{\mathbf{e}_0}^8\in\mathbb{R}^n$.

Proof. We define the error at the n-th step $\mathbf{e}_n=\mathbf{y}_n-\mathbf{y}^*$. To complete the convergence proof, we performed the detailed computations by using Maple and details are provided below in sequence.

$$\begin{aligned}\mathbf{F}'(\mathbf{y}_0)^{-1}=&\Big(\mathbf{I}-2\mathbf{A}_2\mathbf{e}_0+\Big(-3\mathbf{A}_3+4\mathbf{A}_2^2\Big)\mathbf{e}_0^2+\Big(-4\mathbf{A}_4+6\mathbf{A}_3\mathbf{A}_2+6\mathbf{A}_2\mathbf{A}_3-8\mathbf{A}_2^3\Big)\mathbf{e}_0^3+\Big(-5\mathbf{A}_5-12\mathbf{A}_2^2\mathbf{A}_3\\&-12\mathbf{A}_3\mathbf{A}_2^2-12\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2+8\mathbf{A}_4\mathbf{A}_2+9\mathbf{A}_3^2+8\mathbf{A}_2\mathbf{A}_4+16\mathbf{A}_2^4\Big)\mathbf{e}_0^4+\Big(-6\mathbf{A}_6-16\mathbf{A}_2^2\mathbf{A}_4-32\mathbf{A}_2^5\\&-16\mathbf{A}_4\mathbf{A}_2^2-18\mathbf{A}_3^2\mathbf{A}_2-16\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2-18\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3-18\mathbf{A}_2\mathbf{A}_3^2+24\mathbf{A}_3\mathbf{A}_2^3+24\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2+24\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2\\&+10\mathbf{A}_5\mathbf{A}_2+12\mathbf{A}_4\mathbf{A}_3+12\mathbf{A}_3\mathbf{A}_4+10\mathbf{A}_2\mathbf{A}_5+24\mathbf{A}_2^3\mathbf{A}_3\Big)\mathbf{e}_0^5+\cdots+O\left(\mathbf{e}_0^9\right)\Big)\mathbf{A}_1^{-1},\\\mathbf{F}(\mathbf{y}_0)=&\mathbf{A}_1\Big(\mathbf{e}_0+\mathbf{A}_2\mathbf{e}_0^2+\mathbf{A}_3\mathbf{e}_0^3+\mathbf{A}_4\mathbf{e}_0^4+\mathbf{A}_5\mathbf{e}_0^5+\mathbf{A}_6\mathbf{e}_0^6+\mathbf{A}_7\mathbf{e}_0^7+\mathbf{A}_8\mathbf{e}_0^8+O\left(\mathbf{e}_0^9\right)\Big),\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\phi}_1 =& \mathbf{e}_0 - \mathbf{A}_2\mathbf{e}_0^2 + \Big(-2\mathbf{A}_3 + 2\mathbf{A}_2^2\Big)\mathbf{e}_0^3 + \Big(-3\mathbf{A}_4 + 4\mathbf{A}_2\mathbf{A}_3 + 3\mathbf{A}_3\mathbf{A}_2 - 4\mathbf{A}_2^3\Big)\mathbf{e}_0^4 + \Big(-4\mathbf{A}_5 + 6\mathbf{A}_2\mathbf{A}_4 + 4\mathbf{A}_4\mathbf{A}_2 \\
& - 6\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 6\mathbf{A}_3^2 + 8\mathbf{A}_2^4 - 6\mathbf{A}_3\mathbf{A}_2^2 - 8\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(-5\mathbf{A}_6 - 12\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 8\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 8\mathbf{A}_2\mathbf{A}_5 \\
& + 9\mathbf{A}_3\mathbf{A}_4 + 8\mathbf{A}_4\mathbf{A}_3 + 5\mathbf{A}_5\mathbf{A}_2 + 12\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 + 12\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 12\mathbf{A}_3\mathbf{A}_2^3 - 9\mathbf{A}_3^2\mathbf{A}_2 - 12\mathbf{A}_2\mathbf{A}_3^2 - 16\mathbf{A}_2^5 \\
& - 12\mathbf{A}_2^2\mathbf{A}_4 - 8\mathbf{A}_4\mathbf{A}_2^2 + 16\mathbf{A}_2^3\mathbf{A}_3\Big)\mathbf{e}_0^6 + \cdots + O\left(\mathbf{e}_0^9\right), \\
\mathbf{e}_1 =& \mathbf{A}_2\mathbf{e}_0^2 + \Big(2\mathbf{A}_3 - 2\mathbf{A}_2^2\Big)\mathbf{e}_0^3 + \Big(3\mathbf{A}_4 - 4\mathbf{A}_2\mathbf{A}_3 - 3\mathbf{A}_3\mathbf{A}_2 + 4\mathbf{A}_2^3\Big)\mathbf{e}_0^4 + \Big(4\mathbf{A}_5 - 6\mathbf{A}_2\mathbf{A}_4 - 4\mathbf{A}_4\mathbf{A}_2 + 6\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 \\
& - 6\mathbf{A}_3^2 - 8\mathbf{A}_2^4 + 6\mathbf{A}_3\mathbf{A}_2^2 + 8\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(5\mathbf{A}_6 + 12\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 + 8\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 - 8\mathbf{A}_2\mathbf{A}_5 - 9\mathbf{A}_3\mathbf{A}_4 - 8\mathbf{A}_4\mathbf{A}_3 \\
& - 5\mathbf{A}_5\mathbf{A}_2 - 12\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 12\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 12\mathbf{A}_3\mathbf{A}_2^3 + 9\mathbf{A}_3^2\mathbf{A}_2 + 12\mathbf{A}_2\mathbf{A}_3^2 + 16\mathbf{A}_2^5 + 12\mathbf{A}_2^2\mathbf{A}_4 + 8\mathbf{A}_4\mathbf{A}_2^2 \\
& - 16\mathbf{A}_2^3\mathbf{A}_3\Big)\mathbf{e}_0^6 + \cdots + O\left(\mathbf{e}_0^9\right), \\
\mathbf{F}(\mathbf{y}_1) =& \mathbf{A}_1\Big(\mathbf{A}_2\mathbf{e}_0^2 + \Big(2\mathbf{A}_3 - 2\mathbf{A}_2^2\Big)\mathbf{e}_0^3 + \Big(3\mathbf{A}_4 - 4\mathbf{A}_2\mathbf{A}_3 - 3\mathbf{A}_3\mathbf{A}_2 + 5\mathbf{A}_2^3\Big)\mathbf{e}_0^4 + \Big(4\mathbf{A}_5 - 6\mathbf{A}_2\mathbf{A}_4 - 6\mathbf{A}_3^2 - 4\mathbf{A}_4\mathbf{A}_2 \\
& - 12\mathbf{A}_2^4 + 8\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 6\mathbf{A}_3\mathbf{A}_2^2 + 10\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(5\mathbf{A}_6 + 8\mathbf{A}_4\mathbf{A}_2^2 + 12\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 24\mathbf{A}_2^3\mathbf{A}_3 - 19\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\
& - 19\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 11\mathbf{A}_3\mathbf{A}_2^3 + 9\mathbf{A}_3^2\mathbf{A}_2 + 11\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 16\mathbf{A}_2\mathbf{A}_3^2 + 28\mathbf{A}_2^5 - 8\mathbf{A}_2\mathbf{A}_5 - 9\mathbf{A}_3\mathbf{A}_4 - 8\mathbf{A}_4\mathbf{A}_3 \\
& - 5\mathbf{A}_5\mathbf{A}_2 + 15\mathbf{A}_2^2\mathbf{A}_4\Big)\mathbf{e}_0^6 + \cdots + O\left(\mathbf{e}_0^9\right)\Big), \\
\boldsymbol{\phi}_2 =& \mathbf{A}_2\mathbf{e}_0^2 + \Big(-4\mathbf{A}_2^2 + 2\mathbf{A}_3\Big)\mathbf{e}_0^3 + \Big(3\mathbf{A}_4 + 13\mathbf{A}_2^3 - 8\mathbf{A}_2\mathbf{A}_3 - 6\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(4\mathbf{A}_5 - 12\mathbf{A}_2\mathbf{A}_4 - 12\mathbf{A}_3^2 - 8\mathbf{A}_4\mathbf{A}_2 \\
& - 38\mathbf{A}_2^4 + 20\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 18\mathbf{A}_3\mathbf{A}_2^2 + 26\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(5\mathbf{A}_6 + 24\mathbf{A}_4\mathbf{A}_2^2 + 36\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 76\mathbf{A}_2^3\mathbf{A}_3 \\
& - 59\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 55\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 50\mathbf{A}_3\mathbf{A}_2^3 + 27\mathbf{A}_3^2\mathbf{A}_2 + 27\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 40\mathbf{A}_2\mathbf{A}_3^2 + 104\mathbf{A}_2^5 - 16\mathbf{A}_2\mathbf{A}_5 \\
& - 18\mathbf{A}_3\mathbf{A}_4 - 16\mathbf{A}_4\mathbf{A}_3 - 10\mathbf{A}_5\mathbf{A}_2 + 39\mathbf{A}_2^2\mathbf{A}_4\Big)\mathbf{e}_0^6 + \cdots + O\left(\mathbf{e}_0^9\right), \\
\mathbf{e}_2 =& 2\mathbf{e}_0^3\mathbf{A}_2^2 + \Big(-9\mathbf{A}_2^3 + 4\mathbf{A}_2\mathbf{A}_3 + 3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(30\mathbf{A}_2^4 + 6\mathbf{A}_2\mathbf{A}_4 + 6\mathbf{A}_3^2 + 4\mathbf{A}_4\mathbf{A}_2 - 14\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 12\mathbf{A}_3\mathbf{A}_2^2 \\
& - 18\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(47\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 + 43\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 38\mathbf{A}_3\mathbf{A}_2^3 - 16\mathbf{A}_4\mathbf{A}_2^2 - 18\mathbf{A}_3^2\mathbf{A}_2 - 19\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 \\
& - 24\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 28\mathbf{A}_2\mathbf{A}_3^2 + 60\mathbf{A}_2^3\mathbf{A}_3 - 88\mathbf{A}_2^5 + 8\mathbf{A}_2\mathbf{A}_5 + 9\mathbf{A}_3\mathbf{A}_4 + 8\mathbf{A}_4\mathbf{A}_3 + 5\mathbf{A}_5\mathbf{A}_2 - 27\mathbf{A}_2^2\mathbf{A}_4\Big)\mathbf{e}_0^6 \\
& + \cdots + O\left(\mathbf{e}_0^9\right), \\
\mathbf{F}(\mathbf{y}_2) =& \mathbf{A}_1\Big(2\mathbf{A}_2^2\mathbf{e}_0^3 + \Big(-9\mathbf{A}_2^3 + 4\mathbf{A}_2\mathbf{A}_3 + 3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(30\mathbf{A}_2^4 + 6\mathbf{A}_2\mathbf{A}_4 + 6\mathbf{A}_3^2 + 4\mathbf{A}_4\mathbf{A}_2 - 14\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 \\
& - 12\mathbf{A}_3\mathbf{A}_2^2 - 18\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(47\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 + 43\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 38\mathbf{A}_3\mathbf{A}_2^3 - 16\mathbf{A}_4\mathbf{A}_2^2 - 18\mathbf{A}_3^2\mathbf{A}_2 - 19\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 \\
& - 24\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 28\mathbf{A}_2\mathbf{A}_3^2 + 60\mathbf{A}_2^3\mathbf{A}_3 - 84\mathbf{A}_2^5 + 8\mathbf{A}_2\mathbf{A}_5 + 9\mathbf{A}_3\mathbf{A}_4 + 8\mathbf{A}_4\mathbf{A}_3 + 5\mathbf{A}_5\mathbf{A}_2 - 27\mathbf{A}_2^2\mathbf{A}_4\Big)\mathbf{e}_0^6 \\
& + \cdots + O\left(\mathbf{e}_0^9\right)\Big), \\
\boldsymbol{\phi}_3 =& 2\mathbf{A}_2^2\mathbf{e}_0^3 + \Big(-13\mathbf{A}_2^3 + 4\mathbf{A}_2\mathbf{A}_3 + 3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(6\mathbf{A}_2\mathbf{A}_4 + 4\mathbf{A}_4\mathbf{A}_2 - 20\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 56\mathbf{A}_2^4 + 6\mathbf{A}_3^2 - 18\mathbf{A}_3\mathbf{A}_2^2 \\
& - 26\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(-27\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 - 36\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 + 8\mathbf{A}_2\mathbf{A}_5 + 9\mathbf{A}_3\mathbf{A}_4 + 8\mathbf{A}_4\mathbf{A}_3 + 5\mathbf{A}_5\mathbf{A}_2 + 87\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\
& + 79\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 77\mathbf{A}_3\mathbf{A}_2^3 - 24\mathbf{A}_4\mathbf{A}_2^2 - 27\mathbf{A}_3^2\mathbf{A}_2 - 40\mathbf{A}_2\mathbf{A}_3^2 + 112\mathbf{A}_2^3\mathbf{A}_3 - 196\mathbf{A}_2^5 - 39\mathbf{A}_2^2\mathbf{A}_4\Big)\mathbf{e}_0^6 \\
& + \cdots + O\left(\mathbf{e}_0^9\right), \\
\mathbf{e}_3 =& 4\mathbf{A}_2^3\mathbf{e}_0^4 + \Big(-26\mathbf{A}_2^4 + 6\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 6\mathbf{A}_3\mathbf{A}_2^2 + 8\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(-40\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 36\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 39\mathbf{A}_3\mathbf{A}_2^3 \\
& + 8\mathbf{A}_4\mathbf{A}_2^2 + 9\mathbf{A}_3^2\mathbf{A}_2 + 8\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 12\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 + 12\mathbf{A}_2\mathbf{A}_3^2 - 52\mathbf{A}_2^3\mathbf{A}_3 + 108\mathbf{A}_2^5 + 12\mathbf{A}_2^2\mathbf{A}_4\Big)\mathbf{e}_0^6 \\
& + \Big(-80\mathbf{A}_2^2\mathbf{A}_3^2 - 72\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 78\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 + 16\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 + 18\mathbf{A}_3^3 + 16\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 + 18\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 \\
& + 18\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 + 216\mathbf{A}_2^4\mathbf{A}_3 - 78\mathbf{A}_2^3\mathbf{A}_4 - 356\mathbf{A}_2^6 + 10\mathbf{A}_5\mathbf{A}_2^2 + 12\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 + 12\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 + 10\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 \\
& + 168\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 + 150\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 + 148\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 + 168\mathbf{A}_3\mathbf{A}_2^4 + 16\mathbf{A}_2^2\mathbf{A}_5 - 52\mathbf{A}_4\mathbf{A}_2^3 - 54\mathbf{A}_3^2\mathbf{A}_2^2 \\
& - 48\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 - 54\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 - 60\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 54\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2\Big)\mathbf{e}_0^7 + \cdots + O\left(\mathbf{e}_0^9\right),
\end{aligned}
$$

