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A relaxed-inertial CGP algorithm with modified Hager-Zhang-type parameter for constrained nonlinear equations with its application

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Abstract

This paper introduces a generalized relaxed-inertial conjugate gradient projection (CGP) algorithm for solving constrained nonlinear equations, which are widely encountered in practical applications. By integrating the relaxed-inertial mechanism with the projection technique and a modified Hager-Zhang-type conjugate parameter, the proposed algorithm generates the search direction that inherently satisfies sufficient descent condition and trust region feature without requiring any line search. The proposed algorithm is derivative-free, low-memory, and well-suited for large-scale equations. We establish the global convergence of the proposed algorithm under mild hypotheses, notably without the Lipschitz continuity requirement. Numerical results demonstrate the proposed algorithm's efficiency and competitiveness in large-scale constrained nonlinear equations, where it can solve approximately 68.06% of test problems with the fewest iterations, 85.00% with the fewest function evaluations, and 74.72% with the least running time in seconds. Furthermore, it is successfully applied to sparse signal reconstruction applications.

Keywords: Relaxed-inertial mechanism, constrained nonlinear equation, conjugate gradient methods, global convergence, sparse signal reconstruction.

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1. Introduction

Nonlinear equations arise in a wide variety of applications, including neural networks [1], signal processing [7], image recovery [8], chemical equilibrium systems [10], among others. In this paper, we focus on the following class of nonlinear equations with convex constraints:

$$E(x) = 0, \quad x \in \mathcal{C}, \tag{1.1}$$

where $\mathcal{C} \subset R^n$ is a nonempty closed convex set, and $E : R^n \to R^n$ is a monotone function, meaning that for any $x, y \in R^n$, we have

$$\left(E(x) - E(y)\right)^T (x - y) \geqslant 0.$$

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Due to the inherent nonlinearity, closed-form solutions to such problems are typically unattainable. As a result, the development of efficient iterative methods for solving nonlinear equations has attracted significant attention in the literature.

In response to this challenge, a variety of classical iterative methods have been proposed, including Newton methods [13], quasi-Newton methods [14], and Levenberg-Marquardt methods [18]. These methods often exhibit strong local convergence properties. However, their reliance on evaluating or approximating the Jacobian matrix at each iteration makes them computationally impractical for large-scale problems. To address this limitation, some researchers have increasingly turned to derivative-free projection methods [19], particularly those incorporating the hyperplane projection technique. These methods avoid matrix storage entirely and rely solely on function evaluations, making them well-suited for large-scale settings. Among them, the conjugate gradient projection (CGP) method has emerged as one of the most popular and practical methods. Its appeal lies in its minimal memory requirement, algorithmic simplicity, and solid empirical performance, making it a competitive choice for large-scale nonlinear equations (for more details, see [5, 15]).

Despite its practical advantages, the CGP method still faces key challenges in terms of both theoretical analysis and numerical performance. In particular, difficulties remain in selecting appropriate steplength in the line search approach and in designing efficient search directions that balance global convergence and fast local behavior. To address these issues, a series of relaxed-inertial variants of CGP methods have been proposed in recent years, aiming to further accelerate convergence and enhance robustness. By incorporating inertial extrapolation step into CGP frameworks, these methods can effectively mitigate oscillations and stagnation, leading to improved performance in practice. For example, Yuan et al. [20] developed an inertial three-term Polak-Ribiére-Polyak (PRP) CGP method, where the search direction maintains sufficient descent and trust region characteristics. Zheng et al. [22] improved the Fletcher-Reeves (FR) conjugate parameter via a shrinkage multiplier, resulting in a derivative-free two-term search direction and its extended spectral version. Liu et al. [9] proposed a spectral CGP method that integrates inertial factors for solving nonlinear pseudo-monotone equations. Hu [4] introduced an inertial hybrid Wei-Yao-Liu CGP method, specifically designed for large-scale nonlinear monotone equations with convex constraints. Most recently, Li et al. [6] proposed an inertial-based hybrid three-term CGP method tailored for general constrained nonlinear equations.

