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SOME REMARKS ON CONVEXITY OF ČEBYŠEV SETS

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Abstract. In this paper, we study a part of approximation theory that presents the conditions under which a Čebyšev set in a Banach space is convex. To do so, we use Gateaux differentiability of the distance function.

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1. Introduction

In a finite dimensional smooth normed space a Čebyšev set is convex and for infinite dimensional, every weakly closed Čebyšev set in a smooth and uniformly convex Banach space is convex. Every boundedly compact Čebyšev set in a smooth Banach space is convex and in a Banach space, which is uniformly smooth, each approximately compact Čebyšev set is convex (The concept of approximatively compact sets introduced by N. V. Efimov and S. B. Stechkin), and that in a strongly smooth space or in a Banach space X with strictly convex dual X^*, every Čebyšev set with continuous metric projection is convex, ([2]). There are still several open problems concerning convexity of Čebyšev sets. Can we prove that in some Banach spaces, a nonempty subset is a Čebyšev set if and only if it is closed and convex? This is unsolved, even in the special case of infinite-dimensional Hilbert space. As addressed above, it is unknown. In the last part of the paper, we present some conditions under which a Čebyšev subset is convex.

2. Main results

As the first step, let us fix our notation. Through this paper, $(X, \|.\|)$ denotes a real Banach space and $S(X) = \{x \in X; \|x\| = 1\}$.

For an element $x \in X$ and a nonempty subset K in X, we define the distance function $d_K : X \to \mathbb{R}$ by $d_K(x) = inf\{\|y - x\|; y \in K\}$. It is easy to see that the value of $d_K(x)$ is zero if and only if x belongs to $\overline{K}$, the closure of K. The subset K is called proximinal (resp. Čebyšev), if for each $x \in X \setminus K$, the set of best approximations to x from K

$$P_K(x) = \{y \in K; \|y - x\| = d_K(x)\},$$

is nonempty (resp. a singleton). This concept was introduced by S. B. Stechkin and named after the founder of best approximation theory, Čebyšev.

One interesting and fruitful line of research, dating from the early days of Banach space theory, has been to relate analytic properties of a Banach space to various geometrical conditions on the Banach space. The simplest example of such a condition is that of strict convexity. It is often convenient to know whether the triangle inequality is strict for non collinear points in a given Banach space. We say that the norm $\|.\|$ of X is strictly convex (rotund) if,

$$\|x + y\| < \|x\| + \|y\|$$

whenever x and y are not parallel. That is, when they are not multiples of one another.

Related to the notion of strict convexity, is the notion of smoothness.

We say that, the norm $\|.\|$ of X is smooth at $x \in X \setminus \{0\}$ if, there is a unique $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Of course, the Hahn-Banach theorem ensures the existence of at least one such functional f.

The spaces $L^p(\mu), 1 < p < \infty$, are strictly convex and smooth, while the spaces

$L^1(\mu)$ and $C(K)$ are neither strictly convex nor smooth except in the trivial case when they are one dimensional.

If the dual norm of X^* is smooth, then the norm of X is strictly convex and if the dual norm of X^* is strictly convex, then the norm of X is smooth. Note that, The converse is true only for reflexive spaces. There are examples of strictly convex spaces whose duals fail to be smooth.

Let $f : X \to \mathbb{R}$ be a function and $x, y \in X$. Then f is said to be Gateaux differentiable at x if, there exists a functional $A \in X^*$ such that $A(y) = \lim_{t \to 0} \frac{f(x+ty) - f(x)}{t}$. In this case f is called Gateaux differentiable at x with the Gateaux derivative A and A is denoted by $f'(x)$. In this case, the $A(y)$ is denoted usually by $< f'(x), y >$. If the limit above exists uniformly for each $y \in S(X)$, then f is Fréchet differentiable at x with Fréchet derivative A. Similarly, the norm function $\|.\|$ is Gateaux (Fréchet) differentiable at non-zero x if the function $f(x) = \|x\|$ is Gateaux differentiable.