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\begin{aligned}
\mathbf{F}(\mathbf{y}_3) = & \mathbf{A}_1\Big(4\mathbf{A}_2^3\mathbf{e}_0^4 + \Big(-26\mathbf{A}_2^4 + 6\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 6\mathbf{A}_3\mathbf{A}_2^2 + 8\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(-40\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 36\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 39\mathbf{A}_3\mathbf{A}_2^3 \\
& + 8\mathbf{A}_4\mathbf{A}_2^2 + 9\mathbf{A}_3^2\mathbf{A}_2 + 8\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 12\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 + 12\mathbf{A}_2\mathbf{A}_3^2 - 52\mathbf{A}_2^3\mathbf{A}_3 + 108\mathbf{A}_2^5 + 12\mathbf{A}_2^2\mathbf{A}_4\Big)\mathbf{e}_0^6 \\
& + \Big(-80\mathbf{A}_2^2\mathbf{A}_3^2 - 72\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 78\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 + 16\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 + 18\mathbf{A}_3^3 + 16\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 + 18\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 \\
& + 18\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 + 216\mathbf{A}_2^4\mathbf{A}_3 - 78\mathbf{A}_2^3\mathbf{A}_4 - 356\mathbf{A}_2^6 + 10\mathbf{A}_5\mathbf{A}_2^2 + 12\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 + 12\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 + 10\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 \\
& + 168\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 + 150\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 + 148\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 + 168\mathbf{A}_3\mathbf{A}_2^4 + 16\mathbf{A}_2^2\mathbf{A}_5 - 52\mathbf{A}_4\mathbf{A}_2^3 - 54\mathbf{A}_3^2\mathbf{A}_2^2 \\
& - 48\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 - 54\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 - 60\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 54\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2\Big)\mathbf{e}_0^7 + \cdots + O\left(\mathbf{e}_0^9\right)\Big), \\
\mathbf{e}_{31} = & \Big(2\mathbf{A}_2^2 - 2\alpha_2\mathbf{A}_2^2\Big)\mathbf{e}_0^3 + \Big(13\alpha_2\mathbf{A}_2^3 - 4\alpha_2\mathbf{A}_2\mathbf{A}_3 - 3\alpha_2\mathbf{A}_3\mathbf{A}_2 - 9\mathbf{A}_2^3 + 4\mathbf{A}_2\mathbf{A}_3 + 3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(20\alpha_2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 \\
& + 18\alpha_2\mathbf{A}_3\mathbf{A}_2^2 - 6\alpha_2\mathbf{A}_2\mathbf{A}_4 - 6\alpha_2\mathbf{A}_3^2 - 4\alpha_2\mathbf{A}_4\mathbf{A}_2 - 56\alpha_2\mathbf{A}_2^4 + 26\alpha_2\mathbf{A}_2^2\mathbf{A}_3 - 14\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 12\mathbf{A}_3\mathbf{A}_2^2 \\
& + 6\mathbf{A}_2\mathbf{A}_4 + 6\mathbf{A}_3^2 + 4\mathbf{A}_4\mathbf{A}_2 + 30\mathbf{A}_2^4 - 18\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(47\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 + 43\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 38\mathbf{A}_3\mathbf{A}_2^3 - 16\mathbf{A}_4\mathbf{A}_2^2 \\
& - 18\mathbf{A}_3^2\mathbf{A}_2 - 19\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 - 28\mathbf{A}_2\mathbf{A}_3^2 - 24\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 9\alpha_2\mathbf{A}_3\mathbf{A}_4 - 8\alpha_2\mathbf{A}_4\mathbf{A}_3 - 5\alpha_2\mathbf{A}_5\mathbf{A}_2 + 39\alpha_2\mathbf{A}_2^2\mathbf{A}_4 \\
& + 60\mathbf{A}_2^3\mathbf{A}_3 - 88\mathbf{A}_2^5 + 8\mathbf{A}_2\mathbf{A}_5 + 9\mathbf{A}_3\mathbf{A}_4 + 8\mathbf{A}_4\mathbf{A}_3 + 5\mathbf{A}_5\mathbf{A}_2 - 87\alpha_2\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 79\alpha_2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 \\
& - 77\alpha_2\mathbf{A}_3\mathbf{A}_2^3 + 24\alpha_2\mathbf{A}_4\mathbf{A}_2^2 + 27\alpha_2\mathbf{A}_3^2\mathbf{A}_2 + 27\alpha_2\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 40\alpha_2\mathbf{A}_2\mathbf{A}_3^2 + 36\alpha_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 \\
& - 112\alpha_2\mathbf{A}_2^3\mathbf{A}_3 + 196\alpha_2\mathbf{A}_2^5 - 8\alpha_2\mathbf{A}_2\mathbf{A}_5 - 27\mathbf{A}_2^2\mathbf{A}_4\Big)\mathbf{e}_0^6 + \cdots + O\left(\mathbf{e}_0^9\right), \\
\mathbf{F}(\mathbf{y}_{31}) = & \mathbf{A}_1\Big(\Big(2\mathbf{A}_2^2 - 2\alpha_1\mathbf{A}_2^2\Big)\mathbf{e}_0^3 + \Big(13\alpha_1\mathbf{A}_2^3 - 4\alpha_1\mathbf{A}_2\mathbf{A}_3 - 3\alpha_1\mathbf{A}_3\mathbf{A}_2 - 9\mathbf{A}_2^3 + 4\mathbf{A}_2\mathbf{A}_3 + 3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 \\
& + \Big(20\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 18\alpha_1\mathbf{A}_3\mathbf{A}_2^2 - 6\alpha_1\mathbf{A}_2\mathbf{A}_4 - 6\alpha_1\mathbf{A}_3^2 - 4\alpha_1\mathbf{A}_4\mathbf{A}_2 - 56\alpha_1\mathbf{A}_2^4 + 26\alpha_1\mathbf{A}_2^2\mathbf{A}_3 \\
& - 14\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 12\mathbf{A}_3\mathbf{A}_2^2 + 6\mathbf{A}_2\mathbf{A}_4 + 6\mathbf{A}_3^2 + 4\mathbf{A}_4\mathbf{A}_2 + 30\mathbf{A}_2^4 - 18\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(47\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\
& + 43\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 38\mathbf{A}_3\mathbf{A}_2^3 - 16\mathbf{A}_4\mathbf{A}_2^2 - 18\mathbf{A}_3^2\mathbf{A}_2 - 19\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 - 28\mathbf{A}_2\mathbf{A}_3^2 - 24\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 \\
& - 87\alpha_1\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 79\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 77\alpha_1\mathbf{A}_3\mathbf{A}_2^3 + 24\alpha_1\mathbf{A}_4\mathbf{A}_2^2 + 27\alpha_1\mathbf{A}_3^2\mathbf{A}_2 + 27\alpha_1\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 \\
& + 40\alpha_1\mathbf{A}_2\mathbf{A}_3^2 + 36\alpha_1\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 112\alpha_1\mathbf{A}_2^3\mathbf{A}_3 + 188\alpha_1\mathbf{A}_2^5 - 8\alpha_1\mathbf{A}_2\mathbf{A}_5 - 9\alpha_1\mathbf{A}_3\mathbf{A}_4 - 8\alpha_1\mathbf{A}_4\mathbf{A}_3 \\
& - 5\alpha_1\mathbf{A}_5\mathbf{A}_2 + 39\alpha_1\mathbf{A}_2^2\mathbf{A}_4 + 60\mathbf{A}_2^3\mathbf{A}_3 - 84\mathbf{A}_2^5 + 8\mathbf{A}_2\mathbf{A}_5 + 9\mathbf{A}_3\mathbf{A}_4 + 8\mathbf{A}_4\mathbf{A}_3 + 5\mathbf{A}_5\mathbf{A}_2 - 27\mathbf{A}_2^2\mathbf{A}_4 \\
& + 4\alpha_1^2\mathbf{A}_2^5\Big)\mathbf{e}_0^6 + \cdots + O\left(\mathbf{e}_0^9\right)\Big), \\
\mathbf{F}'(\mathbf{y}_{31}) = & \mathbf{A}_1\Big(\mathbf{I} + \Big(-4\alpha_2\mathbf{A}_2^3 + 4\mathbf{A}_2^3\Big)\mathbf{e}_0^3 + \Big(-8\alpha_2\mathbf{A}_2^2\mathbf{A}_3 + 26\alpha_2\mathbf{A}_2^4 - 6\alpha_2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 8\mathbf{A}_2^2\mathbf{A}_3 - 18\mathbf{A}_2^4 \\
& + 6\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(-12\alpha_2\mathbf{A}_2^2\mathbf{A}_4 + 52\alpha_2\mathbf{A}_2^3\mathbf{A}_3 - 112\alpha_2\mathbf{A}_2^5 - 12\alpha_2\mathbf{A}_2\mathbf{A}_3^2 + 36\alpha_2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 \\
& + 40\alpha_2\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 8\alpha_2\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 12\mathbf{A}_2^2\mathbf{A}_4 - 36\mathbf{A}_2^3\mathbf{A}_3 + 60\mathbf{A}_2^5 + 12\mathbf{A}_2\mathbf{A}_3^2 - 24\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 28\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\
& + 8\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2\Big)\mathbf{e}_0^5 + \Big(-56\mathbf{A}_2^2\mathbf{A}_3^2 - 48\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 + 16\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 + 18\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 + 12\alpha_2^2\mathbf{A}_3\mathbf{A}_2^4 \\
& - 36\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 - 38\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 - 32\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 + 120\mathbf{A}_2^4\mathbf{A}_3 + 94\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 + 86\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 + 76\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 \\
& + 12\mathbf{A}_3\mathbf{A}_2^4 - 176\mathbf{A}_2^6 + 16\mathbf{A}_2^2\mathbf{A}_5 + 10\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 - 54\mathbf{A}_2^3\mathbf{A}_4 + 80\alpha_2\mathbf{A}_2^2\mathbf{A}_3^2 + 72\alpha_2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 \\
& - 16\alpha_2\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 - 18\alpha_2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 + 54\alpha_2\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 + 54\alpha_2\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 + 48\alpha_2\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 - 224\alpha_2\mathbf{A}_2^4\mathbf{A}_3 \\
& - 174\alpha_2\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 - 158\alpha_2\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 - 154\alpha_2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 - 24\alpha_2\mathbf{A}_3\mathbf{A}_2^4 + 392\alpha_2\mathbf{A}_2^6 - 16\alpha_2\mathbf{A}_2^2\mathbf{A}_5 \\
& - 10\alpha_2\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 + 78\alpha_2\mathbf{A}_2^3\mathbf{A}_4\Big)\mathbf{e}_0^6 + \cdots + O\left(\mathbf{e}_0^9\right)\Big), \\
\boldsymbol{\phi}_4 = & \Big(2\mathbf{A}_2^2 - 2\alpha_1\mathbf{A}_2^2\Big)\mathbf{e}_0^3 + \Big(-4\alpha_1\mathbf{A}_2\mathbf{A}_3 - 3\alpha_1\mathbf{A}_3\mathbf{A}_2 + 17\alpha_1\mathbf{A}_2^3 + 4\mathbf{A}_2\mathbf{A}_3 + 3\mathbf{A}_3\mathbf{A}_2 - 13\mathbf{A}_2^3\Big)\mathbf{e}_0^4 \\
& + \Big(26\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 6\alpha_1\mathbf{A}_2\mathbf{A}_4 - 4\alpha_1\mathbf{A}_4\mathbf{A}_2 + 24\alpha_1\mathbf{A}_3\mathbf{A}_2^2 - 6\alpha_1\mathbf{A}_3^2 - 90\alpha_1\mathbf{A}_2^4 + 34\alpha_1\mathbf{A}_2^2\mathbf{A}_3 \\
& - 20\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 6\mathbf{A}_2\mathbf{A}_4 + 4\mathbf{A}_4\mathbf{A}_2 - 18\mathbf{A}_3\mathbf{A}_2^2 + 6\mathbf{A}_3^2 + 56\mathbf{A}_2^4 - 26\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(87\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\
& + 79\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 77\mathbf{A}_3\mathbf{A}_2^3 - 24\mathbf{A}_4\mathbf{A}_2^2 - 27\mathbf{A}_3^2\mathbf{A}_2 - 27\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 - 40\mathbf{A}_2\mathbf{A}_3^2 - 36\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 \\