Motivated by these developments, and inspired by the classical work of [3], we propose a relaxed-inertial CGP algorithm with a modified Hager-Zhang-type conjugate parameter, designed specifically for solving constrained nonlinear equations. The main contributions of this paper are summarized as follows.

- ♠ A modified Hager-Zhang-type conjugate parameter is embedded into the search direction, ensuring both the sufficient descent condition and trust region feature without requiring any line search.
- ♠ An inertial extrapolation step is incorporated to accelerate the convergence process and enhance numerical efficiency.
- ♠ The global convergence of the proposed algorithm is established under mild hypotheses, without requiring Lipschitz continuity of nonlinear equations.
- ♠ Numerical experiments on large-scale nonlinear equations and sparse signal reconstruction problems demonstrate the efficacy and competitiveness of the proposed algorithm.

To support these contributions, the structure of this paper is organized are follows. Section 2 introduces the design of the search direction incorporating the modified Hager-Zhang-type conjugate parameter. Section 3 presents the complete algorithm framework of the proposed algorithm. Section 4 is devoted to the global convergence analysis under reasonable hypotheses. Section 5 reports comprehensive numerical results, including applications in sparse signal reconstruction. Finally, Section 6 concludes the paper.

2. Design of search direction

In this section, we aim to develop a relaxed-inertial CGP algorithm for solving constrained nonlinear equations. The proposed algorithm is constructed by designing a new search direction based on the conjugate gradient framework and integrating a relaxed-inertial mechanism.

The conjugate gradient method is well-known for its effectiveness in solving large-scale unconstrained optimization problems of the form $\min_{x\in R^n} f(x)$, where $f : R^n \to R$ is continuously differentiable. Given the current iterative point x_k, the next iterative point is updated by $x_{k+1} = x_k + \alpha_k d_k$, where α_k is the steplength determined by a reasonable line search approach, and d_k is the search direction determined by a suitable conjugate parameter. The sufficient descent condition plays a important role in the analysis of global convergence, which is defined as $g_k^T d_k \leqslant -c_1\|g_k\|^2$, where $g_k := \nabla f(x_k)$, c_1 is a positive constant, and $\|\cdot\|$ denotes the Euclidean norm. Moreover, Yuan et al. [21] have demonstrated that the conjugate gradient method possesses the strong convergence feature if it satisfies the trust region feature, which is defined as $\|d_k\| \leqslant c_2\|g_k\|$, where c_2 is a positive constant. Recently, Hager et al. [3] proposed a new conjugate parameter, defined as:

$$\beta_k^{HZ} = \frac{1}{d_{k-1}^T y_{k-1}}\left(y_{k-1} - 2d_{k-1}\frac{\|y_{k-1}\|^2}{d_{k-1}^T y_{k-1}}\right)^T g_k,$$

where $y_{k-1} = g_k - g_{k-1}$. This method has been shown to satisfy the sufficient descent condition, but it does not inherently satisfy the trust region feature. To overcome this limitation, we propose a modified version of the HZ conjugate parameter that ensures both sufficient descent and trust-region properties. We then incorporate this modified parameter into a relaxed-inertial CGP framework, extending its applicability to constrained nonlinear equations.

In particular, we integrate a relaxed-inertial mechanism into the proposed algorithm to further accelerate convergence and improve stability. The relaxed-inertial mechanism generates an inertial extrapolation point used in the search direction. Specifically, the inertial extrapolation point ν_k is computed as:

$$\nu_k = x_k + t_k(x_k - x_{k-1}), \tag{2.1}$$

where the inertial extrapolation steplength t_k is adaptively chosen by:

$$t_k = \begin{cases} \min\left\{t, \frac{1}{k^2\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ t, & \text{otherwise}, \end{cases} \tag{2.2}$$

where $t \in [0, 1)$. Based on this inertial extrapolation point, we consider the following two-term search direction, defined as:

$$d_0 = -E(\nu_0), \quad \text{and} \quad d_k = -E(\nu_k) + \beta_k d_{k-1}, \quad k \geqslant 1. \tag{2.3}$$