In the general, Gateaux differentiability not imply Fréchet differentiability. For example the canonical norm of l^1 is nowhere Fréchet differentiable and it is Gateaux differentiable at $x = (x_i)_{i \in \mathbb{N}}$ if and only if $x_i \neq 0$ for every $i \in \mathbb{N}$.

The norm of any Hilbert space, is Fréchet differentiable at nonzero points.

Suppose $f : X \to \mathbb{R}$ is a function and $x \in X$. The functional $x^* \in X^*$ is called a subdifferential of f at x if $\langle x^*, y - x \rangle \leq f(y) - f(x)$, for all $y \in X$. The set of all subdifferentials of f at x is denoted by $\partial f(x)$ and we say that f is subdifferentiable at x if $\partial f(x) \neq \emptyset$.

The following theorems presents relationship between various notions of differentiability for norm and the properties of the related space.

Theorem 1. [4] The norm $\|.\|$ is Gateaux differentiable at $x \in X \setminus \{0\}$ if and only if X is smooth in x.

Theorem 2. [4] If the dual norm of X^* is Fréchet differentiable, then X is reflexive.

Theorem 3. [4] Let $f : X \to \mathbb{R}$ be a convex function continuous at $x \in X$ and $\partial f(x)$ is a singleton. Then f is Gateaux differentiable at x.

For a real-valued function ϕ on X and $x \in X$, set

$$F_\phi(x) = \sup_{\|y\|=1} \sup_{z \in X} \limsup_{t \to 0^+} \frac{\phi(x+tz+ty) - \phi(x+tz)}{t}.$$

Lemma 1. [3] Let ϕ is a real-valued function on X, $x \in X$ and $y_0 \in S(X)$ such that the Gateaux derivative of ϕ in x exists and $\langle \phi'(x), y_0 \rangle = F_\phi(x)$. If the norm of X is Gateaux differentiable at y_0 with Gateaux derivative f_{y_0}, then ϕ is Gateaux differentiable at x and for each $y \in X$ we have $\langle \phi'(x), y \rangle = F_\phi(x) f_{y_0}(y)$.

Now the Lemma 1, give us the following corollary, since distance functions are Lipschitz.:

For nonempty closed subset K of X and $x, y \in X$, set

$$d_K^-(x; y) = \liminf_{t \to 0^+} \frac{d_K(x+ty) - d_K(x)}{t}$$

and

$$d_K^+(x; y) = \limsup_{t \to 0^+} \frac{d_K(x+ty) - d_K(x)}{t}.$$

Corollary 1. [1] Suppose $K \subseteq X$ is closed and nonempty, $x \in X \backslash K$, $\overline{x}$ is a nearest point for x in K. If the norm of X is Gateaux differentiable at $(x - \overline{x})$ and $d_K^-(x; x - \overline{x}) = d_K(x)$, then d_K is Gateaux differentiable at x.

Theorem 4. [4] If the dual space of X is strictly convex, then each closed nonempty subset K in X satisfying $\limsup_{\|y\| \to 0} \dfrac{d_K(x+y) - d_K(x)}{\|y\|} = 1$ for all $(x \in X \backslash K)$ is convex.

Remark 1. Suppose that the norm of X and the dual norm of X^* are Fréchet differentiable, $K \subseteq X$ is Čebyšev and $x \in X \backslash K$. Then X is reflexive, since the dual norm of X^* is Fréchet differentiable. Moreover X is smooth, since the norm of X is Fréchet differentiable. Thus X^* is strictly convex. If now d_K is Gateaux differentiable at x, then K is convex.

Remark 2. Suppose that $K \subseteq X$ is Čebyšev, $x \in X \backslash K$ and X^* is strictly convex. By the definition of Čebyšev sets, there is unique $\overline{x} \in K$ such that $\|x - \overline{x}\| = d_K(x)$. If now $d_K^-(x; x - \overline{x}) = d_K(x)$, then by corollary 1 and Remark 1, K is convex.

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