& - 139\alpha_1\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 127\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 128\alpha_1\mathbf{A}_3\mathbf{A}_2^3 + 32\alpha_1\mathbf{A}_4\mathbf{A}_2^2 + 36\alpha_1\mathbf{A}_3^2\mathbf{A}_2 + 35\alpha_1\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 \\
& + 52\alpha_1\mathbf{A}_2\mathbf{A}_3^2 + 48\alpha_1\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 180\alpha_1\mathbf{A}_2^3\mathbf{A}_3 + 368\alpha_1\mathbf{A}_2^5 - 8\alpha_1\mathbf{A}_2\mathbf{A}_5 - 9\alpha_1\mathbf{A}_3\mathbf{A}_4 - 8\alpha_1\mathbf{A}_4\mathbf{A}_3
\end{aligned}
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\begin{aligned}
&-5\alpha_1\mathbf{A}_5\mathbf{A}_2+51\alpha_1\mathbf{A}_2^2\mathbf{A}_4+112\mathbf{A}_2^3\mathbf{A}_3-196\mathbf{A}_2^5+8\mathbf{A}_2\mathbf{A}_5+9\mathbf{A}_3\mathbf{A}_4+8\mathbf{A}_4\mathbf{A}_3+5\mathbf{A}_5\mathbf{A}_2-39\mathbf{A}_2^2\mathbf{A}_4\\
&+4\alpha_1^2\mathbf{A}_2^5\Big)\mathbf{e}_0^6+\cdots+O\left(\mathbf{e}_0^9\right),\\
\boldsymbol{\phi}_5=&\Big(2\mathbf{A}_2^2-2\alpha_1\mathbf{A}_2^2\Big)\mathbf{e}_0^3+\Big(21\alpha_1\mathbf{A}_2^3-4\alpha_1\mathbf{A}_2\mathbf{A}_3-3\alpha_1\mathbf{A}_3\mathbf{A}_2-17\mathbf{A}_2^3+4\mathbf{A}_2\mathbf{A}_3+3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4\\
&+\Big(32\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2+30\alpha_1\mathbf{A}_3\mathbf{A}_2^2-6\alpha_1\mathbf{A}_2\mathbf{A}_4-6\alpha_1\mathbf{A}_3^2-4\alpha_1\mathbf{A}_4\mathbf{A}_2-132\alpha_1\mathbf{A}_2^4+42\alpha_1\mathbf{A}_2^2\mathbf{A}_3\\
&-26\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2-24\mathbf{A}_3\mathbf{A}_2^2+6\mathbf{A}_2\mathbf{A}_4+6\mathbf{A}_3^2+4\mathbf{A}_4\mathbf{A}_2+90\mathbf{A}_2^4-34\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5+\Big(139\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2\\
&+127\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2+128\mathbf{A}_3\mathbf{A}_2^3+8\alpha_2\alpha_1\mathbf{A}_2^5-32\mathbf{A}_4\mathbf{A}_2^2-36\mathbf{A}_3^2\mathbf{A}_2-35\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2-52\mathbf{A}_2\mathbf{A}_3^2-48\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\\
&-203\alpha_1\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2-187\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2-191\alpha_1\mathbf{A}_3\mathbf{A}_2^3+40\alpha_1\mathbf{A}_4\mathbf{A}_2^2+45\alpha_1\mathbf{A}_3^2\mathbf{A}_2+43\alpha_1\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2\\
&+64\alpha_1\mathbf{A}_2\mathbf{A}_3^2+60\alpha_1\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3-264\alpha_1\mathbf{A}_2^3\mathbf{A}_3+624\alpha_1\mathbf{A}_2^5-8\alpha_1\mathbf{A}_2\mathbf{A}_5-9\alpha_1\mathbf{A}_3\mathbf{A}_4-8\alpha_1\mathbf{A}_4\mathbf{A}_3\\
&-5\alpha_1\mathbf{A}_5\mathbf{A}_2+63\alpha_1\mathbf{A}_2^2\mathbf{A}_4+180\mathbf{A}_2^3\mathbf{A}_3-368\mathbf{A}_2^5+8\mathbf{A}_2\mathbf{A}_5+9\mathbf{A}_3\mathbf{A}_4+8\mathbf{A}_4\mathbf{A}_3+5\mathbf{A}_5\mathbf{A}_2-8\alpha_2\mathbf{A}_2^5\\
&-51\mathbf{A}_2^2\mathbf{A}_4+4\alpha_1^2\mathbf{A}_2^5\Big)\mathbf{e}_0^6+\cdots+O\left(\mathbf{e}_0^9\right),\\
\boldsymbol{\phi}_6=&\Big(2\mathbf{A}_2^2-2\alpha_1\mathbf{A}_2^2\Big)\mathbf{e}_0^3+\Big(25\alpha_1\mathbf{A}_2^3-4\alpha_1\mathbf{A}_2\mathbf{A}_3-3\alpha_1\mathbf{A}_3\mathbf{A}_2-21\mathbf{A}_2^3+4\mathbf{A}_2\mathbf{A}_3+3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4\\
&+\Big(38\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2+36\alpha_1\mathbf{A}_3\mathbf{A}_2^2-6\alpha_1\mathbf{A}_2\mathbf{A}_4-6\alpha_1\mathbf{A}_3^2-4\alpha_1\mathbf{A}_4\mathbf{A}_2-182\alpha_1\mathbf{A}_2^4+50\alpha_1\mathbf{A}_2^2\mathbf{A}_3\\
&-32\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2-30\mathbf{A}_3\mathbf{A}_2^2+6\mathbf{A}_2\mathbf{A}_4+6\mathbf{A}_3^2+4\mathbf{A}_4\mathbf{A}_2+132\mathbf{A}_2^4-42\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5+\Big(203\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2\\
&+187\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2+191\mathbf{A}_3\mathbf{A}_2^3+16\alpha_2\alpha_1\mathbf{A}_2^5-40\mathbf{A}_4\mathbf{A}_2^2-45\mathbf{A}_3^2\mathbf{A}_2-43\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2-64\mathbf{A}_2\mathbf{A}_3^2-60\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\\
&-279\alpha_1\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2-259\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2-266\alpha_1\mathbf{A}_3\mathbf{A}_2^3+48\alpha_1\mathbf{A}_4\mathbf{A}_2^2+54\alpha_1\mathbf{A}_3^2\mathbf{A}_2+51\alpha_1\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2\\
&+76\alpha_1\mathbf{A}_2\mathbf{A}_3^2+72\alpha_1\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3-364\alpha_1\mathbf{A}_2^3\mathbf{A}_3+980\alpha_1\mathbf{A}_2^5-8\alpha_1\mathbf{A}_2\mathbf{A}_5-9\alpha_1\mathbf{A}_3\mathbf{A}_4-8\alpha_1\mathbf{A}_4\mathbf{A}_3\\
&-5\alpha_1\mathbf{A}_5\mathbf{A}_2+75\alpha_1\mathbf{A}_2^2\mathbf{A}_4+264\mathbf{A}_2^3\mathbf{A}_3-624\mathbf{A}_2^5+8\mathbf{A}_2\mathbf{A}_5+9\mathbf{A}_3\mathbf{A}_4+8\mathbf{A}_4\mathbf{A}_3+5\mathbf{A}_5\mathbf{A}_2-16\alpha_2\mathbf{A}_2^5\\
&-63\mathbf{A}_2^2\mathbf{A}_4+4\alpha_1^2\mathbf{A}_2^5\Big)\mathbf{e}_0^6+\cdots+O\left(\mathbf{e}_0^9\right),\\
\boldsymbol{\phi}_7=&\Big(2\mathbf{A}_2^2-2\alpha_1\mathbf{A}_2^2\Big)\mathbf{e}_0^3+\Big(29\alpha_1\mathbf{A}_2^3-4\alpha_1\mathbf{A}_2\mathbf{A}_3-3\alpha_1\mathbf{A}_3\mathbf{A}_2-25\mathbf{A}_2^3+4\mathbf{A}_2\mathbf{A}_3+3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4+\Big(44\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\\
&+42\alpha_1\mathbf{A}_3\mathbf{A}_2^2-6\alpha_1\mathbf{A}_2\mathbf{A}_4-6\alpha_1\mathbf{A}_3^2-4\alpha_1\mathbf{A}_4\mathbf{A}_2-240\alpha_1\mathbf{A}_2^4+58\alpha_1\mathbf{A}_2^2\mathbf{A}_3-38\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2-36\mathbf{A}_3\mathbf{A}_2^2\\
&+6\mathbf{A}_2\mathbf{A}_4+6\mathbf{A}_3^2+4\mathbf{A}_4\mathbf{A}_2+182\mathbf{A}_2^4-50\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5+\Big(279\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2+259\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2+266\mathbf{A}_3\mathbf{A}_2^3\\
&+24\alpha_2\alpha_1\mathbf{A}_2^5-48\mathbf{A}_4\mathbf{A}_2^2-54\mathbf{A}_3^2\mathbf{A}_2-51\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2-76\mathbf{A}_2\mathbf{A}_3^2-72\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3-367\alpha_1\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2\\
&-343\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2-353\alpha_1\mathbf{A}_3\mathbf{A}_2^3+56\alpha_1\mathbf{A}_4\mathbf{A}_2^2+63\alpha_1\mathbf{A}_3^2\mathbf{A}_2+59\alpha_1\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2+88\alpha_1\mathbf{A}_2\mathbf{A}_3^2\\
&+84\alpha_1\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3-480\alpha_1\mathbf{A}_2^3\mathbf{A}_3+1452\alpha_1\mathbf{A}_2^5-8\alpha_1\mathbf{A}_2\mathbf{A}_5-9\alpha_1\mathbf{A}_3\mathbf{A}_4-8\alpha_1\mathbf{A}_4\mathbf{A}_3-5\alpha_1\mathbf{A}_5\mathbf{A}_2\\
&+87\alpha_1\mathbf{A}_2^2\mathbf{A}_4+364\mathbf{A}_2^3\mathbf{A}_3-980\mathbf{A}_2^5+8\mathbf{A}_2\mathbf{A}_5+9\mathbf{A}_3\mathbf{A}_4+8\mathbf{A}_4\mathbf{A}_3+5\mathbf{A}_5\mathbf{A}_2-24\alpha_2\mathbf{A}_2^5-75\mathbf{A}_2^2\mathbf{A}_4\\
&+4\alpha_1^2\mathbf{A}_2^5\Big)\mathbf{e}_0^6+\cdots+O\left(\mathbf{e}_0^9\right),\\
\boldsymbol{\phi}_8=&\Big(2\mathbf{A}_2^2-2\alpha_1\mathbf{A}_2^2\Big)\mathbf{e}_0^3+\Big(33\alpha_1\mathbf{A}_2^3-4\alpha_1\mathbf{A}_2\mathbf{A}_3-3\alpha_1\mathbf{A}_3\mathbf{A}_2-29\mathbf{A}_2^3+4\mathbf{A}_2\mathbf{A}_3+3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4+\Big(50\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\\
&+48\alpha_1\mathbf{A}_3\mathbf{A}_2^2-6\alpha_1\mathbf{A}_2\mathbf{A}_4-6\alpha_1\mathbf{A}_3^2-4\alpha_1\mathbf{A}_4\mathbf{A}_2-306\alpha_1\mathbf{A}_2^4+66\alpha_1\mathbf{A}_2^2\mathbf{A}_3-44\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2-42\mathbf{A}_3\mathbf{A}_2^2\\
&+6\mathbf{A}_2\mathbf{A}_4+6\mathbf{A}_3^2+4\mathbf{A}_4\mathbf{A}_2+240\mathbf{A}_2^4-58\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5+\Big(367\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2+343\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2+353\mathbf{A}_3\mathbf{A}_2^3\\
&+32\alpha_2\alpha_1\mathbf{A}_2^5-56\mathbf{A}_4\mathbf{A}_2^2-63\mathbf{A}_3^2\mathbf{A}_2-59\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2-88\mathbf{A}_2\mathbf{A}_3^2-84\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3-467\alpha_1\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2\\
&-439\alpha_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2-452\alpha_1\mathbf{A}_3\mathbf{A}_2^3+64\alpha_1\mathbf{A}_4\mathbf{A}_2^2+72\alpha_1\mathbf{A}_3^2\mathbf{A}_2+67\alpha_1\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2+100\alpha_1\mathbf{A}_2\mathbf{A}_3^2\\
&+96\alpha_1\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3-612\alpha_1\mathbf{A}_2^3\mathbf{A}_3+2056\alpha_1\mathbf{A}_2^5-8\alpha_1\mathbf{A}_2\mathbf{A}_5-9\alpha_1\mathbf{A}_3\mathbf{A}_4-8\alpha_1\mathbf{A}_4\mathbf{A}_3-5\alpha_1\mathbf{A}_5\mathbf{A}_2\\
&+99\alpha_1\mathbf{A}_2^2\mathbf{A}_4+480\mathbf{A}_2^3\mathbf{A}_3-1452\mathbf{A}_2^5+8\mathbf{A}_2\mathbf{A}_5+9\mathbf{A}_3\mathbf{A}_4+8\mathbf{A}_4\mathbf{A}_3+5\mathbf{A}_5\mathbf{A}_2-32\alpha_2\mathbf{A}_2^5-87\mathbf{A}_2^2\mathbf{A}_4\\
&+4\alpha_1^2\mathbf{A}_2^5\Big)\mathbf{e}_0^6+\cdots+O\left(\mathbf{e}_0^9\right),\\
\mathbf{e}_4=&\Big(-16\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2^2\alpha_2^4+48\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^4\alpha_2^4+16\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^3\alpha_2^4-48\mathbf{A}_3\mathbf{A}_2^5\alpha_2^4-816\mathbf{A}_2^7\alpha_2^3+56\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2^2\alpha_2^3\\
&-216\alpha_2^3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^4-56\alpha_2^3\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^3+216\alpha_2^3\mathbf{A}_3\mathbf{A}_2^5+2280\alpha_2^2\mathbf{A}_2^7-72\alpha_2^2\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2^2+360\alpha_2^2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^4\\
&+72\alpha_2^2\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^3-360\alpha_2^2\mathbf{A}_3\mathbf{A}_2^5-2144\alpha_2\mathbf{A}_2^7+40\alpha_2\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2^2-264\alpha_2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^4-40\alpha_2\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^3
\end{aligned}
$$

