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A dynamic model for PERT risk evaluating in fuzzy environment

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Abstract

In this paper we propose a dynamic model for evaluating of time risk in stochastic network, where the activity durations are exponentionaly distributed random variable and independent. We would like to present a new definition of general risk index for project in each time and connect it to the notation of its activity criticalities. We model such networks as finite-state, absorbing, and continuous Marko chain with upper triangular generator matrices. The state space is related to the network structure. The criticality index for each activity will be computed and then we put forward a fuzzy way of measuring the criticality to computing project states criticality. Then by using the probability of absorption in each state severity of criticality will be computed dynamically. The criticality measure obtained may serve as a measure of risk or of the supervision effort needed by senior management. It also by ranking the states before project initiating is able to forecast the critical states in order and help to the project management to developing a proper guideline for resource planning and allocation.

Keywords: Dynamic PERT, dynamic risk management, fuzzy risk, markov chain

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1-introduction

Although project risk management has been investigated by many researchers, one cannot find many models regarding dynamic project risk management in the literature. Several research employed concept of dynamic PERT to compute project completion time. Charnes et al. [1] developed a chance-constrained programming, where the activity durations are assumed to be exponential. For polynomial activity durations, Martin [2] provided a systematic way of analyzing the problem through series-parallel reductions. Schmit and Grossmann [3] developed a new technique for computing the exact overall duration of a project, when activity durations use a probability density function, which combines piecewise polynomial segments and Dirac delta functions, defined over a finite interval. Fatemi Ghomi and Hashemin [4] generalized the Gaussian quadrature formula to compute F (T) or the distribution of the total duration, T. Fatemi Ghomi and Rabbani [5] presented a structural mechanism, which changes the structure of a network to a series parallel network, in order to estimate F(T).Kulkarni and Adlakha [6] developed a continuous-time Markov process approach to PERT problems, with exponentially distributed activity durations. Elmaghraby [7] provided lower bounds for the true expected project completion time. Azaron and modarres [8] developed an analytical method to compute the project completion time in dynamic PERT networks with generating projects.Azaron et al [9] developed a method for approximating the distribution function of the longest path length in the network of queues by constructing a proper continuous-time Markov chain. Fulkerson [10], Robillard [11] and Perry and Creig [12] have done similar work.
Other research also investigated concept of criticality and risk in each activity or path and completion time in project network by using fuzzy and crisp duration time. Zielin´ski [13], Slyeptsov and Tyshchuk [14, 15], Dubois et al. [16], Chanas et al. [17], Chanas and Zielin´ski [18–20], Kuchta [21], and others [22–24], employed the concept of fuzziness [25, 26] to these cases, and developed analysis approaches. Most of these approaches are based on the CPM with formulas for the forward and the backward recursions, in which the deterministic activity times are replaced with the fuzzy activity times.Fatemi and Teimouri [27] presented a exact formula to compute the path critical index and activity critical index for the PERT network by assuming that each activity duration time is a discrete random variable. Bowman [28] described an algorithm for estimating arc and path criticalities in stochastic activity network by combining Mont Carlo simulation with exact analysis conditioned on node release times. Other researchers also done similar cases, but investigating the criticality of a set of activities that are simultaneous progress in project network do not much.

However in this paper we describe a method by combining the concept of continuous-time Markov process approach to PERT problems that are explained by Kulkarni and Adlakha [6] and the approach was proposed by Bowman[28] for estimating arc criticality to obtaining the risk index for a set of activity that are progress in project network. The risk index would help the decision maker to forecast and decide which time project and activity have to be supervised more closely and which time less.

The present paper is organized as follows. In section 2 a full description of transforming PERT network into continuous-time Markov chain from [6] and [8] is outlined. Section 3 deals with the computing risk index for activities and each state by using fuzzy concept and [28].proposed method for dynamic risk estimation in this section is introduced. Numerical example in section 4 is presented and finally we draw the conclusion of the paper in section 5.

2. Transforming PERT network into continuous-time Markov chain

In this section , we review the method of Kulkarni and Adlakha [6] to transforming PERT network into continuous-time Markov chain and obtain the distribution function of absorption to each state of Markov chain by starting from state 1 in every time of project implementation.

Let G = (V, A) be a network with set of nodes $V=\{v_1, v_2, ..., v_m\}$ and set of arcs $A=\{a_1, a_2, ..., a_n\}$.Assume that G is a directed acyclic network with a single source and a single sink. . The source and sink nodes are denoted by s and t, respectively. Length of arc $a \in A$ is an exponentially distributed random variable with parameter λ_a. For $a \in A$, let $s(a)$ be the starting node of arc a and $f(a)$ be the ending node of arc a .

Definition 1. $I(v)$ and $O(v)$ are the sets of arcs ending and starting at node v, respectively, which are defined as follows:

$I(v) = \{a \in A: f(a) = v, v \in V\}$

$O(v) = \{a \in A: s(a) = v, v \in V\}$

Definition 2. If $W \subset V$ such that $s \in W$ and $t \in \overline{W} = V - W$, then an (s, t) cut is defined as:

$(W, \overline{W}) = \{a \in A: s(a) \in W, f(a) \in \overline{W}\}$

An (s, t) cut $(W, \overline{W})$ is called a uniformly directed cut (UDC), if $(\overline{W}, W)$ is empty. Each UDC is clearly a set of arcs, in which the starting node of each arc belongs to W and the ending node of each arc belongs to $\overline{W}$.

