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Solving Nonlinear System of Mixed Volterra-Fredholm Integral Equations by Using Variational Iteration Method

M. Rabbani[1]
Department of Mathematics, Sari branch, Islamic Azad university, Sari, Iran
mrabbani@iausari.ac.ir

R. Jamali
Department of Mathematics, Sari branch, Islamic Azad university, Sari, Iran
jamaliroholla@yahoo.com

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Abstract

In this paper for solving nonlinear system of mixed Volterra-Fredholm integral equations by using variational iteration method, we have used differentiation for converting problem to suitable form such that it can be useful for constructing a correction functional with general lagrange multiplier. The optimum of lagrange multiplier can be found by variational theorem and by choosing of restrict variations properly. By substituting of optimum lagrange multiplier in correction functional, we obtain convergent sequences of functions and by appropriate choosing initial approximation, we can get approximate of the exact solution of the problem with few iterations. Some applications of nonlinear mixed Volterra-Fredholm integral equations arise in mathematical modeling of the Spatio-temporal development of an epidemic. So nonlinear system of mixed Volterra-Fredholm integral equations is important and useful. The above method independent of small parameter in comparison with similar works such as perturbation method. Also this method does not require discretization or linearization. Accuracy of numerical results show that the method is very effective and it is better than Adomian decomposition method since it has faster convergence and it is more simple. Also this method has a closed form and avoids the round of errors for finding approximation of the exact solution. The looking forward the proposed method can be used for solving various kinds of nonlinear problems.

[1] Corresponding author

Keywords: Nonlinear system of mixed; Integral equation; Variational method; Volterra-Fredholm; lagrange multiplier

1 Introduction

In 1987 Inokuti in[17] proposed a general Lagrange multiplier method for solving nonlinear differential equations where he used nonlinear operator forms as follows:

$$Lu + NU = g(x), \tag{1}$$

where L is a linear operator, N is a nonlinear operator, $g(x)$ is a known analytic function and u is an unknown function that to be determined. Also on the supposition that u_0 is the solution of $LU = 0$, in some of the special point such at $x = 1$, the following form was used by Inokuti in[17],

$$u_{cor}(1) = u_0(1) + \int_0^1 \lambda\left(Lu_0 + Nu_0 - g\right)dx, \tag{2}$$

where λ is a general lagrange multiplier and optimum value. It can be found via variational theorem in[11, 17]. The Inokuti method is modified by He which it can be written,

$$u_{n+1}(x_0) = u_n(x_0) + \int_0^{x_0} \lambda\left(Lu_n + N\tilde{u}_n - g\right)ds, \tag{3}$$

where u_0 is an initial approximation and $\tilde{u}_n$ is a restricted variation, in other words $\delta\tilde{u}_n = 0$ in[11]. For arbitrary of x_0, author has converted Eq.(3)to Eq.(4) as follows in[13, 14]:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\left(Lu_n(s) + N\tilde{u}_n(s) - g(s)\right)ds, \tag{4}$$

the above integral in Eq.(4) is called a correction function and index of n denotes the n th approximation. Also Eq.(4)is called variational iteration method(VIM). In [15, 16] this method is used for solving nonlinear problems. The variational iteration method is effective and easy for linear problem because exact solution can be given by only iteration. In the above process Eq.(4) is written in following form after the λ is obtained.

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\left(Lu_n(s) + Nu_n(s) - g(s)\right)ds. \tag{5}$$

In Eq.(5) by using $u_0(x)$ as an initial approximation, we obtain a sequence of approximations of exact solution of Eq.(1). For illustrating of effectively, easily and accurately a large class of nonlinear problems with approximations which converge quickly. we give number of application of variational iteration method. This method is used to solve Burger and coupled Burger equations in [2], (VIM)is used for solving Fokker-Planck equations in [8]. Lane-Emden equations that are kind of Poisson equations for the gravitational potential of a self-gravitating are solved by using(VIM) in [10]. Application of (VIM) to nonlinear Volterra integro-differential equations is investigated in [3]. This method is used for solving nonlinear Helmholtz equation in [20]. Exact solution of integral equations is approximated by (VIM) in [18]. Also in [5], (VIM) is applied for solving nonlinear system of ordinary differential equations. Thus, we can say variational iteration method is a well known method to solve nonlinear equations. The convergence of (VIM) is discussed in [22]. In this work we use (VIM) for solving nonlinear system of mixed Volterra-Fredholm integral equations, For convenience we consider mixed volterra-Fredholm integral equations as follows in [23],