$$+264\alpha_2\mathbf{A}_3\mathbf{A}_2^5+678\mathbf{A}_2^7-8\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2^2+72\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^4+8\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^3-72\mathbf{A}_3\mathbf{A}_2^5\Big)/\Big((2\alpha_2-3)$$
$$\times(\alpha_2^2-2\alpha_2+1)\Big)\mathbf{e}_0^8+O\left(\mathbf{e}_0^9\right).$$

□

4. Numerical simulations

Numerical simulations are the direct way for checking the validity, accuracy and efficiency of an iterative method. In this section, we solved eleven initial and boundary value problems. The discretizations are performed by using four pseudospectral collocation methods. The details of absolute errors are tabulated for different parameters on different pseudospectral collocation methods. The verification of theoretical convergence order is crucial, and we adopt the following definition of computational convergence order (COC)

$$\text{COC}=\frac{\log\left(||\mathbf{F}(\mathbf{x}_{k+1})||_\infty/||\mathbf{F}(\mathbf{x}_k)||_\infty\right)}{\log\left(||\mathbf{F}(\mathbf{x}_k)||_\infty/||\mathbf{F}(\mathbf{x}_{k-1})||_\infty\right)}.$$

4.1. *Verification of computational convergence order*

First we consider the following system of nonlinear equations $\mathbf{F}(\mathbf{x})=[F_1(\mathbf{x}),F_2(\mathbf{x}),F_3(\mathbf{x}),F_4(\mathbf{x})]^T=\mathbf{0}$,

$$\begin{aligned}F_1(\mathbf{x})&=x_2\,x_3+x_4\,(x_2+x_3)=0,\\F_2(\mathbf{x})&=x_1\,x_3+x_4\,(x_1+x_3)=0,\\F_3(\mathbf{x})&=x_1\,x_2+x_4\,(x_1+x_2)=0,\\F_4(\mathbf{x})&=x_1\,x_2+x_3\,(x_1+x_2)=1.\end{aligned}\tag{4.1}$$

Table 1: MFA : verification of convergence order for the problem (4.1).

Iters	$\|\|\mathbf{F}(\mathbf{x}_k)\|\|_\infty$	COC
1	2.50e-1	-
2	5.36e-8	-
3	9.15e-69	9.11
4	1.12e-615	9.00

Table 1 shows that COC is nine which is higher than our claimed convergence order. The verification of theoretical convergence order is hard to verify by solving the system of nonlinear equations associated with IVPs and BVPs.

4.2. *Solving the initial and boundary value problems*

The temporal and spatial discretizations of all IVPs and BVPs are made by employing different J-GL-C methods. The discrete problems are the systems of nonlinear algebraic equations and we used our proposed iterative method DEDF to solve them. In the majority of the solved problems, the vector **0** is the initial guess. The initial and boundary conditions are embedded in the zero vector, and this inclusion makes the initial guess non-smooth. In most of the numerical simulations, the non-smooth initial guess works well. But in some numerical simulations of schrödinger equations, we get divergence and then we make the initial guess smooth to get the convergence.

4.3. *Lane-Emden equation*

$$x''(t) + \frac{2}{t}x'(t) + x(t)^p = 0, \quad x'(0) = 0, \quad x(0) = 1. \tag{4.2}$$

The discrete Lane-Emden equation (4.2) is obtained by using J-GL-C method

$$\mathbf{F}(\mathbf{x}) = \mathbf{S}_{xx}\mathbf{x} + \text{diag}\left(\frac{2}{t}\right)\mathbf{S}_x\,\mathbf{x} + \mathbf{x}^p = \mathbf{0},$$
$$\mathbf{F}'(\mathbf{x}) = \mathbf{S}_{xx} + \text{diag}\left(\frac{2}{t}\right)\mathbf{S}_x + p\,\text{diag}\left(\mathbf{x}^{p-1}\right),$$

where diag$(\cdot)$ means a diagonal matrix, $\mathbf{S}_x = 2/b\,\mathbf{Q}_x$, $\mathbf{S}_{xx} = (2/b\,\mathbf{Q}_x)^2$, $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T$ and $[0, b]$ is the domain of the problem. Figure 1 depicts the numerical solutions of different Lane-Emden equations and initial guess. Table 2 shows infinity norm of residue function for different values of index p of Lane-Emden equation. We performed three iterations (Iters) to show the convergence of our proposed iterative method DEDF.

Table 2: Norm of residue of system of nonlinear equation associated with the problem (4.2) over the domain [0,3], number of grid points = 50, $(\theta, \phi) = (-1/2, -1/2)$.

Iters \ p		2	3	4	5
1	$\|\mathbf{F}(\mathbf{x}_k)\|_\infty$	6.58e-03	3.14e-02	2.39e-02	3.51e-02
2	-	8.73e-35	1.43e-31	91.67e-27	4.07e-25
3	-	5.98e-278	3.96e-255	7.26e-226	3.07e-207

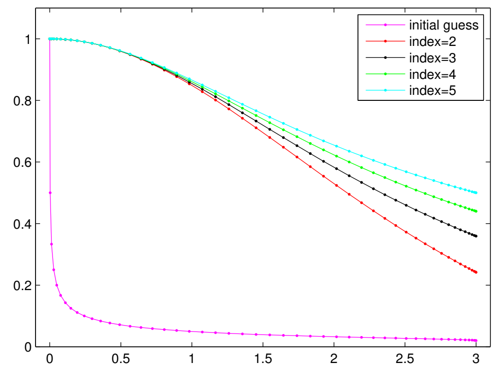

Figure 1: Solution of the Lane-Emden equation (4.2) with different indices, number of gird points=50.

4.4. *Bratu-problem*

$$x''(t) + \alpha\, e^{x(t)} = 0, \quad x(0) = 0, \;\; x(1) = 0, \tag{4.3}$$

where α is a parameter. The associated system of nonlinear equations in (4.3), by using J-GL-C method, can be written as

$$\mathbf{F}(\mathbf{x}) = \mathbf{S}_{xx}\mathbf{x} + \alpha\, e^{\mathbf{x}} = \mathbf{0},$$
$$\mathbf{F}'(\mathbf{x}) = \mathbf{S}_{xx} + \alpha\, \mathrm{diag}(e^{\mathbf{x}}),$$

where $\mathbf{S}_{xx} = (2\mathbf{Q}_x)^2$, $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T$. The Bratu problem has a parameter α and we solved it against the several values of parameter α. Figure 2 shows the solution of the Bratu problem for different values of α and norm of residues of associated system of nonlinear equations are listed in Table 3. We can see that the DEDF iterative method leads to a rapid convergence.