Here, the conjugate parameter β_k is computed as:

$$\beta_k = \frac{h_{k-1}^T E(\nu_k) d_{k-1}^T h_{k-1} - 2\|h_{k-1}\|^2 E(\nu_k)^T d_{k-1}}{\max\left\{\mu\|h_{k-1}\|^2\|d_{k-1}\|(\|d_{k-1}\| + 1), \left(d_{k-1}^T h_{k-1}\right)^2\right\}}, \tag{2.4}$$

where $h_{k-1} = E(\nu_k) - E(\nu_{k-1})$ and $\mu > \frac{1}{2}$. In the following, we demonstrate that the designed search direction satisfies both the sufficient descent condition and trust region feature, without relying on any line search approaches. Starting from the update formula (2.3), we multiply both sides by $E(\nu_k)^T$ to obtain:

$$E(\nu_k)^T d_k = -\|E(\nu_k)\|^2 + \frac{h_{k-1}^T E(\nu_k) d_{k-1}^T h_{k-1} E(\nu_k)^T d_{k-1} - 2\|h_{k-1}\|^2 \left(E(\nu_k)^T d_{k-1}\right)^2}{\max\left\{\mu\|h_{k-1}\|^2\|d_{k-1}\|(\|d_{k-1}\| + 1), \left(d_{k-1}^T h_{k-1}\right)^2\right\}}. \tag{2.5}$$

Applying the inequality $a^T b \leqslant \frac{1}{2}\left(\|a\|^2 + \|b\|^2\right)$ with $a = E(\nu_k)^T h_{k-1} d_{k-1}$ and $b = E(\nu_k)^T d_{k-1} h_{k-1}$, the numerator of the second term in (2.5) can be simplified as:

$$\frac{1}{2}(E(\nu_k)^T h_{k-1})^2\|d_{k-1}\|^2 - \frac{3}{2}(E(\nu_k)^T d_{k-1})^2\|h_{k-1}\|^2 \leqslant \frac{1}{2}\|E(\nu_k)\|^2\|d_{k-1}\|^2\|h_{k-1}\|^2.$$

In addition, the denominator of the second term in (2.5) can be simplified as:

$$\mu\|h_{k-1}\|^2\|d_{k-1}\|^2.$$

Substituting these two relations into (2.5), we obtain:

$$E(\nu_k)^T d_k \leqslant -\|E(\nu_k)\|^2 + \frac{\frac{1}{2}\|E(\nu_k)\|^2\|d_{k-1}\|^2\|h_{k-1}\|^2}{\mu\|h_{k-1}\|^2\|d_{k-1}\|^2} \leqslant -c_1\|E(\nu_k)\|^2, \tag{2.6}$$

with $c_1 = \left(1 - \frac{1}{2\mu}\right)$. Hence, the sufficient descent condition holds. Moreover, the definition of β_k yields that:

$$|\beta_k| \leqslant \frac{3\|h_{k-1}\|^2\|E(\nu_k)\|\|d_{k-1}\|}{\mu\|h_{k-1}\|^2\|d_{k-1}\|^2} = \frac{3\|E(\nu_k)\|}{\mu\|d_{k-1}\|}.$$

Substituting it into (2.3), it further follows that

$$\|d_k\| \leqslant \|E(\nu_k)\| + \frac{3\|E(\nu_k)\|}{\mu\|d_{k-1}\|}\|d_{k-1}\| \leqslant c_2\|E(\nu_k)\|, \tag{2.7}$$

with $c_2 = \left(1 + \frac{3}{\mu}\right)$. Hence, the trust region feature holds.

3. Algorithmic framework

Based on the preceding discussion, we propose a modified Hager-Zhang-type CGP algorithm with a relaxed-inertial mechanism, referred to as IMHZCGP algorithm, for solving constrained nonlinear equations. The complete procedure is detailed in Algorithm 3.1.

