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Another proof for the existence of dominated splitting for robustly ergodic diffeomorphisms

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Abstract

In this paper we show that any robustly ergodic system admits a dominated splitting without using pasting lemma for conservative diffeomorphisms.

Keywords: Ergodic, dominated splitting, conservative diffeomorphism.

1. Introduction

We shall address here the question of how the important concepts of robust ergodicity and dominated splitting are related. Ali Tahzibi in [4] studied the relation between robust transitivity and robust ergodicity for conservative diffeomorphism. This is well known that robustly transitive systems admit a dominated splitting.

Ali Tahzibi mentioned an interesting question for robustly ergodicdiffeomorphism as in the following:

1.1. Question Is it true that any C^1 robustly ergodic conservate diffeomorphism admits dominated splitting?

Using pasting lemma for conservative diffeomorphism, A.Arbieto and C.Matheus in [1] showed that robustly transitive conservative diffeomorphisms admit a non-trivial dominated splitting defined on the whole M . So robustly ergodic diffeomorphisms admit a dominated splitting.

In this paper we give another proof of the existence of dominated splitting for robustly ergodic diffeomorphisms without using pasting lemma for conservative diffeomorphisms.

A Df -invariant splitting $E \oplus F$ of TM is called dominated splitting if the fibers of the bundles have constant dimension on whole manifold and there is $\lambda < 1$ such that:

$$\| Df |_{E_x} \| . \| Df^{-1} | F(f(x)) \| \leq \lambda \, \forall x \in M.$$

Let $Diff_m^1(M)$ denote the set of diffeomorphisms which preserve the Lebesgue measure m induced by the Riemannian metric. We endow this space with the C^1-topology.

Let $Diff_m^{1+\alpha}(M)$ denote the subset of $Diff_m^1(M)$ for which the derivative is α-Holder continuous and put $Diff_m^{1+}(M) = \bigcup_{\alpha>0} Diff_m^{1+\alpha}$.

1.2. Theorem (Main theorem)

Let $f \in Diff_m^1(M)$ be robustly ergodic. Then f admits a dominated splitting.

For the prove of the above theorem we need some notions and lemmas.

For a periodic point p of $f \in Diff_m^1(M)$, we assume that the eigenvalues of $Df^{\pi(p)}$ are $\{\lambda_1, \lambda_2, \cdots, \lambda_d\}$ for witch

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_d| .$$

We say that p is an almost source if $|\lambda_1| = 1$; p is an almost sink if $|\lambda_d| = 1$.

1.3. Lemma

Let $f \in Diff_m^1(M)$ be robustly ergodic. Then f has neither almost sinks nor almost sources.

For the proof of the above lemma we need to conservatve version of Franks lemma [2].

1.4. Proposition (Conservative version of Franks Lemma)

Let f be a diffeomorphism preserving a smooth measure m, p be a periodic point. Assume that B is a conservative ε-perturbation of Df along the orbit of p. Then for every neighborhood V of the orbit of p there is a C^1- perturbation $h \in C^{1+\alpha}$ preserving m and coinciding with f on the orbit of p and out of V, such that Dh is equal to B on the orbit of p.

Proof (lemma 1.3)

Since f is robustly ergodic, then there is a neighbourhood U_f of f in $Diff_m^{1+}$ such that every $g \in U_f$ is ergodic. Assume p is an almost sink for f. By conservative version of Franks lemma there is a $g \in U_f \cap Diff_m^{1+\alpha}(M)$ such that p is a sink for g. which is a contradiction because conservative systems have no sink. This completes the proof of lemma.

1.5. Lemma

For any $\varepsilon > 0$ and for any neighbourhood U of f in $Diff_m^1(M)$, there is a periodic point p of $g \in U \cap Diff_m^{1+}(M)$ such that $d_H(Orb_g(p), M) < \varepsilon$.

For the proof of the above lemma we need the ergodic closing lemma as following;

Theorem B(Ergodic closing lemma).

Consider a diffeomorphism f preserving a smooth volume m. Then there is an f-invariant set $\Sigma(f)$, such that:

(1) $\mu(\Sigma(f)) = 1$ for any invariant probability measure μ.

(2) For every $x \in \Sigma(f)$ and $\varepsilon > 0$ there is a C^1-perturbation $g \in C^{1+\alpha}$ preserving m such that x is a periodic point of g and $d(f^i(x), g^i(x)) < \varepsilon$ for all $i \in [0, \pi_g(x)]$, $\pi_g(x)$ is the period of x with respect to g.

Any $x \in \Sigma(f)$ is called a well closable point.