Table 3: Norm of residue of the Bratu problem (4.3) over the domain [0,1], number of grid points 50, $(\theta, \phi) = (-1/2, -1/2)$.

Iters \ α		1	2	3
1	$\|\mathbf{F}(\mathbf{x}_k)\|_\infty$	6.21e-9	4.45e-7	6.05e-7
2	-	2.72e-75	5.38e-59	1.52e-57
3	-	3.40e-542	5.20e-425	1.00e-411

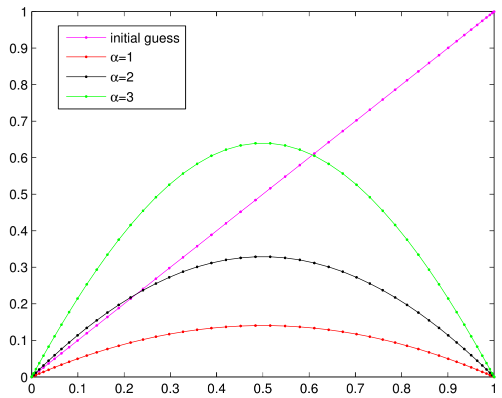

Figure 2: Plot of the solution of Bratu problem (4.3) with different parameter values, number of gird points 50.

4.5. *Frank-Kamenetzkii problem*

Consider the problem

$$x''(t) + \frac{1}{t}x'(t) + \alpha\, e^{x(t)} = 0, \quad x'(0) = 0, \;\; x(1) = 0, \tag{4.4}$$

where α is a parameter. Hence, we have

$$\mathbf{F}(\mathbf{x}) = \mathbf{S}_{xx}\mathbf{x} + \mathrm{diag}\left(\frac{1}{\mathbf{t}}\right)\mathbf{S}_x\mathbf{x} + \alpha\, e^{\mathbf{x}} = \mathbf{0},$$
$$\mathbf{F}'(\mathbf{x}) = \mathbf{S}_{xx} + \mathrm{diag}\left(\frac{1}{\mathbf{t}}\right)\mathbf{S}_x + \alpha\, \mathrm{diag}\left(e^{\mathbf{x}}\right),$$

where $\mathbf{x}^T = [x_1, x_2, \cdots, x_n]^T$. The Frank-Kamenetzkii problem also has a parameter α. The simulated results are depicted in Table 4 and Figure 3.

Table 4: Norm of residue of the problem (4.4) over the domain [0,1], number of grid points 50, $(\theta, \phi) = (-1/2, -1/2)$.

Iters \ α		1	1.1	1.2	1.3
1	$\|\|\mathbf{F}(\mathbf{x}_k)\|\|_\infty$	2.96e-7	5.18e-7	8.31e-7	1.19e-6
2	-	3.11e-62	1.85e-60	2.23e-59	3.03e-60
3	-	1.40e-448	1.80e-435	1.00e-426	1.40e-426

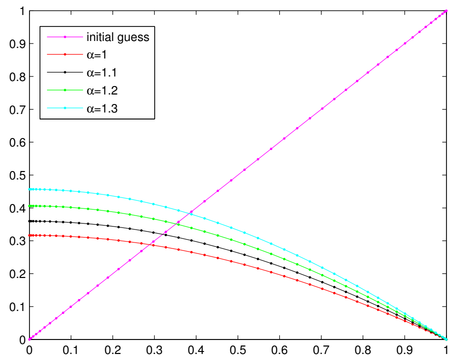

Figure 3: Plot of the solution of Frank-Kamenetzkii problem (4.4) with different parameter values, number of gird points 50.

4.6. 1+1 nonlinear Schrödinger equations

Consider the problem

$$i\,\partial_t\psi(x,t) + \partial_{xx}\psi(x,t) + 2\,\gamma\,|\psi(x,t)|^2\,\psi(x,t) - 2\,\delta\,R(x,t)\,\psi(x,t) = 0\,, \quad (x,t) \in D_x \times D_t\,,$$

where

$$D_x = [a_x, b_x]\,, D_t = [0, t_f]\,,$$

with the initial-boundary conditions

$$\psi(a_x,t) = \eta_1(t)\,, \quad \psi(b_x,t) = \eta_2(t)\,,$$
$$\psi(x,0) = \zeta(x)\,.$$

The complex function $\psi(x,t)$ can be written as $\psi(x,t) = \psi_1(x,t) + i\,\psi_2(x,t)$. The Schrödinger equation in the real function $\psi_1(x,t)$ and $\psi_2(x,t)$ is

$$\begin{bmatrix} \partial_t & \partial_{xx} \\ \partial_{xx} & -\partial_t \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} - 2\delta R(x,t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + 2\gamma\left(\psi_1^2 + \psi_2^2\right) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0\,, \tag{4.5}$$

with initial-boundary conditions

$$\psi_1(a_x,t) + i\,\psi_2(a_x,t) = \eta_{11}(t) + i\,\eta_{12}\,, \quad \psi_1(b_x,t) + i\,\psi_2(b_x,t) = \eta_{21}(t) + i\,\eta_{22}(t),$$
$$\psi_1(x,0) + i\,\psi_2(x,0) = \zeta_1(x) + i\,\zeta_2(x)\,.$$

The discretization of (4.5) by using J-GL-C method leads to

$$\begin{aligned} \mathbf{F}(\boldsymbol{\psi}) = & \begin{bmatrix} \mathbf{S}_t \otimes \mathbf{I}_x & \mathbf{I}_t \otimes \mathbf{S}_{xx} \\ \mathbf{I}_t \otimes \mathbf{S}_{xx} & -\mathbf{S}_t \otimes \mathbf{I}_x \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{bmatrix} - 2\delta \begin{bmatrix} \mathbf{O} & \mathrm{diag}(\mathbf{R}) \\ \mathrm{diag}(\mathbf{R}) & \mathbf{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{bmatrix} \\ & + 2\gamma \begin{bmatrix} \mathbf{O} & \mathrm{diag}\left(\boldsymbol{\psi}_1^2 + \boldsymbol{\psi}_2^2\right) \\ \mathrm{diag}\left(\boldsymbol{\psi}_1^2 + \boldsymbol{\psi}_2^2\right) & \mathbf{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{bmatrix} = \mathbf{0}, \end{aligned} \tag{4.6}$$

$$\mathbf{F}'(\boldsymbol{\psi}) = \begin{bmatrix} \mathbf{S}_t \otimes \mathbf{I}_x & \mathbf{I}_t \otimes \mathbf{S}_{xx} \\ \mathbf{I}_t \otimes \mathbf{S}_{xx} & -\mathbf{S}_t \otimes \mathbf{I}_x \end{bmatrix} - 2\delta \begin{bmatrix} \mathbf{O} & \mathrm{diag}(\mathbf{R}) \\ \mathrm{diag}(\mathbf{R}) & \mathbf{O} \end{bmatrix} + 2\gamma \begin{bmatrix} 2\,\mathrm{diag}(\boldsymbol{\psi}_1 \odot \boldsymbol{\psi}_2) & 3\,\mathrm{diag}(\boldsymbol{\psi}_2^2) \\ 3\,\mathrm{diag}(\boldsymbol{\psi}_1^2) & 2\,\mathrm{diag}(\boldsymbol{\psi}_1 \odot \boldsymbol{\psi}_2) \end{bmatrix},$$

where $\boldsymbol{\psi}_j = [\psi_{j\,(1,1)}, \psi_{j\,(1,2)}, \cdots, \psi_{j\,(1,n_t)}, \cdots, \psi_{j\,(n_x,1)}, \psi_{j\,(n_x,2)}, \cdots, \psi_{j\,(n_x,n_t)}]^T$ for $j = 1, 2$, $\boldsymbol{\psi} = [\boldsymbol{\psi}_1^T\, \boldsymbol{\psi}_2^T]^T$, $\mathbf{R} = [R_{(1,1)}, R_{(1,2)}, \cdots, R_{(1,n_t)}, \cdots, R_{(n_x,1)}, R_{(n_x,2)}, \cdots, R_{(n_x,n_t)}]^T$, $\otimes$ is the Kronecker product, $\odot$ is the element-wise multiplication between two vectors, $\mathbf{O}$ is the zeros matrix, $\mathbf{I}$ is the identity matrix, $\mathbf{S}_t = \left(\frac{2}{t_f}\right) \mathbf{Q}_t$, $\mathbf{S}_{xx} = \left(\frac{2}{b_x - a_x}\mathbf{Q}_x\right)^2$ and $\mathbf{Q}$ is J-GL-C operational matrix. Further the system of nonlinear equations (4.6) can be written as follows

$$\mathbf{F}(\boldsymbol{\psi}) = \mathbf{A}\boldsymbol{\psi} + g(\boldsymbol{\psi}) - \mathbf{p},$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{S}_t \otimes \mathbf{I}_x & \mathbf{I}_t \otimes \mathbf{S}_{xx} \\ \mathbf{I}_t \otimes \mathbf{S}_{xx} & -\mathbf{S}_t \otimes \mathbf{I}_x \end{bmatrix} - 2\delta \begin{bmatrix} \mathbf{O} & \mathrm{diag}(\mathbf{R}) \\ \mathrm{diag}(\mathbf{R}) & \mathbf{O} \end{bmatrix}, \quad g(\boldsymbol{\psi}) = 2\gamma \begin{bmatrix} 2\,\mathrm{diag}(\boldsymbol{\psi}_1 \odot \boldsymbol{\psi}_2) & 3\,\mathrm{diag}(\boldsymbol{\psi}_2^2) \\ 3\,\mathrm{diag}(\boldsymbol{\psi}_1^2) & 2\,\mathrm{diag}(\boldsymbol{\psi}_1 \odot \boldsymbol{\psi}_2) \end{bmatrix},$$

and $\mathbf{p}$ is the vector to incorporate the initial-boundary conditions. A list of four nonlinear Schrödinger equations is given in Table 5, with their corresponding analytical solutions. The numerically computed solutions of four nonlinear Schrödinger equations are tabulated in Tables 6, 7, 8 and 9. In most of the nonlinear Schrödinger equations, the Chebyshev collocation method of first kind shows better accuracy. The numerical solution and absolute errors are visualized in Figures 4, 5, 6 and 7. When the initial guess is non-smooth, due to the inclusion of initial and boundary conditions, the DEDF diverges. Consequently we make the initial guess smooth by applying the following iteration

$$\boldsymbol{\psi} = -\mathbf{A}^{-1}(2g(\boldsymbol{\psi}) - \mathbf{p}),$$

single time for the problem 2 and two times for the problems 1 and 4.

Table 5: Nonlinear Schrödinger equations.