Algorithm 3.1 (IMHZCGP algorithm)**.**

Step 0. Let $x_0 := x_{-1} \in R^n$. Choose the parameters: $\epsilon \in (0,1)$, $t \in [0,1)$, $\mu > \frac{1}{2}$, $\sigma, \rho \in (0,1)$, $0 < \eta_1 < \eta_2$, $\varpi \in (0,2)$. Set iterative counter $k := 0$.

Step 1. If $\|E(x_k)\| \leqslant \epsilon$, stop. Otherwise, compute the inertial extrapolation point ν_k by (2.2) and (2.1).

Step 2. If $\|E(\nu_k)\| \leqslant \epsilon$, stop. Otherwise, compute the search direction d_k by (2.3) and (2.4).

Step 3. Find the smallest integer $i_k \geqslant 0$ such that:

$$-E(\nu_k + \rho^{i_k} d_k)^T d_k \geqslant \sigma\rho^{i_k}\mathcal{M}(i_k)\|d_k\|^2, \tag{3.1}$$

with $\mathcal{M}(i_k) = \max\left\{\eta_1, \min\left\{\|E(\nu_k + \rho^{i_k} d_k)\|, \eta_2\right\}\right\}$. Then, set $\alpha_k = \rho^{i_k}$ and $w_k = \nu_k + \alpha_k d_k$.

Step 4. If $\|E(w_k)\| \leqslant \epsilon$, stop. Otherwise, update the next iterative point by the following projection rule:

$$x_{k+1} = \mathcal{P}_{\mathcal{C}}\left[\nu_k - \varpi\lambda_k E(w_k)\right], \quad \lambda_k = \frac{E(w_k)^T(\nu_k - w_k)}{\|E(w_k)\|^2}.$$

Step 5. Set $k := k + 1$ and return to Step 1.

Remark 3.2. According to the rule defined in (2.2), it holds for all $k \geqslant 0$ that $t_k\|x_k - x_{k-1}\| \leqslant \frac{1}{k^2}$, which implies that

$$\sum_{k=0}^{\infty} t_k\|x_k - x_{k-1}\| < \infty.$$

Remark 3.3. The adaptive line search approach used in the Algorithm 3.1 is inspired by the work of [17]. To regulate the scaling in the line search, a quality measure, denoted as $\mathcal{M}(i_k)$, is introduced to ensure that the right-hand side of (3.1) does not become either excessively small or too large. We assume the existence of an index $k_0 \geqslant 0$ such that (3.1) is not satisfied for any i. Specifically, we have:

$$-E(\nu_{k_0} + \rho^i d_{k_0})^T d_{k_0} < \sigma\rho^i \mathcal{M}(i)\|d_{k_0}\|^2,$$

Taking the limit of this inequality as $i \to +\infty$, we obtain $E(\nu_{k_0})^T d_{k_0} \geqslant 0$, which follows from the continuity of E and $\rho \in (0, 1)$. However, this result contradicts the sufficient descent property discussed in (2.6). Therefore, we conclude that the adaptive line search approach is well-defined.

Remark 3.4. In Step 4th of the Algorithm 3.1, the projection operator $\mathcal{P}_{\mathcal{C}}$ is defined with respect to the close convex set $\mathcal{C}$ as $\mathcal{P}_{\mathcal{C}}[x] = \arg\min\{\|y - x\| \mid y \in \mathcal{C}\}$. This operator possesses the well-known nonexpansive property:

$$\|\mathcal{P}_{\mathcal{C}}[x] - \mathcal{P}_{\mathcal{C}}[y]\| \leqslant \|x - y\|, \quad \forall x, y \in R^n. \tag{3.2}$$

4. Theoretical analysis of convergence

To analyze the global convergence of the Algorithm 3.1, we fist introduce two essential hypotheses that form the theoretical foundation for the subsequent analysis.

Hypotheses H1. The solution set $\mathcal{C}_*$ of problem (1.1) is non-empty.