Proof. (lemma 1.5)

Since f is transitive, there is $x \in M$ such that $\omega(x) = M$. There is $N_1 \in \mathbb{N}$ such that $d_H(\{x, f(x), \cdots, f^{N_1}(x)\},\ Orb_f(x)) < \frac{\varepsilon}{4}$, since M is compact. Since $\omega(x) = M$, there is $N_2 > N_1$ such that $d_H(\{x, \cdots, f^n(x)\}, M) < \frac{\varepsilon}{4}$ for any $n > N_2$. Choose $\delta > 0$ such that if for $y \in M$, $d(x, y) < \delta$ then $d(f^i(x), f^i(y)) < \frac{\varepsilon}{4}$ for $i = 0, \cdots, N_2 + 1$. By Ergodic closing Lemma, since $m(N_{\frac{\delta}{2}}(x)) > 0$, then $\sum(f) \cap N_{\frac{\delta}{2}}(x) \neq \emptyset$. Let $p \in N_{\frac{\delta}{2}}(x)$ be a well closable point. For $\delta > 0$ there is a C^1-perturbation $g \in Diff_m^{1+\alpha}(M)$ such that p is a periodic point of g and $d(f^i(p), g^i(p)) < \frac{\delta}{2}$ for all $i \in [0, \pi_g(p)]$, where $\pi_g(p) > N_2$ is the period of p with respect to g. Since $d(x, p) < \delta$ then, $d(f^i(x), f^i(p)) < \frac{\varepsilon}{4}$ for $i = 0, \cdots, N_2 + 1$. So $d(f^i(x), g^i(p)) < \frac{\varepsilon}{2}$ for $i = 0, \cdots, N_2 + 1$. By the above process we have $d_H(Orb_g(p), M) < \varepsilon$.

As a corollary,

1.6. Corollary

There are a sequence of diffeomorphisms $\{f_n\}$ *in* $Diff_m^{1+}(M)$ *and a sequence of point* $\{p_n\}$ *such that* p_n *is a periodic point of* f_n $\lim f_n = f$ *and* $\lim Orb(p_n) = M$.

For the proof of the main theorem we also need the following Lemma in [2, Lemma 1.4].

1.7. Lemma

Give $c > 0$ *and* $A \in (0,1)$ *. If there is a sequence of diffeomorphisms* $\{f_n\}$ *and a sequence of compact sets* $\{\Lambda_n\}$ *such that* Λ_n *is a compact invariant set of* f_n *and* Λ_n *admits a* (c, A) *-dominated splitting of index* i *with respect to* f_n*, then if* $\Lambda = \lim_{n\to\infty} \Lambda_n$ *exists, then* Λ *admits a* (c, λ) *-dominated splitting of index* i *with respect to* f *.*

Proof. **(main theorem1.2)**

Let $\{p_n\}$ be in the above corollary.

Let $\Sigma = \coprod_{n\in\mathbb{N}} \{p_n, f(p_n), \cdots, f^{\pi(p_n)-1}(p_n)\}$. One can define a natural d-dimensional vector bundle E on Σ as following:

for any $x \in \Sigma$, the fiber on x is $T_x M$. For any $i \in [0, \pi(p_n) - 1 \cap \mathbb{N}]$, we define $h(f_n^i(p_n)) = f_n^{i+1}(p_n)$ and $A|_{E(f_n^i(p_n))} = Df_n(f_n^i(p_n))$. Thus $\mathcal{A} = (\Sigma, h, E, A)$ is a bounded large periodic systems as in [3]. Then by [3,Theorem 2.2] either there is an infinite subset $\Sigma' \subset \Sigma$ which is invariant by h such that the periodic linear cocycle $\mathcal{A}' = (\Sigma', h, E|_{\Sigma'}, A)$ admits a dominated splitting or there is a perturbation $\mathcal{B}$ of $\mathcal{A}$ and an in finite invariant subset Σ' of Σ such that for any $x \in \Sigma$, all eigenvalues of $B(h^{\pi(x)-1}(x)) \circ B(h^{\pi(x)-2}(x)) \circ \cdots \circ B(x)$ are real, with same modulus.

By Remark 7.2 in [2] we can consider the perturbation A' such that $detA'(x) = 1$ for every $x \in \Sigma'$. Then by proposition 1.4 we can translate the above statement for $Diff_m^1(M)$.

either there are constant $c > 0$ and $A \in (0,1)$ such that there is a $c, \lambda)$ -dominated splitting on the orbit $\{p_n\}$; or there is a sequence of diffeomorphism $\{g_n\}$ in $Diff_m^{1+}(M)$ such that $\lim_{n\to\infty} g_n = f$ and $Orb_{f_n}(p_n)$ is also a periodic orbit of g_n and all eigenvalues of $Dg_n^{\pi(p_n)}(p_n)$ are all real, with same modulus.

So by an perturbation there is an almost sink or an almost source for a $g \in Diff_m^{1+}(M)$ near f which contradicts to the fact that f is robustly ergodic. Thus the second case of the above statement is false for f.

Now by corollary1.6 and letting $\Lambda_n = M$ in the Lemma1.7, the proof of the main theorem is complete.

Pengfel Zhang in [5] showed that if f has a dominated splitting $TM = E \oplus F$, then f can not be minimal. Recall that the map f is said to be minimal if for each $x \in M$, the orbit $O(x) = \{f^n(x) : n \in Z\}$ is a dense subset in M. So we have the following corollary.

1.8. Corollary

If f is robustly ergodic, then f is not minimal.

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