Problem	Schrödinger equation	Analytical solution
1	$i\,\partial_t\psi + \partial_{xx}\psi + 2\lvert\psi\rvert^2\psi = 0$	$\psi(x,t) = e^{i\,(x+t)}$
2	$i\,\partial_t\psi + \partial_{xx}\psi - 2\lvert\psi\rvert^2\psi = 0$	$\psi(x,t) = e^{i\,(x-3t)}$
3	$i\,\partial_t\psi + \frac{1}{2}\partial_{xx}\psi - \lvert\psi\rvert^2\psi - \cos(x)^2\psi = 0$	$\psi(x,t) = e^{\frac{-3it}{2}}\sin(x)$
4	$i\,\partial_t\psi + \partial_{xx}\psi + 2\lvert\psi\rvert^2\psi - 2\,\delta\psi = 0$	$\psi(x,t) = e^{i\,(x+(1-2\delta)t)}$

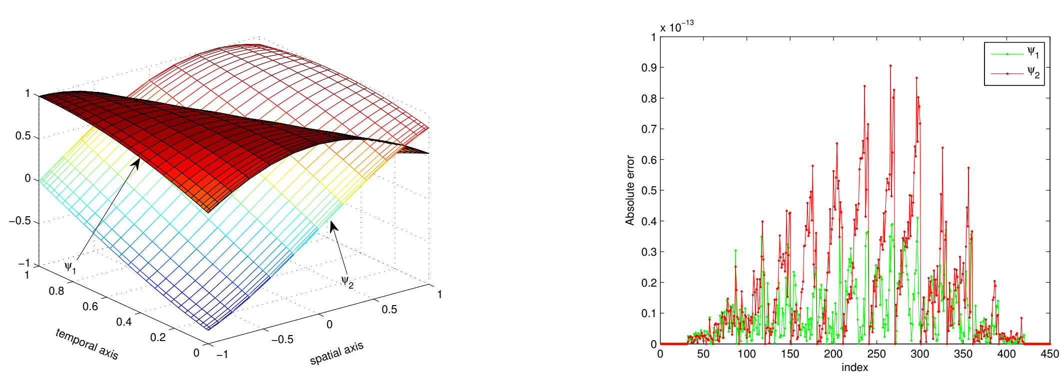

Figure 4: Plot of the solution of nonlinear Schrödinger equation problem 1, $(\theta, \phi) = (-1/2, -1/2)$, $n_x = 15$, $n_t = 30$.

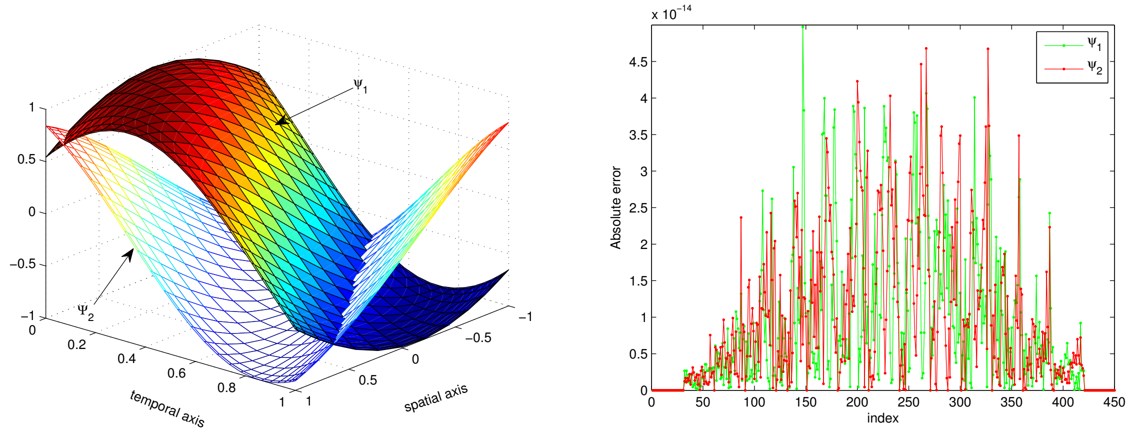

Figure 5: Plot of the solution of nonlinear Schrödinger equation problem 2, $(\theta, \phi) = (-1/2, -1/2)$, $n_x = 15$, $n_t = 30$.

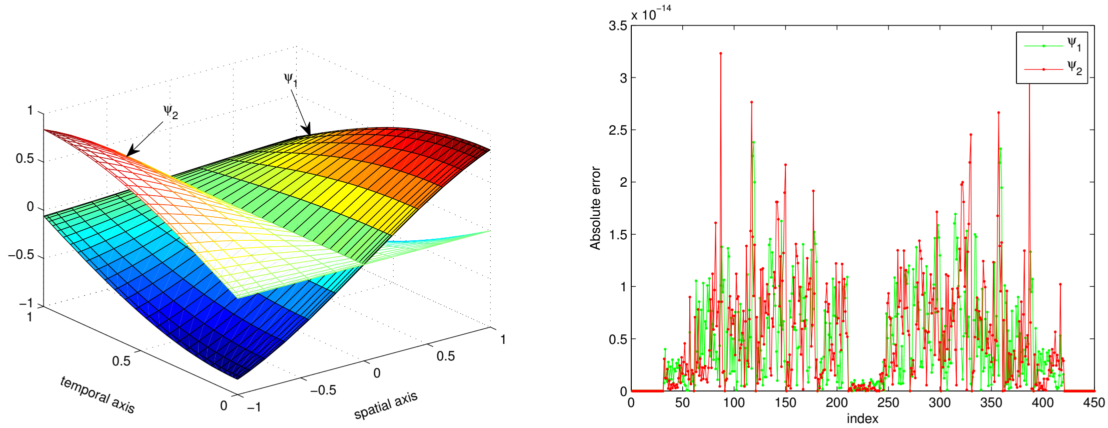

Figure 6: Plot of the solution of nonlinear Schrödinger equation problem 3, $(\theta, \phi) = (-1/2, -1/2)$, $n_x = 15$, $n_t = 30$.

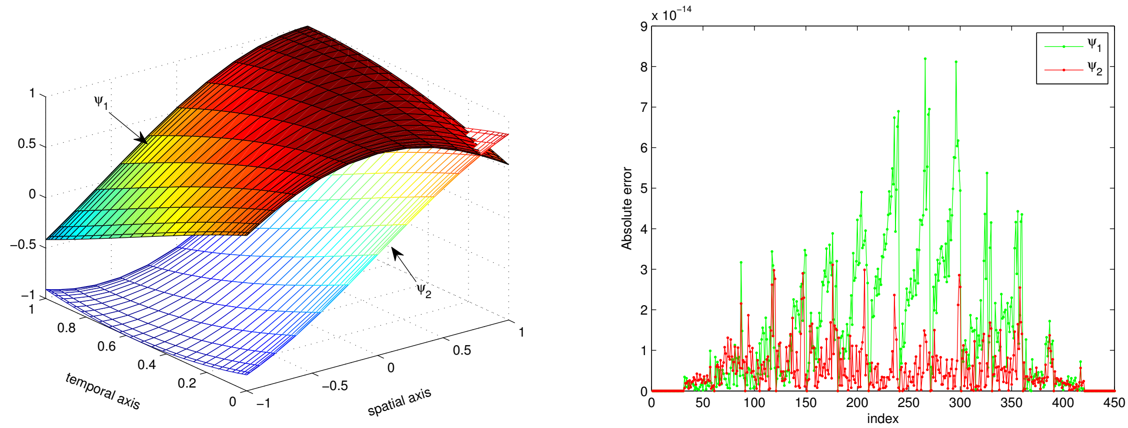

Figure 7: Plot of the solution of nonlinear Schrödinger equation problem 4, $(\theta, \phi) = (-1/2, -1/2)$, $n_x = 15$, $n_t = 30$.

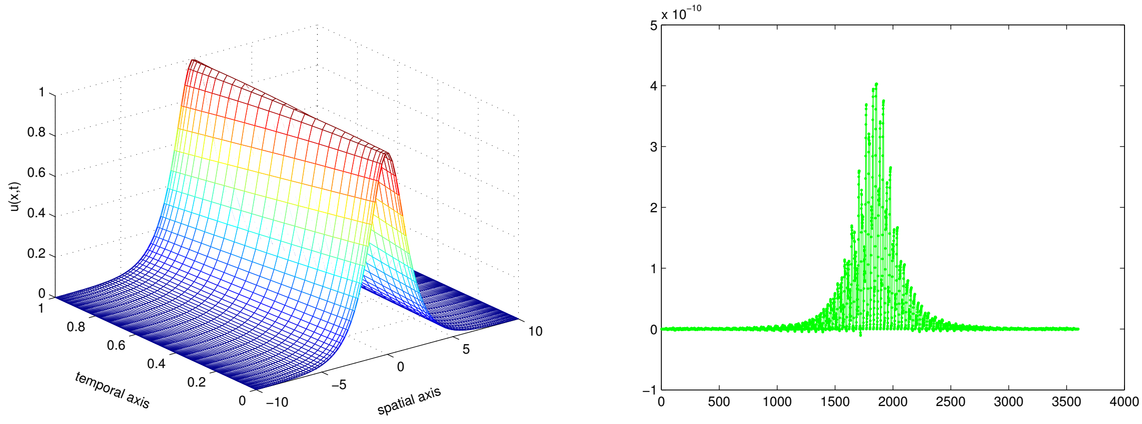

Figure 8: Plot of the solution of nonlinear Klein Gorden equation, $(\theta, \phi) = (-1/2, -1/2)$, $n_x = 120$, $n_t = 30$.

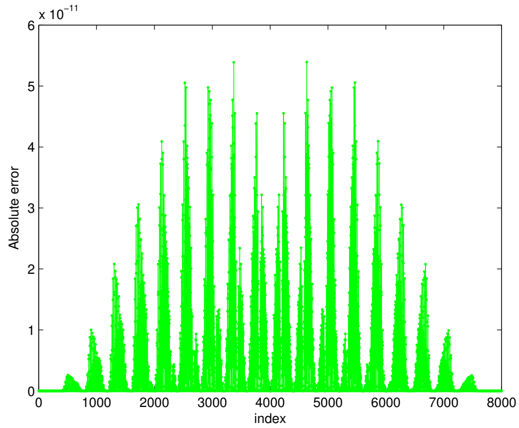

Figure 9: Plot of absolute error for the nonlinear 2-D wave equation, $(\theta, \phi) = (-1/2, -1/2)$, $n_x = 20$, $n_y = 20$, $n_t = 20$.

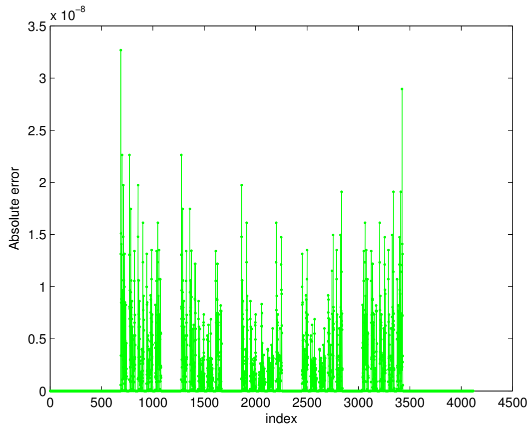

Figure 10: Plot of absolute error for the nonlinear 3-D wave equation, $(\theta, \phi) = (-1/2, -1/2)$, $n_x = 7$, $n_y = 7$, $n_z = 7$, $n_t = 12$.

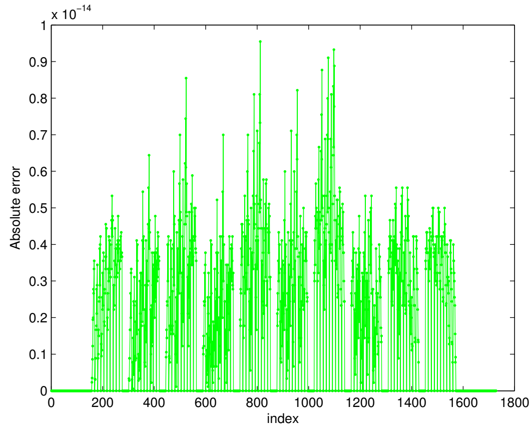

Figure 11: Plot of absolute error for the nonlinear 3-D Poisson equation, $(\theta, \phi) = (-1/2, -1/2)$, $n_x = 20$, $n_y = 20$, $n_z = 20$.

Table 6: Problem 1 in Table 5 over the domain $(x, t) \in [-1, 1] \times [0, 1]$, number of iterations=1, initial guess is $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_2 = \mathbf{0}$.