Definition 3. Let $D = E \bigcup F$ be a uniformly directed cut (UDC) of a network. Then, it is called an admissible 2-partition, if:

1. $E \bigcup F = D$
2. $E \bigcap F = \phi$
3. $\forall a \in F, I(f(a)) \not\subset F$

Definition 4. During the project execution and at time t, each activity can be in one of the active, dormant or idle states, which are defined as follows:

(i) Active: activity a is active at time t, if it is being executed at time t.

(ii) Dormant: activity a is dormant at time t, if it has finished but there is at least one unfinished activity in $I(f(a))$.If activity a is dormant at time t, then its successor activities in $O(f(a))$ cannot begin.

(iii) Idle: activity a is idle at time t if it is neither active nor dormant at time t.

The sets of active and dormant states are denoted by $Y(t), Z(t)$ respectively, and $X(t) = (Y(t), Z(t))$.

Let S denote the set of all admissible 2-partition UDCs of the network, and $\bar{S} = S \cup \{(\phi,\phi)\}$. Note that $X(t) = (\phi,\phi)$ implies that $Y(t) = \phi$ and $Z(t) = \phi$, i.e., all activities are idle at time t and hence the project is completed by time t. $\{X(t) = (Y(t), Z(t)); t \geq 0\}$ Is a continuous-time Markov process with the state space $\bar{S}$. The elements of the infinitesimal generator matrix of this process denoted by $Q = [q\{(E,F),(E',F')\}]$ & $(E,F),(E',F') \in \bar{S}$ and are calculated as follows:

$$q\{(E,F),(E',F')\} = \lambda_a \; if a \in E, I(f(a)) \not\subset F \cup \{a\}, E' = E - \{a\}, F' = F \cup \{a\}; \quad (1)$$

$$= \lambda_a ; if a \in E, I(f(a)) \subset F \cup \{a\}, E' = E - \{a\} \cup O(f(a)), F' = F - I(f(a));$$

$$= -\sum_{a \in E} \lambda_a \; if E' = E, F' = F$$

$$= 0 \qquad \text{otherwise}$$

$\{X(t); t \geq 0\}$ Is a finite-state absorbing continuous-time Markov process and since $q\{(\phi,\phi),(\phi,\phi)\} = 0$, it is concluded that this state is an absorbing one and obviously the other states are transient. Furthermore, we number the states in $\bar{S}$ such this Q matrix are an upper triangular one. We assume that the states are numbered $1,2,3,\ldots,N = |\bar{S}|$. State 1 is the initial state, namely $(O(s),\phi)$, and state N is the absorbing state, namely (ϕ, ϕ). Chapman–Kolmogorov forward equations can be applied to compute the distribution function of the absorption to each state of Markov chain by starting from state 1 in the stochastic network. Applying the forward algorithm, define:

$$P_i(t) = P\{X(t) = i / X(0) = 1\}, i = 1,2,\ldots, N \quad (2)$$

Using the forward algorithm, the system of differential equations for the vector $P(t) = [P_1(t), P_2(t), \ldots, P_N(t)]$ is given by:

$$P'(t) = P(t)Q, \quad (3)$$

$$P(0) = [1,0,\ldots,0]^T$$

By using the Laplace transform we can solve the system of differential equations for the vector $P(t) = [P_1(t), P_2(t), \ldots, P_N(t)]$.

3. Arcs and states criticality (risk) estimation

The criticality of each activity is defined the probability that each activity is on the critical path of the network. In this section first we present the method of Bowman [28] to obtain the arcs criticality in project network. In this method the arcs criticality are computed by using following theorem and conditioning on the node release times (the time at which all arcs coming into a node have been completed and all arcs emanating from the node can begin).

The theorem refers to Figure 1. Figure 1 focuses the view of a network on a single node

(Node *). Similarly, Theorem focuses on a single node at a time, thus providing a basis for arc criticality estimation for the entire network. The following notation is used in addition to that which has already been introduced.
Let T_i = the length of the longest path to node i (with realizations t_i),
$C_n(k)$ = the probability that node k is on the longest path,
$F_i(x)$ = the cdf for the length of arc i, and
$f_{T_1 \dots T_n}(t_1,\dots,t_n)$ = the joint PDF for the node release times and the activity durations are exponentionaly distributed.
$C_a(i)$ = criticality of activity i (the probability that the activity i is on the critical path of the network).

THEOREM1.

$$C_a(i)=\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty} C_a(i|T_1=t_1,\dots,T_n=t_n)\times f_{T_1\dots T_n}(t_1,\dots,t_n)dt_1\dots dt_n \quad \text{For } i=1,\dots,q \tag{4}$$

Where

$$C_a(i|T_1=t_1,\dots,T_n=t_n)=\frac{f_i(t_*-t_i)\prod_{\substack{j=1\\ j\neq i}}^{q}F_j(t_*-t_j)}{\sum_{k=1}^{q}f_k(t_*-t_k)\prod_{\substack{j=1\\ j\neq k}}^{q}F_j(t_*-t_j)}\times C_n(*|T_1=t_1,\dots,T_n=t_n) \tag{5}$$

And

$$C_n(*|T_1=t_1,\dots,T_n=t_n)=\sum_{j=q+1}^{q+s}C_a(j|T_1=t_1,\dots,T_n=t_n) \tag{6}$$

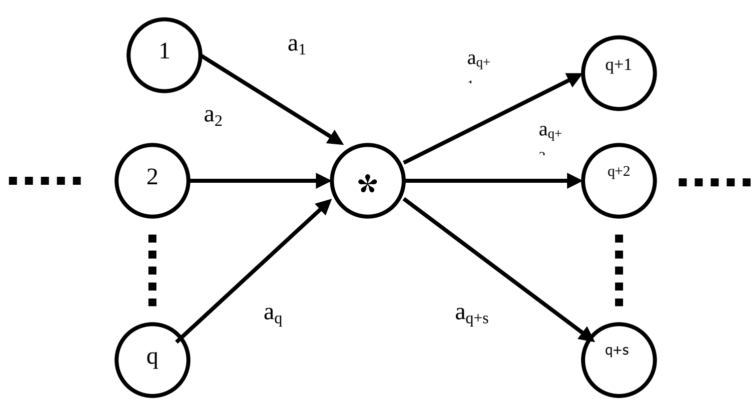

Figure 1

Now by using theorem1 we can compute the criticality index of each state of Markov chain as follows:
Definition 5. the criticality index of each state is the criticality of the set of active activities in each state.