$$u(x,y) = f(x,y) + \int_o^y \int_\Omega G(x,y,s,t,u(s,t))dsdt, (x,y) \in \Omega \times [0,T] \tag{6}$$

where, we suppose that Ω is a closed subset of R, $f(x,y)$ and $G(x,y,s,t,u(s,t))$ are analytic on $D = \Omega \times [0,T]$, $\hat{D}$ respectively such that:

$$\hat{D} = \{(x,y,s,t); 0 \le t \le y \le T, (x,s) \in \Omega^2\} \times R,$$

$u(x,y)$ is an unknown function that to be determined. In[1] homotopy perturbation method is applied to solve Eq.(6) and in the case of linear form is solved by continuous time collocation method in [21]. Also in[4], Eq.(6) is solved by Adomian decomposition method. In [23], the exact solution of Eq.(6) is approximated Adomian modified decomposition method and this method is applied for integral equations in the case of unmixed in [4, 6]. Also collocation method is used in[7] to solve Eq.(6). By choosing suitable norm the existence and uniqueness for Eq.(6)can be found in[9, 12].

2 Nonlinear System of Mixed Volterra-Fredholm Integral Equations

The system of nonlinear integral or differential equations are important in applied sciences and engineering. In [5] linear and nonlinear system of ordinary equations are solved by using He's variational iteration method and in[19] the exact solution of nonlinear system of mixed Volterra-Fredholm integral equations, is approximated by Adomian decomposition method. Now we consider this problem as follows:

$$U(x,y) = F(x,y) + \int_0^y \int_\Omega G(x,y,s,t,U(s,t))dsdt, (x,y) \in \Omega \times [0,T] \tag{7}$$

where

$$U(x,y) = (u_1(x,y), u_2(x,y), ..., u_n(x,y))^T,$$
$$F(x,y) = (f_1(x,y), f_2(x,y), ..., f_n(x,y))^T,$$

$$G(x,y,s,t,U(s,t)) = (g_1(x,y,s,t,U(s,t)), ..., g_n(x,y,s,t,U(s,t))),$$

consider i th equation of Eq.(7) as

$$u_i(x,y) = f_i(x,y) + \int_0^y \int_\Omega g_i(x,y,s,t,u_1(s,t),...,u_n(s,t))dsdt, i = 1,...,n \tag{8}$$

We try to obtain an effective method for solving nonlinear system of mixed Volterra-Fredholm integral equations, so we suppose $\Omega = [0,1]$ and by differentiating from both sides of Eq.(8) respect to y, we have

$$\frac{\partial u_i(x,y)}{\partial y} = \frac{\partial f_i(x,y)}{\partial y} + \int_0^1 g_i(x,y,s,y,u_1(s,y),...,u_n(s,y))ds + \int_0^y \int_0^1 \frac{\partial g_i}{\partial y} dsdt,$$

in other words we can write

$$\frac{\partial u_i}{\partial y} - \frac{\partial f_i}{\partial y} - \int_0^1 g_i(x,y,s,y,u_1(s,y),...,u_n(s,y))ds - \int_0^y \int_0^1 \frac{\partial g_i}{\partial y} dsdt = 0, \tag{9}$$

now, we use (VIM) for Eq.(9), so we obtain the following system of iteration sequences

$$u_{i,n+1}(x,y) = u_{i,n}(x,y) + \int_0^y \lambda_i \{ \frac{\partial f_i(x,\tau)}{\partial \tau} - \frac{\partial u_{i,n}(x,\tau)}{\partial \tau} -$$

