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GRAM-SCHMIDT APPROACH FOR LINEAR SYSTEM OF EQUATIONS WITH FUZZY PARAMETERS

S.H.Nasseri[1],*, M.Sohrabi

[1] Department of Mathematics and Computer Sciences, Mazandaran University, P.O.Box 47415-1468, Babolsar, Iran.

* *E-mail:* nasseri@umz.ac.ir

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ABSTRACT
In this paper, we focus on solving linear system of equations with fuzzy parameters. We employ Dubois and Prades approximate arithmetic operators on LR fuzzy numbers to find a positive fuzzy vector $\tilde{x}$ which satisfies $\tilde{A}\otimes\tilde{x}=\tilde{b}$, where $\tilde{A}$ and $\tilde{b}$ are the fuzzy matrix and vector, respectively. We shall illustrate our method by solving some numerical examples.

Keywords: Fully fuzzy linear system, Fuzzy number, QR-decomposition, Gram-Schmidt method.

1. Introduction

The term of fuzzy matrix, which is the most important concept in this paper, has various meanings. For definition of a fuzzy matrix we follow the definition of Dubois and Prade, i.e. a matrix with fuzzy numbers as its elements [5]. These classes of fuzzy matrices consist of applicable matrices, which can model uncertain aspects and the works on them are too limited. Some of the most interesting works on these matrices can be seen in [2, 3, 4, 5]. A general model for solving a fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy vector, first proposed by Friedman et al. [6]. Friedman and his colleagues used the embedding method and replaced the original fuzzy linear system by a crisp linear system and then they solved it. A review of some methods for solving these systems can be found in [7, 8]. In addition, another important kind of fuzzy linear systems are including fuzzy numbers in whose all parameters and is named fully fuzzy linear systems (see in [3, 4, 7]). Dehghan et.al in [3] and [4] proposed two numerical methods for solving these systems. In [7], authors used a new method for solving these systems based on QR decomposition. Here, based on Gram-Schmidt approach we intend to solve $\tilde{A}\otimes\tilde{x}=\tilde{b}$, where $\tilde{A}$ is the fuzzy matrix and $\tilde{x}$ and $\tilde{b}$ are fuzzy vectors with appropriate sizes. This paper is organized in 5 sections:

In next section, we give some preliminaries and definition concerning to the fuzzy sets theory and in particular fuzzy arithmetic. In Section 3, we describe Gram- Schmidt process to obtain a QR-decomposition coefficient of the matrix of the linear systems. In Section 4, we first define the linear system of equations with fuzzy numbers in all parameters. Numerical examples are given in Section 5 to illustrate our method.

2. Preliminaries

In this section, we review some necessary backgrounds and notions of fuzzy sets theory (taken from [5, 7]).

Definition 2.1. A fuzzy subset of R is defined by its membership function

$$\mu_{\underset{\sim}{A}} : R \to [0,1]$$

which assigns a real number $\mu_{\underset{\sim}{A}}$ in the interval [0, 1], to each element $x \in R$, where the value of $\mu_{\underset{\sim}{A}}$ at x shows the grade of membership of x in $\tilde{A}$. Indeed, a fuzzy subset $\tilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\mu_{\underset{\sim}{A}}$ and is often written

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in R\}.$$

Definition 2.2. The α-level set of a fuzzy set $\tilde{A}$ is defined as an ordinary set $[\tilde{A}]_\alpha$ for which the degree of its membership function exceeds the level α:

$$[\tilde{A}]_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha, \alpha \in [0,1]\}.$$

Definition 2.3. A fuzzy number M is called positive (negative), denoted by M > 0 (M < 0), if its membership function $\mu_M(x)$ satisfies $\mu_M(x) = 0 \quad \forall x \leq 0 (\forall x > 0)$.

Definition 2.4. A fuzzy number M is said to be an LR fuzzy number, if

$$\mu_M(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right), x \leq m, \alpha > 0, \\ R\left(\dfrac{x-m}{\beta}\right), x \geq m, \beta > 0, \end{cases}$$

where m is the mean value of M and α and β are left and right spreads, respectively, and a function L(·) the left shape function satisfying:

$L(x) = L(-x)$,
$L(0) = 1$ and $L(1) = 0$,
$L(x)$ is non-increasing on $[0, \infty)$.

Naturally, a right shape function R(·) is similarly defined as $L(\cdot)$.

Using its mean value and left and right spreads, and shape functions, such an LR fuzzy number M is symbolically written $M = (m, \alpha, \beta)_{LR}$, that, $M = (m, \alpha, \beta)_{LR}$ is positive, if and only if $m - \alpha \geq 0$.