Let $Y_i(t)$ =the set of active activities in state i in time t

n_i =number of active activities in state i (number of member of $Y_i(t)$)

C_{ij} =the criticality of activity j in state i

Define:

$$l_i = \min\{C_{ij};\ j = 1,\dots,\ n_i\} \qquad (7)$$

$$u_i = \max\{C_{ij};\ j = 1,\dots,\ n_i\} \qquad (8)$$

$$\overline{R}_i = \frac{\sum_{j=1}^{n_i} C_{ij}}{n_i} \qquad (9)$$

l_i, u_i and $\overline{R}_i$ can be candid for state i criticality index. But selecting the l_i and u_i, respectively is optimistic and pessimistic selection. Using of the $\overline{R}_i$ is suitable, but is not very good estimate. Because the different states with variety activity criticality can be have similar amount. Therefore using of fuzzy triangular number can be proper in this case. Hence let us to define:

$$\tilde{x}_i = \left(l_i - \varepsilon, \overline{R}_i, u_i + \varepsilon\right) \quad 0 \le \varepsilon \le 0.1 \qquad (10)$$

$\tilde{x}_i$ Is presented in figure 2.

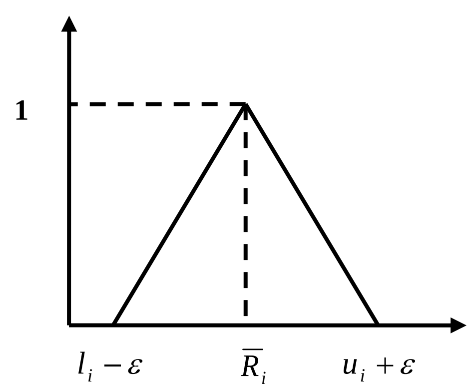

Figure 2.

ε Is amount of the decision maker's attitude or understanding of criticality that takes into account to l_i and u_i have a chance for selecting as a state criticality index.

The membership function of $\tilde{x}_i$ is defined as follow:

$$\mu_i(x) = \frac{1}{\overline{R}_i - l_i + \varepsilon}\left(x - \overline{R}_i\right) + 1 \quad \text{If} \quad l_i - \varepsilon \le x \le \overline{R}_i \qquad (11)$$

$$= \frac{1}{\overline{R}_i - u_i - \varepsilon}\left(x - \overline{R}_i\right) + 1 \quad \text{If} \quad \overline{R}_i \le x \le u_i + \varepsilon$$

Now by the centroid defuzzy method we can obtain the criticality (risk) index of each state as follow:

$$R_i = \frac{\int_{l_i}^{u_i} x\mu_i(x)dx}{\int_{l_i}^{u_i} \mu_i(x)dx} \qquad (12)$$

Now we are ready to introduce dynamic risk estimation for each state. Let us define:

$$SR_i(t) = P_i(t) \times R_i \qquad (13)$$

$SR_i(t)$ =the severity of risk in state i at time t

$P_i(t)$ = the probability of absorption at time t in state i when starting in state 1

R_i = the risk index for state i

By using (13) we are able to forecast and estimate the severity of project risk in every time of project implementation.

4. Numerical example

To illustrate the proposed method we investigate the example that it was studied by Elmaghrabi [29] to compute project completion time.
In figure 3 assume all arc lengths are exponentially distributed and C_1,C_2 and C_3 are uniformly direct cuts.

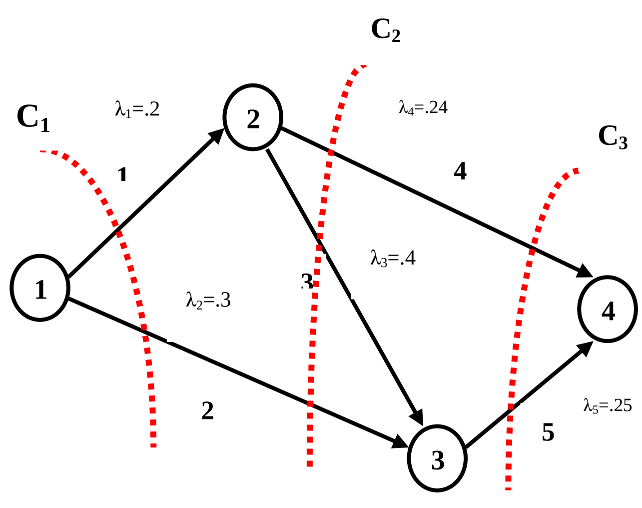

Figure 3.