$$\int_0^1 g_i(x,\tau,s,\tau,u_1(s,\tau),...,u_n(s,\tau))ds - \int_0^\tau \int_0^1 \frac{\partial g_i}{\partial \tau} dsdt \} d\tau, \qquad (10)$$

$$i = 1,2,...,n \quad n = 0,1,2,...,$$

by using the variational theorem and effect δ in both sides of Eq.(10) and also with assume $\delta u_{i,n+1}(x,y) = 0$, and

$$\delta \left\{ -\int_0^1 g_i(x,y,s,y,u_1(s,y),...,u_n(s,y))ds - \int_0^y \int_0^1 \frac{\partial g_i}{\partial y} dsdt \right\} = 0,$$

we have

$$\delta u_{i,n+1}(x,y) = (1 + \lambda_i(y)) \delta u_{i,n}(x,y) - \int_0^1 \lambda_i^{'}(\tau) \delta u_{i,n}(s,\tau) d\tau = 0.$$

By considering $\delta u_{i,n+1}(x,y) = 0$, optimal values of λ_i is given by the following differential equation:

$$1 + \lambda_i(y) = 0,$$
$$\lambda_i^{'}(\tau)|_{\tau=y} = 0. \qquad (11)$$

So, we get the lagrange multiplier as follows:

$$\lambda_i = -1,$$

by substituting $\lambda_i = -1$ in Eq.(10), we obtain an iteration algebraic system for finding solution of Eq.(7).

3 Numerical Examples

In this section, we solve two examples of the nonlinear system of mixed Volterra-Fredholm integral equations which have solved in[19] by using Adomian decomposition method. Numerical results show that our proposed method has a high accuracy and also it's better than Adomian decomposition method.

Example 1. Consider the following nonlinear system of mixed Volterra-Fredholm integral equations:

$$u(x,y) = \frac{-1}{6}(x^2+y^2)(ycosy - siny - \frac{1}{2}xsiny) + \int_0^y \int_0^1 (x^2+y^2) stu(s,t) dsdt,$$

$$v(x,y) = 0.14726 y^3 (y-x) + ytanx + \int_0^y \int_0^1 (s(x-y) v^2(s,t)) dsdt,$$

with the exact solution $u(x,y) = -\frac{x}{2} siny$, $v(x,y) = ytanx$. For solving the above system we use the iteration algebraic system of (10) when $\lambda_i = -1$. So we find the iteration sequences as follows:

$$u_{n+1}(x,y)=u_n(x,y)-\int_0^y\{\frac{\partial u_n(x,\tau)}{\partial\tau}-\frac{\partial(-\frac{1}{6}(x^2+\tau^2)(\tau cos\tau-sin\tau-\frac{1}{2}xsin\tau))}{\partial\tau}$$
$$-\int_0^1\left[(x^2+\tau^2)s\tau u_n(s,\tau)\right]ds-\int_0^\tau\int_0^1\frac{\partial}{\partial\tau}\left[(x^2+\tau^2)st\,u_n(s,t)\right]dsdt\}\,d\tau,$$

$$v_{n+1}(x,y)=v_n(x,y)-\int_0^y\{\frac{\partial v_n(x,\tau)}{\partial\tau}-\frac{\partial(0.14726\tau^3(\tau-x)+\tau\,tagx)}{\partial\tau}$$
$$-\int_0^1\left[s(x-\tau)v_n^2(s,\tau)\right]ds-\int_0^\tau\int_0^1\frac{\partial[s(x-\tau)v_n^2(s,t)]}{\partial\tau}dsdt\}\,d\tau,$$

$n=0,1,2,...$

by choosing $v_0=u_0=0$ as an initial approximation we show absolute errors by our proposed method in some different points in table1, results compared with Adomian decomposition method. Numerical results for example1.