Remark 2.1. We consider $\tilde{0} = (0,0,0)$ as the zero fuzzy number.

Remark 2.2. We show the set of all fuzzy numbers by F(R).

Definition 2.5 (Equality in fuzzy numbers). Two LR fuzzy numbers $M = (m, \alpha, \beta)_{LR}$ and $N = (n, \gamma, \delta)_{LR}$ are said to be equal, if and only if $m = n$, $\alpha = \gamma, \beta = \delta$.

Definition 2.6. For two LR fuzzy numbers $M = (m, \alpha, \beta)_{LR}$ and $N = (n, \gamma, \delta)_{LR}$, therefore:

$$M \oplus N = (m, \alpha, \beta)_{LR} \oplus (n, \gamma, \delta)_{LR} = (m+n, \alpha+\gamma, \beta+\delta)_{LR}$$

$$-M = -(m, \alpha, \beta)_{LR} = (-m, \beta, \alpha)_{RL}$$

$$M - N = (m, \alpha, \beta)_{LR} - (n, \gamma, \delta)_{RL} = (m-n, \alpha+\delta, \beta+\gamma)_{LR}$$

If M > 0 and N > 0, then

$$M \otimes N = (m, \alpha, \beta)_{LR} \otimes (n, \gamma, \delta)_{LR} \cong (mn, n\alpha + m\gamma, n\beta + m\delta)_{LR}$$

Definition 2.7. A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix, if each element of $\tilde{A}$ is a fuzzy number.

A fuzzy matrix $\tilde{A}$ will be positive and denoted by $\tilde{A} > \tilde{0}$, if each element of $\tilde{A}$ be positive. We may represent the fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, such that $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})$, with the new notation $\tilde{A} = (A, M, N)$, where $A = (a_{ij})_{n\times n}$, $M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are three $n \times n$ crisp matrices, that, the matrix A is a center matrix, and M and N are the right and left spread matrices.

3. QR-decomposition

In the following theorem we first give the main condition about QR-decomposition.

Theorem 3.1. If A is an $m \times k$ matrix with full column rank, then A can be factored as

$$A = QR \qquad (3.1)$$

where Q is an $m \times k$ matrix whose column vectors form an orthonormal basis for the column space of A and R is a $k \times k$ invertible upper triangular matrix .

Proof: see [1].

This theorem guarantees that every matrix A with full column rank has a QR-decomposition, in particular, if A is invertible. The fact that Q has orthonormal column implies that $Q^T Q = I$, so multiplying both sides of (3.1) by on the left side

$$R = Q^T A.$$

Thus, one method for finding a QR-decomposition of a matrix A with full rank is to apply the Householder process to the column vectors of A, then form the matrix Q from the resulting orthonormal basis vectors, and then find R.

3.1. Gram-Schmidt method for QR-decomposition.

Theorem 3.2. Every nonzero subspace of R^n has an orthonormal basis.
Proof: see [1].

Let W be a nonzero subspace of R^n and $w_1, w_2, \ldots, w_n$ be a basis for W. The following sequence of steps will produce an orthogonal basis $v_1, v_2, \ldots, v_n$ for W:

Step 1: let $v_1 = w_1$.

Step 2: let $v_2 = w_2 - \dfrac{w_2.v_1}{\| v_1 \|^2}.v_1$.

Step 3: let $v_3 = w_3 - \dfrac{w_3.v_1}{\| v_1 \|^2}.v_1 - \dfrac{w_3.v_2}{\| v_2 \|^2}.v_2$.

Step 4: let $v_4 = w_4 - \dfrac{w_4.v_1}{\| v_1 \|^2}.v_1 - \dfrac{w_4.v_2}{\| v_2 \|^2}.v_2 - \dfrac{w_4.v_3}{\| v_3 \|^2}.v_3$.

Step 5 to k: continuing in this way produces an orthogonal set $v_1, v_2, \ldots v_n$ after k step.

If the resulting orthogonal vectors are normalized to produce an orthonormal basis for subspace, then the algorithm is called the Gram-Schmidt process [1].

Example 3.1. Find orthonormal basis for R^3.

$$W_1 = (1,1,1), \qquad W_2 = (0,1,1), \qquad W_3 = (0,0,1).$$

Step 1: let $v_1 = w_1 = (1,1,1)$.

Step 2: let $v_2 = w_2 - \dfrac{w_2.v_1}{\| v_1 \|^2}.v_1 = (1,1,1) - \dfrac{2}{3}(1,1,1) = \left(\dfrac{-2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$.