Hypotheses H2. The function E is monotone on R^n.

These hypotheses play a crucial role in establishing the convergence behavior of the Algorithm 3.1, as they ensure that the iterative process will converge to a solution within the feasible region of problem (1.1). The following lemma is adapted from the work [12].

Lemma 4.1. *Suppose that the sequences $\{p_k\}$ and $\{q_k\}$ are nonnegative real numbers and satisfy the relation $p_{k+1} \leqslant p_k + q_k$. If the series $\sum_{k=1}^{\infty} q_k < \infty$, then the sequence $\{p_k\}$ is convergent.*

Lemma 4.2. *Suppose that Hypotheses* H1 *and* H2 *hold. If the sequences $\{x_k\}$, $\{\nu_k\}$, and $\{w_k\}$ are generated by the Algorithm 3.1, then the sequence $\{\|x_k - x_*\|\}$ is convergent for any $x_* \in \mathcal{C}_*$.*

Proof. From the fact that $E(x_*) = 0$ with $x_* \in \mathcal{C}_*$, this immediately implies that $E(x_*)^T(w_k - x_*) = 0$. By the monotonicity of E, it follows that $E(w_k)^T(w_k - x_*) \geqslant 0$. Utilizing this inequality, it yields that

$$E(w_k)^T(\nu_k - x_*) = E(w_k)^T(\nu_k - w_k) + E(w_k)^T(w_k - x_*) \geqslant E(w_k)^T(\nu_k - w_k). \tag{4.1}$$

Together with this inequality and (3.1), we derive the inequality:

$$E(w_k)^T(\nu_k - x_*) \geqslant \sigma\eta_1\|\nu_k - w_k\|^2. \tag{4.2}$$

For any $x_* \in \mathcal{C}_*$, from the definition of $\mathcal{P}_{\mathcal{C}}$ and (3.2), we obtain

$$\begin{aligned}\|x_{k+1} - x_*\|^2 &= \|\mathcal{P}_{\mathcal{C}}\left[\nu_k - \varpi\lambda_k E(w_k)\right] - \mathcal{P}_{\mathcal{C}}[x_*]\|^2 \\ &\leqslant \|\nu_k - \varpi\lambda_k E(w_k) - x_*\|^2 = \|\nu_k - x_*\|^2 - 2\varpi\lambda_k E(w_k)^T(\nu_k - x_*) + \varpi^2\lambda_k^2\|E(w_k)\|^2.\end{aligned}$$

By substituting (4.1) and the expression for λ_k into the above inequality, we derive:

$$\begin{aligned}\|x_{k+1} - x_*\|^2 &\leqslant \|\nu_k - x_*\|^2 - 2\varpi\lambda_k E(w_k)^T(\nu_k - w_k) + \varpi^2\lambda_k^2\|E(w_k)\|^2 \\ &= \|\nu_k - x_*\|^2 - \varpi(2 - \varpi)\frac{\left[E(w_k)^T(\nu_k - w_k)\right]^2}{\|E(w_k)\|^2}.\end{aligned}$$

Finally, using the inequality (4.2), this yields

$$\|x_{k+1}-x_*\|^2 \leqslant \|v_k-x_*\|^2 - \varpi(2-\varpi)\frac{\sigma^2\eta_1^2\|v_k-w_k\|^4}{\|E(w_k)\|^2}. \tag{4.3}$$

Moreover, we recall the definition of v_k and use the fact that $\varpi \in (0,2)$. This leads to:

$$\|x_{k+1}-x_*\| \leqslant \|v_k-x_*\| = \|x_k+t_k(x_k-x_{k-1})-x_*\| \leqslant \|x_k-x_*\|+t_k\|x_k-x_{k-1}\| \leqslant \|x_k-x_*\|+\frac{1}{k^2}.$$

Hence, applying Lemma 4.1, we conclude that the sequence $\{\|x_k-x_*\|\}$ is convergent. □