Where:
C_1= {1, 2}, C_2= {2, 3, 4} and C_3= {4, 5}
Note that we did not include cutset {1, 3, 5} since it is not a UDC: arcs (1, 2) and (3, 4) are directed from W into $\overline{W}$, but arc (2, 3) is directed from $\overline{W}$ into W .
When the project is initiated cutset C_1 would come into play, and activities 1 and 2 are

both active. Change in the state of the project can come about in one of two shapes: either (i) activity 1 completes before activity 2, in which case cutset C_2 comes into play, or
(ii) activity 2 completes, in which case it remains dormant until activity 1 completes, at which time the cutset changes to C_2 with activities 3 and 4 active and activity 2 remaining dormant. Denote an active activity by the letter 'a' and a dormant activity by the letter 'd'. Then we may represent the possible states thus far discussed as follows:

$(1a,2a) \rightarrow (1a,2d)$ Or $(2a,3a,4a)$

According to similar process other states will be obtained and S is presented as follow:

$$S = \begin{Bmatrix} (1a,2a),(1a,2d),(2a,3a,4a),(2d,3a,4a),(2a,3d,4a), \\ (2a,3d,4d),(2d,3a,4d),(2a,3a,4d),(4a,5a),(4d,5a),(4a,5d) \end{Bmatrix}$$

And
$\bar{S} = S \cup \{(\phi,\phi)\}$ Is defined as follow:

$$\bar{S} = \begin{Bmatrix} (1a,2a),(1a,2d),(2a,3a,4a),(2d,3a,4a),(2a,3d,4a),(2a,3d,4d), \\ (2d,3a,4d),(2a,3a,4d),(4a,5a),(4d,5a),(4a,5d),(\phi,\phi) \end{Bmatrix}$$

The state space of the continuous-time Markov process is presented in figure 4 and table 1.

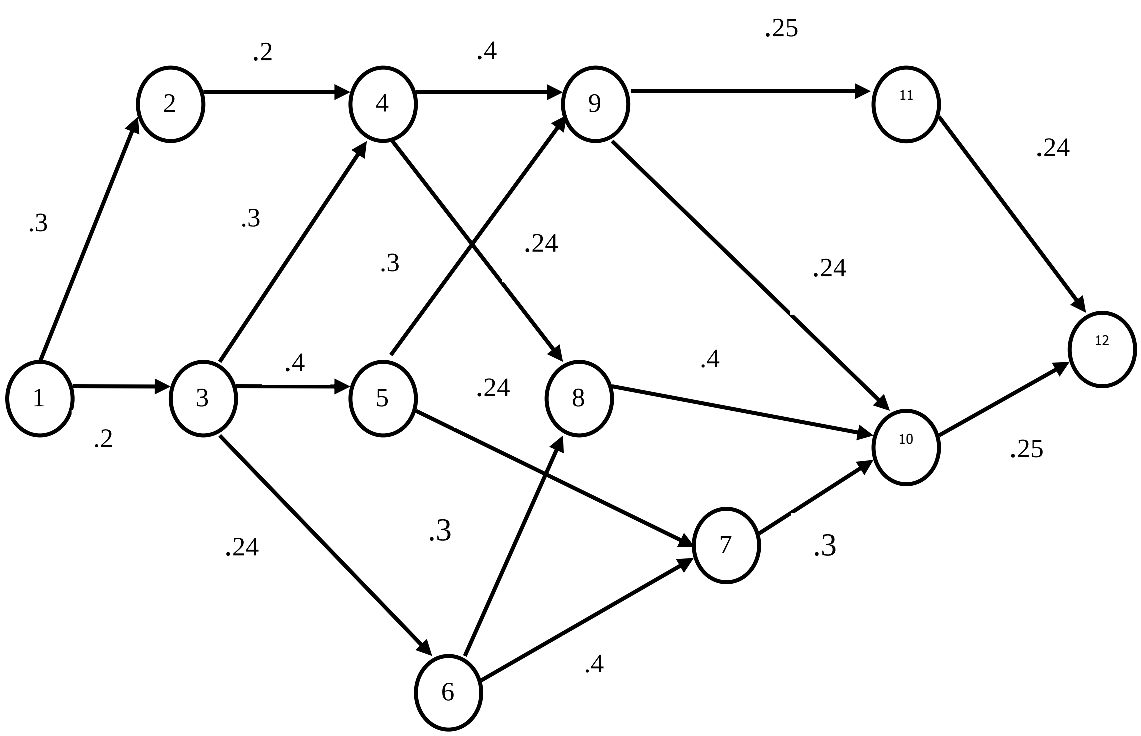

Figure 4.

number	state
1	$(\{1a,2a\},\phi)$
2	$(\{1a\},\{2d\})$
3	$(\{2a,3a,4a\},\phi)$
4	$(\{3a,4a\},\{2d\})$
5	$(\{2a,4a\},\{3d\})$
6	$(\{2a,3a\},\{4d\})$
7	$(\{2a\},\{3d,4d\})$
8	$(\{3a\},\{2d,4d\})$
9	$(\{4a,5a\},\phi)$
10	$(\{5a\},\{4d\})$
11	$(\{4a\},\{5d\})$
12	(ϕ,ϕ)

Table 1.
The states of continuous-time Markov process

The elements of the infinitesimal generator matrix of this process appear as follow:

$$Q=\begin{bmatrix} -.5 & .3 & .2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.2 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -.94 & .3 & .4 & .24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.64 & 0 & 0 & 0 & .24 & .4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.54 & 0 & .24 & 0 & .3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.7 & .4 & .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -.3 & 0 & 0 & .3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.4 & 0 & .4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.49 & .24 & .25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.25 & 0 & .25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.24 & .24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The differential equations corresponding to the rate matrix Q from (3) are:

$$P'_{1,1}(t)=-.5P_{1,1}(t) \qquad (*)$$

$$P'_{1,2}(t)=.3P_{1,1}(t)-.2P_{1,2}(t)$$

$$P'_{1,3}(t)=.2P_{1,1}(t)-.94P_{1,3}(t)$$

$$P'_{1,4}(t)=.2P_{1,2}(t)+.3P_{1,3}(t)-.64P_{1,4}(t)$$

$$P'_{1,5}(t)=.4P_{1,3}(t)-.54P_{1,5}(t)$$

$$P'_{1,6}(t)=.24P_{1,3}(t)-.7P_{1,6}(t)$$

$$P'_{1,7}(t)=.24P_{1,5}(t)+.4P_{1,6}(t)-.3P_{1,7}(t)$$

$$P'_{1,8}(t)=.24P_{1,4}(t)+.3P_{1,6}(t)-.4P_{1,8}(t)$$

$P'_{1,9}(t) = .4P_{1,4}(t) + .3P_{1,5}(t) - .49P_{1,9}(t)$
$P'_{1,10}(t) = .3P_{1,7}(t) + .4P_{1,8}(t) + .24P_{1,9}(t) - .25P_{1,10}(t)$
$P'_{1,11}(t) = .25P_{1,9}(t) - .24P_{1,11}(t)$
$P'_{1,12}(t) = .25P_{1,10}(t) + .24P_{1,11}(t)$
$P(0) = [1,0,...,0]^T$

Note that $P_{1,i}(t)$ is the probability of absorption at time t in state i when starting in state 1.to solve this system differential equation by using Laplace transform we will have:

From (*): $sP_{1,1}(s) - P_{1,1}(0) = .5P_{1,1}(s) \Rightarrow P_{1,1}(s)(s+.5) = P_{1,1}(0) \Rightarrow P_{1,1}(s) = \frac{1}{s+.5}$

Now inversion of this Laplace transform would give:

$P_{1,1}(t) = e^{-.5t}$

By using this method for other differential equations the solutions appears as follow:

$\mathrm{P}_{1,2}(\mathrm{t}) = e^{-.2t} - e^{-.5t}$

$\mathrm{P}_{1,3}(\mathrm{t}) = 0.454545e^{-.5t} - 0.454545e^{-.94t}$

$\mathrm{P}_{1,4}(\mathrm{t}) = 0.454545e^{-.2t} - 0.454545e^{-.5t} + 0.454545e^{-.94t} - 0.454545e^{-.64t}$

$\mathrm{P}_{1,5}(\mathrm{t}) = 4.54545e^{-.5t} - 4.999995e^{-.54t} + 0.454545e^{-.94t}$

$\mathrm{P}_{1,6}(\mathrm{t}) = 0.545454e^{-.5t} - 0.999999e^{-.7t} + 0.454545e^{-.94t}$

$\mathrm{P}_{1,7}(\mathrm{t}) = 0.999999e^{-.3t} - 6.545448e^{-.5t} + 4.999995e^{-.54t} - 0.454545e^{-.94t} + 0.999999e^{-.7t}$

$\mathrm{P}_{1,8}(\mathrm{t}) = -0.999999e^{-.4t} + 0.545454e^{-.2t} - 0.545454e^{-.5t} - 0.454545e^{-.94t} + 0.454545e^{-.64t} +$
$0.999999e^{-.7t}$

$\mathrm{P}_{1,9}(\mathrm{t}) = 87.049722e^{-.49t} + 0.626958e^{-.2t} - 118.1817e^{-.5t} - 0.70707e^{-.94t} + 1.21212e^{-.64t} +$
$29.99997e^{-.54t}$

$\mathrm{P}_{1,10}(\mathrm{t}) = -7.1111004e^{-.25t} - 5.999994e^{-.3t} + 122.181696e^{-.5t} - 29.99997e^{-.54t} + 0.70707e^{-.94t} -$
$1.555554e^{-.7t} + 2.666664e^{-.4t} + 7.3730304e^{-.2t} - 1.21212e^{-.64t} - 87.049722e^{-.49t}$

$\mathrm{P}_{1,11}(\mathrm{t}) = -4.999977e^{-.24t} - 87.049722e^{-.49t} + 3.918475e^{-.2t} + 113.63625e^{-.5t} + 0.252524e^{-.94t} -$
$0.757575e^{-.64t} - 24.999975e^{-.54t}$

$\mathrm{P}_{1,12}(\mathrm{t}) = 0.9999963 + 7.1111004e^{-.25t} + 4.999995e^{-.3t} - 115.636248e^{-.5t} + 24.999975e^{-.54t} -$
$0.2525247e^{-.94t} + 0.555555e^{-.7t} - 1.666665e^{-.4t} - 13.918458e^{-.2t} + 0.757575e^{-.64t} +$
$87.049722e^{-49t} + 4.999977e^{-.24t}$

Now we estimate the activities criticality by using theorem 1.

$$\mu_1 = \frac{1}{\lambda_1} = \frac{1}{.2} = 5$$

$$\mu_2 = \frac{1}{\lambda_2} = \frac{1}{.3} = 3.333$$

$$\mu_3 = \frac{1}{\lambda_3} = \frac{1}{.4} = 2.5$$

$$\mu_4 = \frac{1}{\lambda_4} = \frac{1}{.24} = 4.167$$

$$\mu_5 = \frac{1}{\lambda_5} = \frac{1}{.25} = 4$$

Therefore:

$$T_1 = 0, T_2 = 5, T_3 = 7.5, T_4 = 11.5$$

And by (5):

$$C_{34} = \frac{(.25\times 4\times e^{-1})(1-e^{-9.167\times .24})\times 6.5}{\left[(.25\times 4\times e^{-1})(1-e^{-9.167\times .24})\times 6.5\right]+\left[(.24\times 6.5\times e^{-1})(1-e^{-11.5\times .25})\times 4\right]}\times 1 = \frac{2.126}{4.293} = .495$$