Step 3: let $v_3 = w_3 - \dfrac{w_3.v_1}{\| v_1 \|^2}.v_1 - \dfrac{w_3.v_2}{\| v_2 \|^2}.v_2 = (0,1,1) - \dfrac{1}{3}(1,1,1) - \dfrac{1}{3}(1,1,1) - \dfrac{1}{2}(\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3}) = \left(0, \dfrac{-1}{2}, \dfrac{1}{2}\right)$.

Thus, the vectors

$$v_1 = (1,1,1), \quad v_2 = (\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}), \quad v_3 = (0, \frac{-1}{2}, \frac{1}{2}).$$

are orthogonal basis for R^3. The norms of these vectors are

$$\| v_1 \| = \sqrt{3}, \qquad \| v_2 \| = \frac{\sqrt{6}}{3}, \qquad \| v_3 \| = \frac{1}{\sqrt{2}}.$$

So, an orthonormal basis for R^3 is given by

$$q_1 = \frac{v_1}{\| v_1 \|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), q_2 = \frac{v_2}{\| v_2 \|} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), q_3 = \frac{v_3}{\| v_3 \|} = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Example 3.2. Find QR-decomposition of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

The matrix A has full column rank, so it is guaranteed to have QR-decomposition.
Applying Gram-Schmidt process to

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

And forming the matrix Q that has resulting orthonormal basis vectors as column yields

$$q = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

It follows that

$$R = Q^T A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Thus, we obtain the QR-decomposition as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

4. A new method for solving FFLS

In this section, we first give the definition of fuzzy linear system of equations and then deal with on solving theses systems.

Definition 4.1. Consider the $n \times n$ fuzzy linear system of equations [3, 7]:

$$\begin{cases} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1 \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2 \\ . \\ . \\ . \\ (\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{b}_n \end{cases}$$

The matrix form of the above equations is $\tilde{A} \otimes \tilde{x} = \tilde{b}$, where the coefficient matrix $\tilde{A} = \left(\tilde{a}_{ij}\right)$ $1 \le i, j \le n$, is an $n \times n$ fuzzy matrix and $\tilde{x}_j, \tilde{b}_j \in F(R)$. This system is called the fully fuzzy linear system (FFLS).

Theorem 4.1. The unique solution X is a fuzzy vector for arbitrary Y, if S^{-1} is nonnegative

$$(S^{-1})_{ij} \geq 0, \qquad 1 \leq i, j \leq 2n .$$

Proof: see[6].

Here, we are going to obtain a positive solution of FFLS $\tilde{A} \otimes \tilde{x} = \tilde{b}$, where $\tilde{A} = (A, M, N) > \tilde{0}, \tilde{b} = (b, h, g) > \tilde{0}$ *and* $\tilde{x} = (x, y, z) > \tilde{0}$. So we have

$$(A, M, N) \otimes (x, y, z) = (b, h, g)$$

Then by using Eq.(2.5) we have

$$(Ax, Ay + Mx, Az + Nx) = (b, h, g)$$

Therefore, Definition 2.5 concludes that

$$\begin{cases} Ax = b, \\ Ay + Mx = h, \\ Az + Nx = g. \end{cases}$$

Thus, we easily have

$$Ax = b \Rightarrow x = A^{-1}b,$$

and then by this representation in the second and the third equations, we have

$$y = A^{-1}h - A^{-1}Mx,$$

and

$$z = A^{-1}g - A^{-1}Nx,$$

We use QR-decomposition for matrix A, and so, we have

$$\begin{cases} Ax = b, & \Rightarrow x = R^{-1}Q^T b \\ Ay + Mx = h, & \Rightarrow y = R^{-1}Q^T (h - Mx) \\ Az + Nx = g, & \Rightarrow z = R^{-1}Q^T (g - Nx) \end{cases}$$

5. Numerical examples

In this section, we examine the proposed method in the last section.

Example 5.1. Consider the following FFSL:

$$\begin{pmatrix} (15,1,4) & (5,2,9) \\ (10,5,6) & (25,3,4) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} (10,15,25) \\ (20,30,40) \end{pmatrix}$$

First we obtain QR-decomposition for matrix A as follows:

$$A = \begin{pmatrix} 15 & 5 \\ 10 & 25 \end{pmatrix} = QR = \begin{pmatrix} -0.832050 & -0.554700 \\ -0.554700 & 0.832050 \end{pmatrix}$$

$$= \begin{pmatrix} -18.027756 & -18.027756 \\ 0 & 18.027756 \end{pmatrix}$$

So, we have

$$x = R^{-1}Q^T b = (0.461538, 0.615384),$$

$$y = R^{-1}Q^T (h - Mx) = (0.626035, 0.783461),$$

$$z = R^{-1}Q^T (g - Nx) = (1.009467, 0.494674).$$

Therefore, the solution is achieved as follows

$$\tilde{x} = (0.461538, 0.626035, 1.009467) \qquad , \qquad \tilde{y} = (0.615384, 0.783461, 0.494674)$$