	proposed method		in[19]	
(x,y)	$\lvert u-u_3\rvert$	$\lvert v-v_3\rvert$	$\lvert u-u_n\rvert$	$\lvert v-v_n\rvert$
(0,0)	0	0	0	0
(0.1,0.1)	2.0×10^{-13}	0	8.7×10^{-7}	0
(0.2,0.2)	1.1×10^{-10}	0	1.9×10^{-5}	0
(0.3,0.3)	4.0×10^{-9}	0	8.8×10^{-5}	0
(0.4,0.4)	7.5×10^{-8}	0	1.2×10^{-4}	0
(0.5,0.5)	6.7×10^{-7}	0	3.7×10^{-4}	0
(0.6,0.6)	4.2×10^{-6}	0	2.8×10^{-3}	0
(0.7,0.7)	2.1×10^{-5}	0	1.0×10^{-2}	0
(0.8,0.8)	8.8×10^{-5}	0	3.0×10^{-2}	0

Table 1: Numerical results for example 1

Example 2. Consider the following system of nonlinear mixed Volterra-Fredholm integral equations:

$$u(x,y)=-y^2-\frac{1}{90}y^3(130-45y+18y^2)+2xy+\int_0^y\int_0^1(y^2+u^2(s,t))dsdt,$$

$$v(x,y)=1-0.0166y(-60-18.3879y^3-302711y^4)(y^2-x)+y^2sinx+$$
$$\int_0^y\int_0^1(x-y^2)v^2(s,t)dsdt,$$

where the exact solution is $u(x,y)=-y^2+2yx$ and $v(x,y)=1+y^2 sinx$. By considering the iteration algebraic system of (10) for $\lambda_i=-1$, we obtain the iteration sequences as follows:

$$u_{n+1}(x,y)=u_n(x,y)-\int_0^y\{\frac{\partial u_n(x,\tau)}{\partial\tau}-\frac{\partial(-\tau^2-\frac{1}{90}\tau^3(130-45\tau+18\tau^2))+2x\tau}{\partial\tau}$$
$$-\int_0^1\left[\tau^2+u_n^2(s,\tau)\right]ds-\int_0^\tau\int_0^1\frac{\partial[\tau^2u_n^2(s,t)]}{\partial\tau}dsdt\}d\tau,$$

$$v_{n+1}(x,y)=v_n(x,y)-$$
$$\int_0^y\{\frac{\partial v_n(x,t)}{\partial\tau}-\frac{\partial[1-0.0166\tau[(-60-18.3879\tau^3-3.2711\tau^4)(\tau^2-x)]+\tau^2 sinx]}{\partial\tau}$$
$$-\int_0^1\left[(x-\tau^2)v_n^2(s,\tau)\right]ds-\int_0^\tau\int_0^1\frac{\partial[(x-\tau^2)v_n^2(s,t)]}{\partial\tau}dsdt\}d\tau,$$

$$n=0,1,2,...,$$

by choosing $u_0=v_0=0$ as an initial approximation we have shown absolute error in table 2 in some different points and our results is compared with Adomian decomposition method in[19].

	proposed method		in[19]	
(x,y)	$\lvert u-u_3\rvert$	$\lvert v-v_3\rvert$	$\lvert u-u_n\rvert$	$\lvert v-v_n\rvert$
(0.125,0.125)	1.40×10^{-10}	6.8×10^{-5}	7.2×10^{-4}	1.3×10^{-5}
(0.250,0.250)	5.1×10^{-8}	9.1×10^{-4}	4.3×10^{-3}	5.4×10^{-4}
(0.500,0.500)	1.1×10^{-8}	1.0×10^{-3}	4.3×10^{-3}	2.5×10^{-2}
(0.625,0.625)	4.0×10^{-5}	1.8×10^{-5}	8.8×10^{-5}	2.6×10^{-6}
(1.000,1.000)	75×10^{-4}	0	1.2×10^{-4}	0

Table 2: Numerical results for example 2

4 Conclusion

In this paper, we have used differentiation for converting problem to a suitable form that can be useful for constructing a correction functional with general lagrange multiplier. By substituting the optimum of lagrange multiplier in correction functional, we have obtained convergent sequences of functions and by appropriate choosing initial approximation we have gotten approximate of the exact solution of the problem with few iterations. So we can say our proposed method has a good accuracy and it's better than Adomian decomposition method.

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