Iters \ Grid size		5×30	10×30	15×30
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	4.33e-02	7.11e-02	7.33e-02
2	Legendre polynomials	2.85e-04	7.53e-06	7.44e-06
3	$(\theta, \phi) = (0, 0)$	2.85e-04	1.06e-07	1.86e-09
4	-	2.85e-04	3.72e-09	2.57e-12
5	-	2.85e-04	9.47e-11	5.92e-13
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	7.21e-02	divergent	7.27e-02
2	$(\theta, \phi) = (1/2, 0)$	1.59e-03		7.39e-06
3	-	1.59e-03		1.75e-09
4	-	1.59e-03		9.07e-12
5	-	1.59e-03		2.45e-13
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	4.83e-02	7.60e-02	7.84e-02
2	Chebyshev polynomials of second kind	1.14e-03	8.33e-06	9.63e-06
3	$(\theta, \phi) = (1/2, 1/2)$	1.14e-03	4.95e-09	5.52e-09
4	-	1.14e-03	1.90e-09	3.01e-12
5	-	1.14e-03	1.90e-09	5.09e-13
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	3.06e-02	6.88e-02	7.15e-02
2	Chebyshev polynomials of first kind	1.51e-03	7.01e-06	6.93e-06
3	$(\theta, \phi) = (-1/2, -1/2)$	1.52e-03	1.00e-09	8.15e-10
4	-	1.52e-03	4.90e-10	3.54e-13
5	-	1.52e-03	4.90e-10	8.68e-14

Table 7: Problem 2 in Table 5 over the domain $(x,t) \in [-1,1] \times [0,1]$, number of iterations=1, initial guess is $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_2 = \mathbf{0}$.

Iters \ Grid size		5×30	10×30	15×30
1	$\lVert\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\rVert_\infty$	4.33e-02	7.11e-02	7.33e-02
2	Legendre polynomials	2.85e-04	7.53e-06	7.44e-06
3	$(\theta, \phi) = (0,0)$	2.85e-04	1.06e-07	1.86e-09
4	-	2.85e-04	3.72e-09	2.57e-12
5	-	2.85e-04	9.47e-11	5.92e-13
1	$\lVert\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\rVert_\infty$	7.21e-02	divergent	7.27e-02
2	$(\theta, \phi) = (1/2,0)$	1.59e-03	divergent	7.39e-06
3	-	1.59e-03	divergent	1.75e-09
4	-	1.59e-03	divergent	9.07e-12
5	-	1.59e-03	divergent	2.45e-13
1	$\lVert\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\rVert_\infty$	4.83e-02	7.60e-02	7.84e-02
2	Chebyshev polynomials of second kind	1.14e-03	8.33e-06	9.63e-06
3	$(\theta, \phi) = (1/2,1/2)$	1.14e-03	4.95e-09	5.52e-09
4	-	1.14e-03	1.90e-09	3.01e-12
5	-	1.14e-03	1.90e-09	5.09e-13
1	$\lVert\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\rVert_\infty$	3.06e-02	6.88e-02	7.15e-02
2	Chebyshev polynomials of first kind	1.51e-03	7.01e-06	6.93e-06
3	$(\theta, \phi) = (-1/2,-1/2)$	1.52e-03	1.00e-09	8.15e-10
4	-	1.52e-03	4.90e-10	3.54e-13
5	-	1.52e-03	4.90e-10	8.68e-14

4.7. Klein Gorden equation

We consider the problem

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) + k\,u - \gamma\,u(x,t)^3 = 0\,, \quad (x,t) \in D_x \times D_t\,, \tag{4.7}$$

where

$$D_x = [a_x, b_x], \quad D_t = [0, t_f]\,,$$

with the initial-boundary conditions

$$\begin{aligned} u(a_x,t) &= \rho_1(t)\,, \quad u(b_x,t) = \rho_2(t), \\ u(x,0) &= \xi_1(x)\,, \quad u_t(x,0) = \xi_2(x)\,. \end{aligned}$$

Consequently we find,

$$\begin{aligned} \mathbf{F}(\mathbf{u}) &= \mathbf{S}_{tt}\mathbf{u} - c^2\mathbf{S}_{xx}\mathbf{u} + k\,\mathbf{u} - \gamma\,\mathbf{u}^3 = \mathbf{0}, \\ \mathbf{F}'(\mathbf{u}) &= \mathbf{S}_{tt} - c^2\mathbf{S}_{xx} + k\,\mathbf{I} - 3\,\gamma\,\mathrm{diag}\left(\mathbf{u}^2\right)\,, \end{aligned}$$

where

$$\mathbf{u} = [u_{(1,1)}, u_{(1,2)}, \cdots, u_{(1,n_t)}, \cdots, u_{(n_x,1)}, u_{(n_x,2)}, \cdots, u_{(n_x,n_t)}]^T,$$

$$\mathbf{S}_{tt} = \left(\frac{2}{t_f}\mathbf{Q}_t\right)^2, \quad \mathbf{S}_{xx} = \left(\frac{2}{b_x - a_x}\mathbf{Q}_x\right)^2.$$

The analytical solution of (4.7) is $u(x,t) = \delta \operatorname{sech}(\kappa(x - \nu t))$, where $\kappa = \sqrt{\frac{k}{c^2 - \nu^2}}$ and $\delta = \sqrt{\frac{2k}{\gamma}}$. The parameters with their numerical values are $c = 1$, $\gamma = 1$, $\nu = 0.5$ and $k = 0.5$. Table 10 shows that the Chebyshev collocation method of first kind exhibits has the best accuracy in the numerical solution for finer grid. The numerical solution with achieved best absolute error is plotted in Figure 8.

Table 8: Problem 3 in Table 5 over the domain $(x,t) \in [-1,1] \times [0,1]$, number of iterations=1, initial guess is $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_2 = \mathbf{0}$.

Iters	\ Grid size	5×30	10×30	15×30
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	4.33e-02	7.11e-02	7.33e-02
2	Legendre polynomials	2.85e-04	7.53e-06	7.44e-06
3	$(\theta, \phi) = (0, 0)$	2.85e-04	1.06e-07	1.86e-09
4	-	2.85e-04	3.72e-09	2.57e-12
5	-	2.85e-04	9.47e-11	5.92e-13
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	7.21e-02	divergent	7.27e-02
2	$(\theta, \phi) = (1/2, 0)$	1.59e-03		7.39e-06
3	-	1.59e-03		1.75e-09
4	-	1.59e-03		9.07e-12
5	-	1.59e-03		2.45e-13
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	4.83e-02	7.60e-02	7.84e-02
2	Chebyshev polynomials of second kind	1.14e-03	8.33e-06	9.63e-06
3	$(\theta, \phi) = (1/2, 1/2)$	1.14e-03	4.95e-09	5.52e-09
4	-	1.14e-03	1.90e-09	3.01e-12
5	-	1.14e-03	1.90e-09	5.09e-13
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	3.06e-02	6.88e-02	7.15e-02
2	Chebyshev polynomials of first kind	1.51e-03	7.01e-06	6.93e-06
3	$(\theta, \phi) = (-1/2, -1/2)$	1.52e-03	1.00e-09	8.15e-10
4	-	1.52e-03	4.90e-10	3.54e-13
5	-	1.52e-03	4.90e-10	8.68e-14

Table 9: Problem 4 in Table 6 over the domain $(x,t) \in [-1,1] \times [0,1]$, number of iterations=1, initial guess is $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_2 = \mathbf{0}$.

Iters \ Grid size		5×30	10×30	15×30
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	4.33e-02	7.11e-02	7.33e-02
2	Legendre polynomials	2.85e-04	7.53e-06	7.44e-06
3	$(\theta,\phi) = (0,0)$	2.85e-04	1.06e-07	1.86e-09
4	-	2.85e-04	3.72e-09	2.57e-12
5	-	2.85e-04	9.47e-11	5.92e-13
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	7.21e-02	divergent	7.27e-02
2	$(\theta,\phi) = (1/2,0)$	1.59e-03		7.39e-06
3	-	1.59e-03		1.75e-09
4	-	1.59e-03		9.07e-12
5	-	1.59e-03		2.45e-13
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	4.83e-02	7.60e-02	7.84e-02
2	Chebyshev polynomials of second kind	1.14e-03	8.33e-06	9.63e-06
3	$(\theta,\phi) = (1/2,1/2)$	1.14e-03	4.95e-09	5.52e-09
4	-	1.14e-03	1.90e-09	3.01e-12
5	-	1.14e-03	1.90e-09	5.09e-13
1	$\|\boldsymbol{\psi} - \boldsymbol{\psi}_{\text{analytical}}\|_\infty$	3.06e-02	6.88e-02	7.15e-02
2	Chebyshev polynomials of first kind	1.51e-03	7.01e-06	6.93e-06
3	$(\theta,\phi) = (-1/2,-1/2)$	1.52e-03	1.00e-09	8.15e-10
4	-	1.52e-03	4.90e-10	3.54e-13
5	-	1.52e-03	4.90e-10	8.68e-14

4.8. *2-D nonlinear wave equation*

We consider the problem

$$u_{tt}(x,y,t) - c^2\Big(u_{xx}(x,y,t) + u_{yy}(x,y,t)\Big) + f(u) = p(t,x,y)\,, \quad (x,y,t) \in D_x \times D_y \times D_t\,, \tag{4.8}$$

where

$$D_x = [a_x, b_x], \quad D_y = [a_y, b_y], \quad D_t = [0, t_f],$$

with the initial-boundary conditions

$$\begin{aligned} &u(a_x,y,t) = \rho_{11}(y,t)\,, \quad u(b_x,y,t) = \rho_{12}(y,t),\\ &u(x,a_y,t) = \rho_{21}(x,t)\,, \quad u(x,b_y,t) = \rho_{22}(x,t),\\ &u(x,y,0) = \zeta(x,y). \end{aligned}$$

As a consequence we have

$$\begin{aligned} &\mathbf{F}(\mathbf{u}) = (\mathbf{S}_{tt} \otimes \mathbf{I}_x \otimes \mathbf{I}_y)\,\mathbf{u} - c^2\,(\mathbf{I}_t \otimes \mathbf{S}_{xx} \otimes \mathbf{I}_y + \mathbf{I}_t \otimes \mathbf{I}_x \otimes \mathbf{S}_{yy})\,\mathbf{u} + f(\mathbf{u}) - \mathbf{p} = \mathbf{0},\\ &\mathbf{F}'(\mathbf{u}) = \mathbf{S}_{tt} \otimes \mathbf{I}_x \otimes \mathbf{I}_y - c^2\,(\mathbf{I}_t \otimes \mathbf{S}_{xx} \otimes \mathbf{I}_y + \mathbf{I}_t \otimes \mathbf{I}_x \otimes \mathbf{S}_{yy}) + \mathrm{diag}(f'(\mathbf{u}))\,, \end{aligned}$$

where $\mathbf{u} = [u_{(1,1,1)}, u_{(1,1,2)}, \cdots, u_{(1,1,n_t)}, \cdots, u_{(n_x,n_y,1)}, u_{(n_x,n_y,2)} \cdots, u_{(n_x,n_y,n_z)}]^T$, $\mathbf{S}_{tt} = \left(\dfrac{2}{t_f}\mathbf{Q}_t\right)^2$,

$\mathbf{S}_{xx} = \left(\frac{2}{b_x - a_x}\mathbf{Q}_x\right)^2$ and $\mathbf{S}_{yy} = \left(\frac{2}{b_y - a_y}\mathbf{Q}_y\right)^2$. We assume $e^{-t}\sin(x+y)$ as the analytical solution of (4.8) with $c = 1$ and $f(u) = u^3$. The Chebyshev collocation method of first kind over the grid $20 \times 20 \times 20$ achieve best accuracy. Table 11 and Figure 9 represent the results of numerical simulations for the 2-D nonlinear wave equation.

Table 10: Klein Gorden equation (4.7) over the domain $(x, t) \in [-10, 10] \times [0, 1]$, number of iterations=1, initial guess is $\mathbf{u} = \mathbf{0}$.