Example 5.2. Consider the following FFSL (taken from [3]):

$$\begin{pmatrix} (6,1,4) & (5,2,2) & (3,2,1) \\ (12,8,20) & (14,12,15) & (8,8,10) \\ (24,10,34) & (32,30,30) & (20,19,24) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} (58,30,60) \\ (142,139,257) \\ (316,297,514) \end{pmatrix}$$

First we obtain QR-decomposition for matrix A as follows:

$$A = \begin{pmatrix} 6 & 5 & 3 \\ 12 & 14 & 8 \\ 24 & 32 & 20 \end{pmatrix} = QR = \begin{pmatrix} -0.2182 & -0.8164 & -0.5345 \\ -0.4364 & -0.4082 & 0.8017 \\ -0.8728 & 0.4082 & -0.2672 \end{pmatrix} \begin{pmatrix} -27.4954 & -35.1330 & -21.6035 \\ 0 & 3.2659 & 2.4494 \\ 0 & 0 & -0.5345 \end{pmatrix}$$

So, we can calculate:

$$x = R^{-1}Q^{T}b = (4.000,5.000,3.000),$$
$$y = R^{-1}Q^{T}(h - Mx) = (1.000,0.4999,0.5000),$$
$$z = R^{-1}Q^{T}(g - Nx) = (3.000,1.999,1.000).$$

Therefore,

$$\tilde{x} = (4.00,1.00,3.00) \quad , \quad \tilde{y} = (5.00,0.49,1.99) \quad , \quad \tilde{z} = (3.00,0.50,1.00)$$

6. Conclusion

In this paper, we used a certain decomposition of the coefficient matrix of the fully fuzzy linear system of equations to construct a new algorithm for solving fully fuzzy linear systems.

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References

1. H. Anton, R.C. Busby, Contemporary Linear Algebra, John Wiley, 2003.
2. J.J. Buckley, Y. Qu, Solving systems of linear fuzzy equations, Fuzzy Sets and Systems, 43 (1991) 33-43.
3. M. Dehghan, B. Hashemi, M. Ghatee, Computational methods for solving fully fuzzy linear systems, Applied Mathematics and Computation, 179 (2006) 328-343
4. M. Dehghan, B. Hashemi, Solution of the fully fuzzy linear systems using the decomposition prodecure , Applied Mathematics and Computation, 182 (2006) 1568-1580.
5. D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.
6. M. Friedman, M. Ming, A. Kandel, Fuzzy linear systems, Fuzzy Sets and Systems, 96 (1998) 201-209.
7. M. Matinfar, S.H. Nasseri, M. Sohrabi, Solving fuzzy linear system of equations by using Householder decomposition method, Applied Mathematical Sciences Journal, 51 (2008) 2569-2575.
8. S.H. Nasseri, Solving fuzzy linear system of equations by use of the matrix decomposition, International Journal of Applied Mathematics, 21 (2008) In prees.
9. S.H. Nasseri, M. Khorramizadeh, A new method for solving fuzzy linear systems, International Journal of Applied Mathematics, 20 (2007) 507-516.**Abstract**

In this paper first a series of basic transformation such integral, Rising and Falling has been
defined. then the integrals have been proved. So falling and rising planes have been studied and a theorem about it has been proved. At the end, operations fuzzy time planes is shown and related proposition to it is proved.

Keywords: fuzzy plane, Y-function, operations fuzzy time planes, Extend, Shift, Exp, Integrate.

Introduction

[1] Basic transformation about fuzzy interval time has been studied. First, basic concepts in fuzzy plane time have been studied and we argue a series of operations on fuzzy time planes by using [1], [20], [4, 7 and 18].we define summary of formula of basic unary transformation such as integral, Rising and Falling. Then we continue to argue about integrals and we prove some theorems. Time planes usually don't appear from nowhere, but they are constructed from other time planes. Plane operators are more general construction functions. They take one or more fuzzy time planes and construct a new one out of them.

We distinguish two ways of constructing new fuzzy time planes, first by means of Y-functions and then by means of plane operators. Y-functions map fuzzy values to fuzzy values. They can therefore be used to construct a new plane from a given one by applying the y-function point by point to the membership function values. Plane operators are more general construction functions.

In fact, our gold for presenting of this paper is that there are fuzzy planes which can be defined 2-dimension basic transformation for them, be defined some theorems for them.