By similar calculation other arcs criticalities are obtain as follow:

$$C_{24} = .505, C_{23} = .233, C_{13} = .262, C_{12} = .738$$

And then by (9)

$$\overline{R}_1 = \frac{\sum_{j=1}^{2} C_{1j}}{2} = \frac{.738 + .262}{2} = .5$$

And respectively:

$$\overline{R}_2 = .738, \overline{R}_3 = .333, \overline{R}_4 = .369, \overline{R}_5 = .384, \overline{R}_6 = .262, \overline{R}_7 = .233,$$
$$\overline{R}_8 = .248, \overline{R}_9 = .5, \overline{R}_{10} = .505, \overline{R}_{11} = .495, \overline{R}_{12} = 0$$

For simplify let us assume $\varepsilon = 0$.now by using (7), (8) and (9) the fuzzy triangular numbers are made .results are presented in the following table.

i	state	$\overline{R}_i$	l_i	u_i	$\tilde{x}_i$
1	$(1a,2a)$	.500	.262	.738	(.262,.500,.738)
2	$(1a,2d)$	.738	.738	.738	(.738,.738,.738)

3	$(2a,3a,4a)$	.333	.233	.505	(.233,.333,.505)
4	$(2d,3a,4a)$	.369	.233	.505	(.233,.369,.505)
5	$(2a,3d,4a)$	.384	.262	.505	(.262,.384,.505)
6	$(2a,3a,4d)$	.248	.233	.262	(.233,.248,.262)
7	$(2a,3d,4d)$	.262	.262	.262	(.262,.262,.262)
8	$(2d,3a,4d)$	.233	.233	.233	(.233,.233,.233)
9	$(4a,5a)$	.500	.495	.505	(.495,.500,.505)
10	$(4d,5a)$	.505	.505	.505	(.505,.505,.505)
11	$(4a,5d)$	.495	.495	.495	(.495,.495,.495)
12	(ϕ,ϕ)	0	0	0	(0,0,0)

Table 2.

Now by the centroid defuzzy method the criticality (risk) index of each state is obtained as follow:

$R_1 = 0.5,\ R_2 = 0.738,\ R_3 = 0.357,\ R_4 = 0.369,\ R_5 = 0.343,\ R_6 = 0.248,$

$R_7 = 0.262,\ R_8 = 0.233,\ R_9 = 0.498,\ R_{10} = 0.505,\ R_{11} = 0.495,\ R_{12} = 0$

Now the severity of risk for each state will be calculated from (13).in the remaining paper SR is used instead "severity of risk". To show the results of the proposed method, present example is investigated at interval time [0-15]. (We assume that the time step is equal to 0.1).amount of SR for some times is presented in table 3.

SR For some times in each state

Time / state	T=0.1	T=5	T=7.5	T=10	T=12.5	T=15
1	0.47561471	0.04104250	0.01175887	0.00336897	0.00096523	0.00027654
2	0.02137931	0.21091630	0.14731396	0.09490483	0.05915405	0.03633468
3	0.00664452	0.01184423	0.00367553	0.00107996	0.00031198	0.00008963
4	0.00020916	0.04262407	0.03224555	0.02130445	0.01338915	0.00824664
5	0.00012844	0.01413854	0.00692216	0.00277203	0.00100292	0.00034186
6	0.00006347	0.00531377	0.00226479	0.00079547	0.00025506	0.00007795
7	0.00000158	0.01255955	0.01137865	0.00763502	0.00442552	0.00236683
8	0.00000159	0.01517855	0.01577093	0.01245568	0.00868858	0.00569315
9	0.00000564	0.05013567	0.04941036	0.03696320	0.02453516	0.01542563
10	0.00000009	0.05477884	0.09169653	0.10119975	0.09055081	0.07181572
11	0.00000004	0.02120638	0.03567689	0.03960991	0.03570870	0.02858365
12	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

Table 3.

Treat of the amount of SR for each state is shown in the following figures at times T= 0.1, T=5, T=10 and T=15.

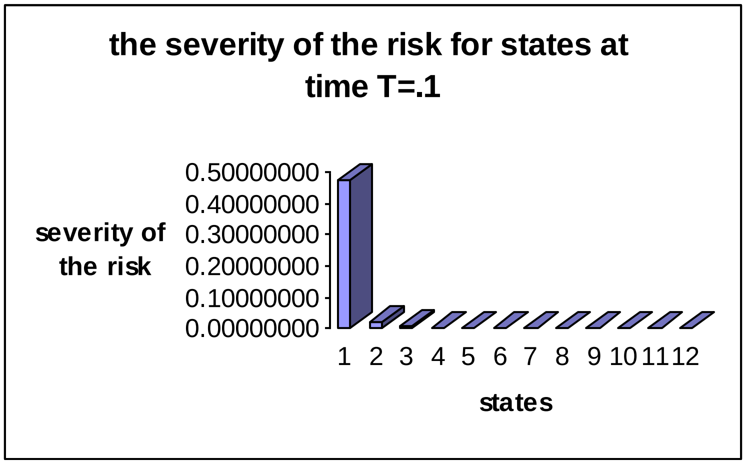

Figure 5.

In initial times the state no 1 have the SR almost equal to 0.475 and other states own less SR.