Iters \ Grid size		30×30	60×30	120×30
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	8.54e-03	1.43e-05	3.29e-06
2	Legendre polynomials	8.53e-03	1.42e-05	4.54e-09
3	$(\theta, \phi) = (0, 0)$	8.53e-03	1.42e-05	4.55e-09
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	7.67e-03	3.55e-05	3.24e-06
2	$(\theta, \phi) = (1/2, 0)$	7.67e-03	3.55e-05	1.45e-09
3	-	7.67e-03	3.55e-05	1.45e-09
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	7.85e-03	1.29e-05	3.30e-06
2	Chebyshev polynomials of second kind	7.85e-03	1.29e-05	2.25e-09
3	$(\theta, \phi) = (1/2, 1/2)$	7.85e-03	1.29e-05	2.25e-09
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	9.28e-03	1.58e-05	3.30e-06
2	Chebyshev polynomials of first kind	9.28e-03	1.58e-05	3.55e-10
3	$(\theta, \phi) = (-1/2, -1/2)$	9.28e-03	1.58e-05	3.26e-10

Table 11: 2-D nonlinear wave equation (4.8) over the domain $(x, y, t) \in [-1, 1] \times [-1, 1] \times [0, 2]$, number of iterations=1, initial guess is $\mathbf{u} = \mathbf{0}$.

Iters \ Grid size		$5 \times 5 \times 20$	$10 \times 10 \times 20$	$20 \times 20 \times 20$
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	1.00e-03	divergent	2.73e-01
2	Legendre polynomials	1.00e-03		8.57e-10
3	$(\theta, \phi) = (0, 0)$	1.00e-03		1.48e-11
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	1.32e-03	2.45e-07	1.54e-03
2	$(\theta, \phi) = (1/2, 0)$	1.32e-03	1.95e-07	1.69e-11
3	-	1.32e-03	1.95e-07	1.69e-11
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	1.17e-03	1.21e-07	4.42e-02
2	Chebyshev polynomials of second kind	1.17e-03	2.74e-07	3.73e-06
3	$(\theta, \phi) = (1/2, 1/2)$	1.17e-03	2.74e-07	2.57e-11
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	7.58e-04	8.23e-07	3.58e-05
2	Chebyshev polynomials of first kind	7.58e-04	2.01e-07	5.39e-11
3	$(\theta, \phi) = (-1/2, -1/2)$	7.58e-04	2.01e-07	5.39e-11

4.9. 3-D nonlinear wave equation

We consider the problem

$$u_{tt}(x,y,z,t)-c^2\Big(u_{xx}(x,y,z,t)+u_{yy}(x,y,z,t)+u_{zz}(x,y,z,t)\Big)+f(u)=p(x,y,z)\,,\\ (x,y,z,t)\in D_x\times D_y\times D_z\times D_t\,, \tag{4.9}$$

where

$$D_x=[a_x,b_x],\quad D_y=[a_y,b_y],\quad D_z=[a_z,b_z],\quad D_t=[0,t_f],$$

with the initial-boundary conditions

$$\begin{aligned} &u(a_x,y,z,t)=\rho_{11}(y,z,t)\,, \quad u(b_x,y,z,t)=\rho_{12}(y,z,t),\\ &u(x,a_y,z,t)=\rho_{21}(x,z,t)\,, \quad u(x,b_y,z,t)=\rho_{22}(x,z,t),\\ &u(x,y,a_z,t)=\rho_{31}(x,y,t)\,, \quad u(x,y,b_z,t)=\rho_{32}(x,y,t),\\ &u(x,y,z,0)=\zeta(x,y,z)\,. \end{aligned}$$

Therefor we find

$$\begin{aligned} \mathbf{F}(\mathbf{u})&=(\mathbf{S}_{tt}\otimes\mathbf{I}_x\otimes\mathbf{I}_y\otimes\mathbf{I}_z)\,\mathbf{u}-c^2\,(\mathbf{I}_t\otimes\mathbf{S}_{xx}\otimes\mathbf{I}_y\otimes\mathbf{I}_z+\mathbf{I}_t\otimes\mathbf{I}_x\otimes\mathbf{S}_{yy}\otimes\mathbf{I}_z+\mathbf{I}_t\otimes\mathbf{I}_x\otimes\mathbf{I}_y\otimes\mathbf{S}_{zz})\,\mathbf{u}\\ &\quad+f(\mathbf{u})-\mathbf{p}=\mathbf{0},\\ \mathbf{F}'(\mathbf{u})&=(\mathbf{S}_{tt}\otimes\mathbf{I}_x\otimes\mathbf{I}_y\otimes\mathbf{I}_z)-c^2\,(\mathbf{I}_t\otimes\mathbf{S}_{xx}\otimes\mathbf{I}_y\otimes\mathbf{I}_z+\mathbf{I}_t\otimes\mathbf{I}_x\otimes\mathbf{S}_{yy}\otimes\mathbf{I}_z+\mathbf{I}_t\otimes\mathbf{I}_x\otimes\mathbf{I}_y\otimes\mathbf{S}_{zz})\\ &\quad+\mathrm{diag}(f'(\mathbf{u}))\,, \end{aligned}$$

where $\mathbf{u}=[u_{(1,1,1,1)},u_{(1,1,1,2)},\cdots,u_{(1,1,1,n_t)},\cdots,u_{(n_x,n_y,n_z,1)},u_{(n_x,n_y,n_z,2)},\cdots,u_{(n_x,n_y,n_z,n_t)}]^T$, $\mathbf{S}_{tt}=\left(\frac{2}{t_f}\mathbf{Q}_t\right)^2$, $\mathbf{S}_{xx}=\left(\frac{2}{b_x-a_x}\mathbf{Q}_x\right)^2$, $\mathbf{S}_{yy}=\left(\frac{2}{b_y-a_y}\mathbf{Q}_y\right)^2$ and $\mathbf{S}_{zz}=\left(\frac{2}{b_z-a_z}\mathbf{Q}_z\right)^2$. We assume $e^{-t}\sin(x+y+z)$ as the solution of (4.9) with $c=0.8$ and $f(u)=u^2$. The Legendre collocation method produces the best numerical results. The absolute error is displayed with respect to different girds in Table 12. The numerical solution and absolute error can be visualized in Figure 10.

4.10. 3-D nonlinear Poisson equation

We consider the problem

$$u_{xx}(x,y,z)+u_{yy}(x,y,z)+u_{zz}(x,y,z)+f(u)=p(x,y,z)\,,\quad (x,y,z)\in D_x\times D_y\times D_z\,, \tag{4.10}$$

where

$$D_x=[a_x,b_x],\quad D_y=[a_y,b_y],\quad D_z=[a_z,b_z],$$

with the initial-boundary conditions

$$\begin{aligned} &u(a_x,y,z)=\rho_{11}(y,z)\,, \quad u(b_x,y,z)=\rho_{12}(y,z),\\ &u(x,a_y,z)=\rho_{21}(x,z)\,, \quad u(x,b_y,z)=\rho_{22}(x,z),\\ &u(x,y,a_z)=\rho_{31}(x,y)\,, \quad u(x,y,b_z)=\rho_{32}(x,y). \end{aligned}$$

Consequently we have

$$\begin{aligned} \mathbf{F}(\mathbf{u})&=(\mathbf{S}_{xx}\otimes\mathbf{I}_y\otimes\mathbf{I}_z+\mathbf{I}_x\otimes\mathbf{S}_{yy}\otimes\mathbf{I}_z+\mathbf{I}_x\otimes\mathbf{I}_y\otimes\mathbf{S}_{zz})\,\mathbf{u}+f(\mathbf{u})-\mathbf{p}=\mathbf{0},\\ \mathbf{F}'(\mathbf{u})&=(\mathbf{S}_{xx}\otimes\mathbf{I}_y\otimes\mathbf{I}_z+\mathbf{I}_x\otimes\mathbf{S}_{yy}\otimes\mathbf{I}_z+\mathbf{I}_x\otimes\mathbf{I}_y\otimes\mathbf{S}_{zz})+\mathrm{diag}(f'(\mathbf{u}))\,, \end{aligned} \tag{4.11}$$

where $\mathbf{u}=[u_{(1,1,1)},u_{(1,1,2)},\cdots,u_{(1,1,n_z)},\cdots,u_{(n_x,n_y,1)},u_{(n_x,n_y,2)}\cdots,u_{(n_x,n_y,n_z)}]^T$, $\mathbf{S}_{xx}=\left(\frac{2}{b_x-a_x}\mathbf{Q}_x\right)^2$, $\mathbf{S}_{yy}=\left(\frac{2}{b_y-a_y}\mathbf{Q}_y\right)^2$ and $\mathbf{S}_{zz}=\left(\frac{2}{b_z-a_z}\mathbf{Q}_z\right)^2$. We assume $\sin(x+y+z)$ as the solution of (4.10) with $f(u)=u^4$. In the case of 3-D nonlinear Poisson equation, the Legendre collocation method offers best accuracy over different sizes grids. Figure 11 depicts the absolute error in the numerically computed solution.

Table 12: 3-D nonlinear wave equation (4.9) over the domain $(x, y, z, t) \in [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$, number of iterations=1, initial guess is $\mathbf{u} = \mathbf{0}$.

Iters \ Grid size		$5 \times 5 \times 5 \times 12$	$6 \times 6 \times 6 \times 12$	$7 \times 7 \times 7 \times 12$
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	5.54e-06	7.52e-08	9.54e-09
2	Legendre polynomials	5.54e-06	7.55e-08	3.92e-09
3	$(\theta, \phi) = (0, 0)$	5.54e-06	7.55e-08	3.92e-09
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	2.24e-05	2.11e-06	1.45e-01
2	$(\theta, \phi) = (1/2, 0)$	2.24e-05	2.11e-06	1.16e-07
3	-	2.24e-05	2.11e-06	1.16e-07
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	1.81e-05	1.76e-06	9.51e-07
2	Chebyshev polynomials	1.81e-05	1.76e-06	5.35e-08
3	of second kind	1.81e-05	1.76e-06	5.35e-08
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	1.92e-05	1.51e-02	3.24e-08
2	Chebyshev polynomials of first kind	1.92e-05	1.08e-06	3.27e-08
3	$(\theta, \phi) = (-1/2, -1/2)$	1.92e-05	1.08e-06	3.27e-08

Table 13: 3-D nonlinear Poisson equation (4.11) over the domain $(x, y, z) \in [0, 1] \times [0, 1] \times [0, 1]$, number of iterations=1, initial guess is $\mathbf{u} = \mathbf{0}$.

Iters \ Grid size		$8 \times 8 \times 8$	$10 \times 10 \times 10$	$12 \times 12 \times 12$
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	6.83e-08	7.52e-08	7.91e-08
2	Legendre polynomials	2.95e-11	1.55e-14	6.99e-15
3	$(\theta, \phi) = (0, 0)$	2.95e-11	1.54e-14	7.22e-15
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	7.49e-08	8.07e-08	8.32e-08
2	$(\theta, \phi) = (1/2, 0)$	1.33e-09	9.39e-13	6.33e-15
3	-	1.33e-09	9.39e-13	6.44e-15
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	6.93e-08	7.62e-08	7.98e-08
2	Chebyshev polynomials	9.59e-10	6.78e-13	1.75e-14
3	of second kind	9.59e-10	6.78e-13	1.81e-14
1	$\|\|\mathbf{u} - \mathbf{u}_{\text{analytical}}\|\|_\infty$	6.65e-08	7.40e-08	7.83e-08
2	Chebyshev polynomials of first kind	5.63e-10	3.26e-13	9.55e-15
3	$(\theta, \phi) = (-1/2, -1/2)$	5.63e-10	3.26e-13	9.77e-15

5. Conclusions

We consider higher order frozen Jacobian iterative methods which are computationally efficient. We compute once the frozen Jacobian LU factors. We use them to solve eight lower and upper triangular systems per iteration. The numerical solution of lower and the upper triangular system is computational economic that makes the entire method computationally efficient. We can see that in most of the simulations, we achieved good accuracy in almost three iterations. Different discretizations collocation methods are used that are derived from J-GL-C method, by changing the parameters. We conclude that in the majority of the cases, the Chebyshev collocation method of first kind gives best accuracy for different sizes grids. Eleven IVPs and BVPs are solved to show the validity, accuracy and efficiency of our higher order iterative method DEDF.

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