[1]Corresponding author Tel/Fax: +98-112-5342460

Basic Unary Transformations

Definition (Basic Unary Transformations**)**

Let $p \in F_{R^k}$ be a fuzzy plane. We define the following (parameterized) plane operators**:**

$\hat{S} = \sup(p(x, y))$ [1]

$f_m = \text{first maximom}$ [1]

$l_m = \text{last maximom}$ [1]

$\text{identity}(p) = p$

$$\text{integrate}^+(p)(f(x)) \overset{\text{def}}{=} \lim_{a\to\infty}\lim_{b\to\infty}\frac{\int_{-a}^{x}\int_{-b}^{x} p(f(y_1, y_2))\ dy_1\, dy_2}{\int_{-a}^{+a}\int_{-b}^{+b} p(f(y_1, y_2))\ dy_1\, dy_2}$$

$$\text{integrate}^-(p)(f(x)) \overset{\text{def}}{=} \lim_{a\to\infty}\lim_{b\to\infty}\frac{\int_{x}^{+a}\int_{x}^{+b} p(f(y_1, y_2))\ dy_1\, dy_2}{\int_{-a}^{+a}\int_{-b}^{+b} p(f(y_1, y_2))\ dy_1\, dy_2}$$

Integrate

This operator integrates over the membership function and normalizes the integral to values≤ 1. The two integration operators integrate^+ and integrate^- can be simplified for finite fuzzy time planes.

Proposition (Integration for Finite planes)

If the fuzzy plane p is finite then

$\text{integrate}^+(p)(f(x)) \overset{\text{def}}{=} \frac{\int_{-a}^{x}\int_{-b}^{x} p(f(y_1,y_2))\ dy_1\, dy_2}{|p|}$ And

$\text{integrate}^-(p)(f(x)) \overset{\text{def}}{=} \frac{\int_{x}^{+a}\int_{x}^{+b} p(f(y_1,y_2))\ dy_1\, dy_2}{|p|}$

The proofs are straightforward [1].

Proposition (Integration for planes with Finite Kernel)

If the infinite fuzzy plane p has a finite kernel with $p_1 \overset{\text{def}}{=} p(-\infty, -\infty)$ and $p_2 \overset{\text{def}}{=} p(+\infty, +\infty)$ then $\text{integrate}^+(p)(f(x)) = \frac{p_1}{p_1+p_2}$ and $\text{integrate}^-(p)(f(x)) = \frac{p_2}{p_1+p_2}$.

Proof: by using [2]

$$integrate^+(p)(f(x)) = \lim_{a\to\infty}\lim_{b\to\infty}\frac{\int_{-a}^{x}\int_{-b}^{x} p(f(y_1, y_2))\ dy_1\, dy_2}{\int_{-a}^{+a}\int_{-b}^{+b} p(f(y_1, y_2))\ dy_1\, dy_2}$$

$$= \lim_{a\to\infty}\lim_{b\to\infty}\frac{|p|_{-a}^{i^{fk}} + |p|_{i^{fk}}^{x} + |p|_{-b}^{i^{fk}} + |p|_{i^{fk}}^{x}}{|p|_{-a}^{i^{fk}} + |p|_{i^{fk}}^{i^{lk}} + |p|_{i^{lk}}^{a} + |p|_{-b}^{i^{fk}} + |p|_{i^{fk}}^{i^{lk}} + |p|_{i^{lk}}^{b}}$$

$$= \lim_{a\to\infty}\lim_{b\to\infty}\frac{|p|_{-a}^{i^{fk}} + |p|_{-b}^{i^{fk}}}{|p|_{-a}^{i^{fk}} + |p|_{i^{lk}}^{a} + |p|_{-b}^{i^{fk}} + |p|_{i^{lk}}^{b}}$$

$$= \lim_{a\to\infty}\lim_{b\to\infty}\frac{(i^{fk}+a).i_1 + (i^{fk}+b).i_1}{(i^{fk}+a).i_1 + (a-i^{lk}).i_2 + (i^{fk}+b).i_1 + (b-i^{lk}).i_2}$$

$$= \lim_{a\to\infty}\lim_{b\to\infty}\frac{a.i_1 + b.i_1}{a.i_1 + a.i_2 + b.i_1 + b.i_2}$$

$$= \lim_{a\to\infty}\lim_{b\to\infty}\frac{(a+b)i_1}{a(i_1+i_2) + b(i_1+i_2)} = \lim_{a\to\infty}\lim_{b\to\infty}\frac{(a+b)i_1}{(a+b)(i_1+i_2)} = \frac{i_1}{(i_1+i_2)}$$

$$integrate^-(p)(f(x)) \overset{\text{def}}{=} \lim_{a\to\infty}\lim_{b\to\infty}\frac{\int_{x}^{+a}\int_{x}^{+b} p(f(y_1, y_2))\ dy_1\, dy_2}{\int_{-a}^{+a}\int_{-b}^{+b} p(f(y_1, y_2))\ dy_1\, dy_2}$$