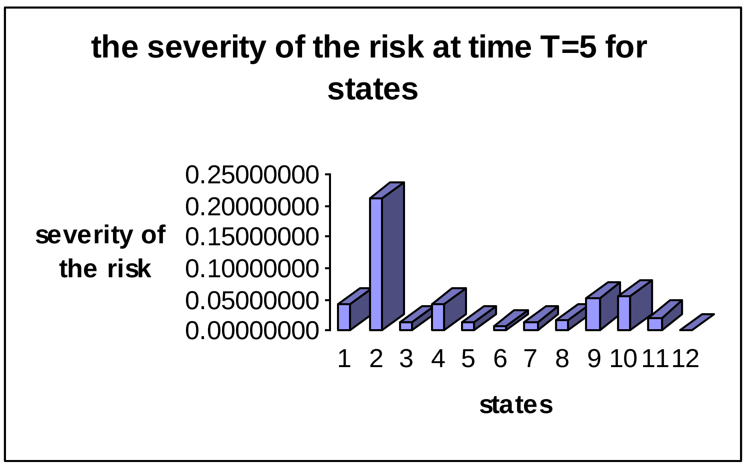

Figure 6.

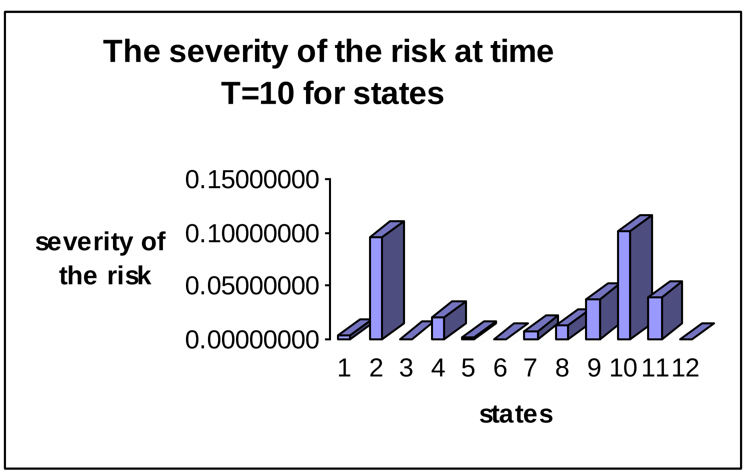

Figure 7.

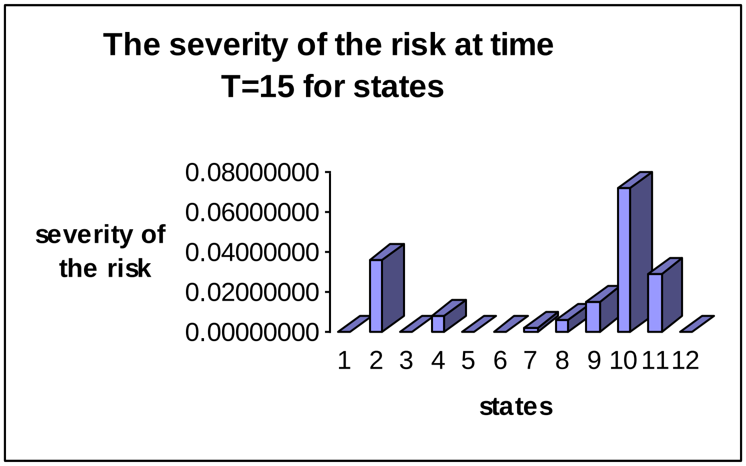

Figure 8.

Figures 6,7and 8 imply that some states such as 6 and 7 own less SR during project implementation.

Respectively, treat of SR for some states such as 1,6and 11 in interval time [0-15] are presented in following figures:

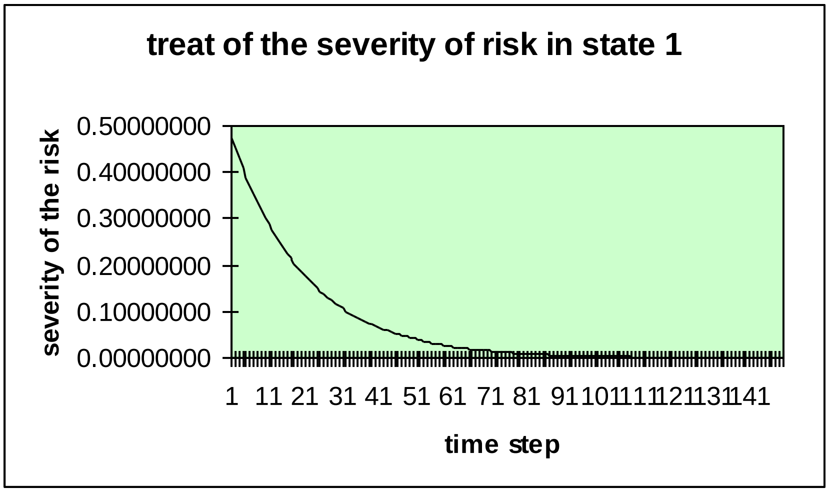

Figure 9.

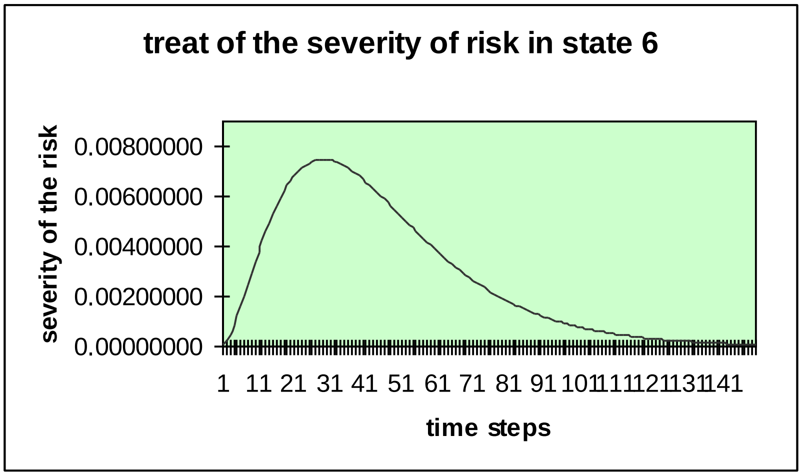

Figure 10.