$$= \lim_{a\to\infty}\lim_{b\to\infty}\frac{|p|_{x}^{i^{lk}} + |p|_{i^{lk}}^{a} + |p|_{x}^{i^{lk}} + |p|_{i^{lk}}^{b}}{|p|_{-a}^{i^{fk}} + |p|_{i^{fk}}^{i^{lk}} + |p|_{i^{lk}}^{a} + |p|_{-b}^{i^{fk}} + |p|_{i^{fk}}^{i^{lk}} + |p|_{i^{lk}}^{b}}$$

$$= \lim_{a\to\infty}\lim_{b\to\infty}\frac{|p|_{i^{lk}}^{a} + |p|_{i^{lk}}^{b}}{|p|_{-a}^{i^{fk}} + |p|_{i^{lk}}^{a} + |p|_{-b}^{i^{fk}} + |p|_{i^{lk}}^{b}}$$

$$= \lim_{a\to\infty}\lim_{b\to\infty}\frac{(a-i^{lk}).i_2 + (b-i^{lk}).i_2}{(i^{fk}+a).i_1 + (a-i^{lk}).i_2 + (i^{fk}+b).i_1 + (b-i^{lk}).i_2}$$

$$= \lim_{a\to\infty}\lim_{b\to\infty}\frac{a.i_2+b.i_2}{a.i_1+a.i_2+b.i_1+b.i_2}$$
$$= \lim_{a\to\infty}\lim_{b\to\infty}\frac{(a+b)i_2}{a(i_1+i_2)+b(i_1+i_2)} = \lim_{a\to\infty}\lim_{b\to\infty}\frac{(a+b)i_2}{(a+b)(i_1+i_2)} = \frac{i_2}{(i_1+i_2)}$$

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Rising and Falling Fuzzy planes

Definition (Rising and Falling Fuzzy planes and plane Operators**)**

A fuzzy set p is rising $\boldsymbol{iff}$ for its membership function $p(x,y)=(1,1)$ for all $(x,y)>p^{fm}$. P is falling $\boldsymbol{iff}$ for its membership function $p(x,y)=(1,1)$ for all $(x,y)<p^{lm}$.

Proposition

The basic unary transformations $extend^{+}$and $int^{+}$are rising plane operators and the unary transformations $extend^{-}$ and int^{-} are falling plane operators.

Proof: Any composition $f_1\circ\ldots\circ f_n\circ f$ where f is a rising (falling) plane operator is again a Rising (falling) plane operator. The proofs are straightforward [1].

Linear Y-Functions

A small, but important class of y-functions are linear y-functions. They are important firstly because very natural operators, like standard complement, intersection and union of fuzzy time planes can be described with linear y-functions. Secondly they are important because they allow us to transform planes represented by polygons in a very efficient way: only the vertices of the polygons need to be transformed.

The main characterization of linear y-functions is therefore that they map non intersecting straight plane segments to straight plane segments.

Definition (Y-Functions)

$Y-FCT^{n} \overset{\text{def}}{=} \{f:[(0,0),(1,1)]^{n}\longrightarrow[(0,0),(1,1)]\}$ Is the set of n-place y-functions.

They map fuzzy values to fuzzy values.

$Y-FCT \overset{\text{def}}{=} \bigcup_{n\geq1} Y-FCT^{n}$.

Definition (plane Operators)

$S-OP\mu^{n}=\{g:F_{R^k}^{n}\longrightarrow F_{R^k}\}$Is the set of n-place plane operators.

They map fuzzy planes to fuzzy planes.

$S-OP\mu \overset{\text{def}}{=} \bigcup_{n\geq0} S-OP\mu^{n}$.

Every y-function can be used to construct a new fuzzy time plane from given ones by applying the y-function to the fuzzy values.

Definition (Associated plane Operators)

If $f\in Y-FCT^{n}$is a y-function then $g_f\in S-OP\mu^{n}$ defined $g_f(s_1,s_2,\ldots,s_n)(x,y)=f\big(s_1(x,y),\ldots,s_n(x,y)\big)$is the associated plane operator.

Definition (Linear Y-Function)

A y-function $f\in Y-FCT^{n}$ is linear if the mapping

$f'\Big(\big(z,(x_1,y_1)\big),\ldots,\big(z,(x_n,y_n)\big)\Big) \overset{\text{def}}{=} \Big(z,f\big((x_1,y_1),\ldots,(x_n,y_n)\big)\Big)$

Maps non-intersecting plane segments

$\big(z_1,(x_{11},y_{11})\big)-\big(z_2,(x_{12},y_{12})\big),\ldots,\big(z_1,(x_{n1},y_{n1})\big)-\big(z_2,(x_{n2},y_{n2})\big)$

To a line segment

$\Big(z_1,f\big((x_{11},y_{11}),\ldots,(x_{n1},y_{n1})\big)\Big)-\Big(z_2,f\big((x_{12},y_{12}),\ldots,(x_{n2},y_{n2})\big)\Big)$.

One-place linear y-functions can be characterized in the following way

Proposition (Characterization of One-Place Linear y-Functions)

A one-place y-function f is linear if and only if $f(x,y)=f(0,0)+\big(f(1,1)-f(0,0)\big).(x,y)$ holds.

Proof: Suppose f is linear. We take the straight plane segment between $\big((0,0),(0,0)\big)$ and $\big((1,1),(1,1)\big)$. The mapping $f'\big(z,(x,y)\big)=\big(z,f(x,y)\big)$ maps this plane segment to a plane segment between $\big((0,0),f(0,0)\big)$ and $\big((1,1),f(1,1)\big)$.

Therefore

$f(x,y)=f(0,0)+\frac{f(1,1)-f(0,0)}{(1,1)-(0,0)}.\big((x,y)-(0,0)\big)$ **(Line equation)**

$\quad = f(0,0)+\big(f(1,1)-f(0,0)\big).(x,y)$

Other direction: clearly.

An example for a one-place linear y-function is the standard negation $n(x,y)=1-(x,y)$.

The characterization of two-place linear y-functions

Proposition (Characterization of Two-Place Linear y-Functions)

A two-place y-function f is linear if and only if the following condition holds:

$f\big((x_1,y_1),(x_2,y_2)\big)=$

$$= \begin{cases} f\big((0,0),(0,0)\big) + \left(f\left(\frac{(x_1,y_1)}{(x_2,y_2)},(1,1)\right) - f\big((0,0),(0,0)\big)\right).(x_2,y_2) & if\ (x_1,y_1) \le (x_2,y_2) \\ f\left((0,0),\frac{(x_1,y_1)-(x_2,y_2)}{(1,1)-(x_2,y_2)}\right) + \left(f(1,1) - f\left((0,0),\frac{(x_1,y_1)-(x_2,y_2)}{(1,1)-(x_2,y_2)}\right)\right).(x_2,y_2) & \text{otherwise} \end{cases}$$

Proof: Suppose f is linear. We consider the case $(x_1,y_1) \le (x_2,x_2)$ first. To this end we take the straight plane segment between $\big((0,0),(0,0)\big)$ and$\big((1,1),(1,1)\big)$. The line equation for this curve is just$\|y\| = \|x\|$. Now take an arbitrary $(x_2,y_2) \in [(0,0),(1,1)]$ and an arbitrary $(x_1,y_1) \le (x_2,y_2)$. The line equation for the plane segment starting At $\big((0,0),(0,0)\big)$ and crossing $\big((x_2,y_2),(x_1,y_1)\big)$ is $(x,y) = \frac{((x_1,y_1)-(0,0))}{((x_2,y_2)-(0,0))}.(w_1,w_2)$. For $(w_1,w_2) = (1,1)$ We get $(z_1,z_2) = \frac{(x_1,y_1)}{(x_2,y_2)}$.

Since f is linear we have

$$f\big((x_1,y_1),(x_2,y_2)\big) = f\big((0,0),(0,0)\big) + \frac{f\big((z_1,z_2),(1,1)\big) - f\big((0,0),(0,0)\big)}{(0,0)-(1,1)}.(x_2,y_2)$$

$$= f\big((0,0),(0,0)\big) + \left(f\left(\frac{(x_1,y_1)}{(x_2,y_2)},(1,1)\right) - f\big((0,0),(0,0)\big)\right).(x_2,y_2)$$

Now consider the case$(x_1,y_1) \ge (x_2,x_2)$.

The plane starting at $\big((1,1),(1,1)\big)$ and crossing $\big((x_2,y_2),(x_1,y_1)\big)$ crosses the y-axis At $(z_1,z_2) = \frac{((x_1,y_1)-(x_2,y_2))}{((1,1)-(x_2,y_2))}$.

Since f is linear we have

$$f\big((x_1,y_1),(x_2,y_2)\big) = f\big((0,0),(z_1,z_2)\big) + \frac{f\big((1,1),(1,1)\big) - f\big((0,0),(z_1,z_2)\big)}{(1,1)-(0,0)}.(x_2,y_2)$$

$$= f\left((0,0),\frac{(x_1,y_1)-(x_2,y_2)}{(1,1)-(x_2,y_2)}\right) + \left(f\big((1,1),(1,1)\big) - f\left((0,0),\frac{(x_1,y_1)-(x_2,y_2)}{(1,1)-(x_2,y_2)}\right)\right).(x_2,y_2)$$

The other direction, showing that the two conditions imply linearity, is again straightforward.