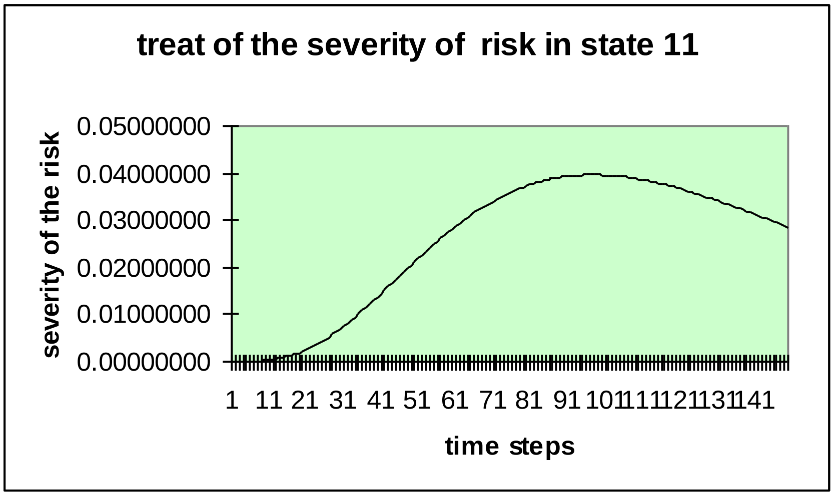

Figure 11.

Curve of maximum SR at each time of project implementation in interval [0-15] is present in figure 12.

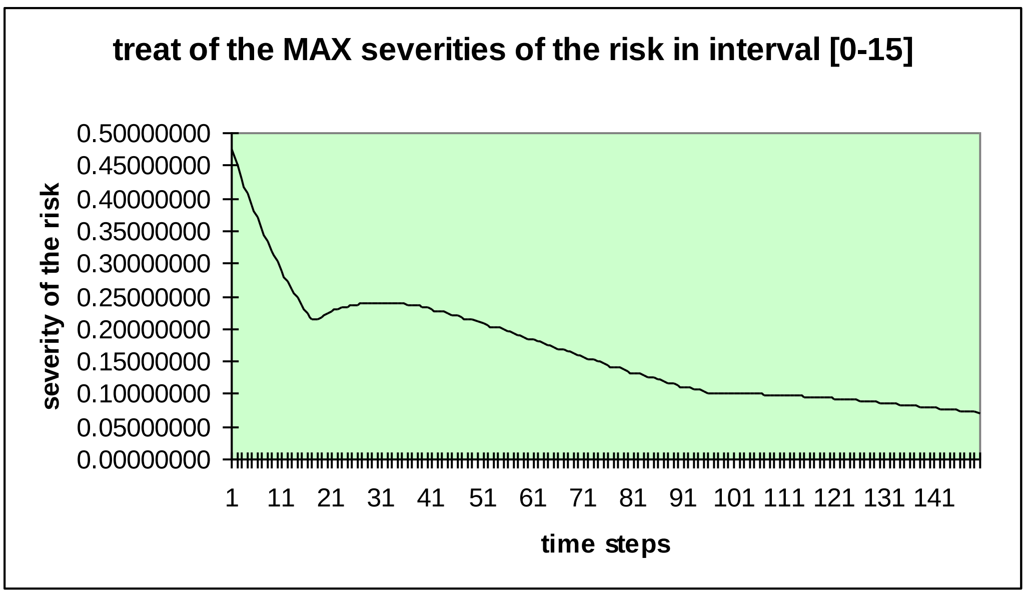

Figure 12.

5. Conclusion

In this paper an analytical method is developed to evaluating severity of the risk in PERT network, where the activity durations are exponentionaly distributed random variable and independent. In the proposed method, the PERT network was transformed into continuous-time Markov process. Then we presented a method for computing the risk index for each state of Markov process based on fuzzy theory and by using absorption probability in each state the severity of risk for each state was computed. It connects a new concept of project risk to its activity criticalities and proposes a new method to evaluating the risk in stochastic networks. The measure obtained would help the decision maker to forecast and decide which time project and activity have to be supervised more closely and which time less. It also can serve as indicator of general risk of whole of the project to reporting to the senior management or others in each specific time. In other word it can be use as a tool to identify states that needs to more attention and keep in managing and resources allocation. This model is being introduced here for the first time, and it would be impossible to compare our result to other paper results. Our estimation can be considered as a good index for general risk of project and a start to develop some analytical or approximation approaches to deal with overall project risk management.

However, this paper can be considered as an introduction for the development of proper dispatching rules in dynamic risk management, analytically. For further research, it is necessary to understand the limitations of our method and then develop some approaches to overcome the problems, which result from the limitations. The state space of the continuous-time Markov process grows exponentially with the network size. As the worst case example, for a complete transformed network with l nodes and $l(l-1)/2$ arcs, the size of the state space is given by $N(l) = U_l - U_{l-1}$, where

$$U_l = \sum_{k=0}^{l} 2^{k(l-k)} \tag{14}$$

(Refer to [6]).

Therefore, the major limitation of the method described in this paper is its nature of NP hardness. In practice, the number of arcs is generally much less than $l(l-1)/2$, and it should also be noted that for large networks, any alternate method of producing reasonably accurate answers will be prohibitively expensive.

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