Simple examples for linear two-place y-functions are the minimum and maximum function. The minimum function is used to realize standard intersection of two fuzzy time planes, and the maximum function is used to realize standard union of two fuzzy time planes.

References

[1] H.J. ohlbach - University of Munich - 31 Augast 2004.

[2] Hans Jurgen Ohlbach. Calendrical calculations with time partitionings and fuzzy time intervals. In H. J. Ohlbach and S. Schaffert, editors, Proc. of PPSWR04, number 3208 in LNCS. Springer Verlag, 2004.

[3] Hans Jurgen Ohlbach. Fuzzy time intervals and relations-the FuTIRe library. Technical report, Inst. f. Informatik, LMU Munchen, 2004. See http://www.pms.informatik.unimuenchen.de/mitarbeiter/ohlbach/systems/FuTIRe.

[4] Hans Jurgen Ohlbach. Relations between fuzzy time intervals. In C. Combi and G. Ligozat, editors, Proc. of the 11th International Symposium on Temporal Representation and Reasoning, pages 44-51, Los Alamitos, California, 2004. IEEE.

[5] Hans Jurgen Ohlbach. The role of labelled partitionings for modeling periodic temporal notions. In C. Combi and G. Ligozat, editors, Proc. of the 11th International Symposium on Temporal Representation and Reasoning, pages 60-63, Los Alamitos, California, 2004.IEEE.

[6] Franois Bry, Bernhard Lorenz, Hans Jurgen Ohlbach, and Stephanie Spranger. On reasoning on time and location on the web. In N. Henze F. Bry and J. Malusynski, editors, Principles and Practice of Semantic Web Reasoning, volume 2901 of LNCS, pages 69-83.Springer Verlag, 2003.

[7] James. F. Allen. Maintaining knowledge about temporal intervals. Communication of the Acm, 832 - 843, 1983.

[8] Fronz Baader, Diego Calvanese, Deborah Me Guinness, Daniele Nardi, and peter patel Schneider, editors. The Description logic Han dbook. Theary, Implementation and Applicanso Cambridge University press, 2003.

[9] T. Bernerz - Lee, M. Fishchetti andM.Dertouzos.Weaving the web: The original Design and Ultimate Desting of the word wid web. Harper, son Froncisco, September 1999.

[10] Diana R. Cukierman. A Formalization of structured temporal objects and Repetition. PhD thesis, siman Franser University, Vancouver, Canada, 2003.

[11] Didier Dubois and Henri prade, editors. Fundamentals of fuzzy sets. kluwer Academi publisher, 2000.

[12] Joseph o Rouke. Computational Geometry in C. Cambridge University press. 1998.

[13] Gabor Nagypal and Boris Motik. A fuzzy model for representing uncertain, subjective and vague temporal knowledge in ontologies. In Proceedings of the International Conference onOntologies, Databases and Applications of Semantics, (ODBASE), volume 2888 of LNCS.Springer-Verlag, 2003.

[14] Klaus U. Schulz and Felix Weigel. Systematic and architecture for a resource representing knowledge about named entities. In Jan Maluszynski Francois Bry, Nicola Henze, editor, Principles and Practice of SemanticWeb Reasoning, pages 189-208, Berlin, 2003. Springer-Verlag.
[15] The ACM Compating Classification System, 2001
Http: // www.acm . Ogr/class/1998/home page.hutml
[16] Nachum Dershowitz and Edward M. Reingold. Calendrical Calculations. Cambridge University Press, 1997.
[17] Hans Jurgen Ohlbach. About real time, calendar systems and temporal notions. In H. Barringer and D. Gabbay, editors, Advances in Temporal Logic, pages 319-338. Kluwer Academic Publishers, 2000.
[18] Hans Jurgen Ohlbach. Calendar logic. In I. Hodkinson D.M. Gabbay and M. Reynolds, editors, Temporal Logic: Mathematical Foundations and Computational Aspects, pages489 586. Oxford University Press, 2000.
[19] Hans Jurgen Ohlbach and Dov Gabbay. Calendar logic. Journal of Applied Non-Classical Logics, 8(4), 1998.
[20] L. A. Zadeh. Fuzzy sets. Information and Control, 8:338-353, 1965.
[21] Jacob E. Goodman and Joseph O'Rourke, editors. Handbook of Discrete and Computational Geometry. CRC Press, 1997.

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