Lemma 4.3. *Suppose that Hypotheses* H1 *and* H2 *hold. If the sequences* $\{x_k\}$, $\{v_k\}$, *and* $\{w_k\}$ *are generated by the Algorithm* 3.1, *then all these sequences are bounded.*

Proof. We begin by recalling from Lemma 4.2 that the sequence $\{\|x_k-x_*\|\}$ is convergent for any $x_* \in \mathcal{C}_*$. As a direct consequence, both $\{\|x_k-x_*\|\}$ and $\{x_k\}$ must be bounded. In particular, there exists two positive constants M_1 and M_2 such that $\|x_k-x_*\| \leqslant M_1$, $\|x_k\| \leqslant M_2$. With this boundedness of $\{x_k\}$ established, we now estimate the norm of the difference between successive iterative points:

$$\|x_k-x_{k-1}\| \leqslant \|x_k-x_*\|+\|x_{k-1}-x_*\| \leqslant 2M_1.$$

Using the above bound and the definition of t_k, we obtain

$$\|v_k\| = \|x_k+t_k(x_k-x_{k-1})\| \leqslant \|x_k\|+t_k\|x_k-x_{k-1}\| \leqslant M_2+1,$$

which shows that the sequence $\{v_k\}$ is also bounded. Next, by the continuity of E, it follows that $\{E(v_k)\}$ is bounded as well. Furthermore, the trust region feature described in (2.7) ensures that the sequence $\{d_k\}$ remains bounded. Since $w_k = v_k+\alpha_k d_k$ and $\alpha_k \in (0,1)$, we conclude that $\{w_k\}$ is bounded. □

Lemma 4.4. *Suppose that Hypotheses* H1 *and* H2 *hold. If the sequences* $\{x_k\}$, $\{v_k\}$, *and* $\{w_k\}$ *are generated by the Algorithm* 3.1, *then the following conclusions hold:*

$$\lim_{k\to\infty}\alpha_k\|d_k\| = \lim_{k\to\infty}\|v_k-w_k\| = 0.$$

Proof. To establish this result, we make use of the boundedness of the sequences derived in Lemma 4.3, as well as the continuity of E. Specifically, the boundedness of $\{w_k\}$ and continuity of E imply that $\{E(w_k)\}$ is bounded. That is, there exists a positive constant M_3 such that $\|E(w_k)\| \leqslant M_3$ for all k. Next, considering the definition of v_k, $t \in (0,1)$, and Remark 3.4, we can estimate the norm as follows:

$$\begin{aligned}
\|v_k-x_*\|^2 &= \|x_k+t_k(x_k-x_{k-1})-x_*\|^2 \\
&= \|x_k-x_*\|^2+2t_k(x_k-x_{k-1})^T(x_k-x_*)+t_k^2\|x_k-x_{k-1}\|^2 \\
&\leqslant \|x_k-x_*\|^2+t_k\|x_k-x_{k-1}\|(2\|x_k-x_*\|+t_k\|x_k-x_{k-1}\|) \leqslant \|x_k-x_*\|^2+(2M_1+\frac{1}{k^2})\frac{1}{k^2}.
\end{aligned}$$

Now, substituting this estimate into (4.3), we obtain:

$$\|x_{k+1}-x_*\|^2 \leqslant \|x_k-x_*\|^2+(2M_1+\frac{1}{k^2})\frac{1}{k^2}-\varpi(2-\varpi)\frac{\sigma^2\eta_1^2\|v_k-w_k\|^4}{M_3^2}.$$

Summing both sides of the inequality from $k=0$ to ∞, we derive

$$\frac{\varpi(2-\varpi)\sigma^2\eta_1^2}{M_3^2}\sum_{k=0}^{\infty}\|v_k-w_k\|^4 \leqslant \sum_{k=0}^{\infty}\left(\|x_k-x_*\|^2-\|x_{k+1}-x_*\|^2+(2M_1+\frac{1}{k^2})\frac{1}{k^2}\right) < \infty.$$

This implies that $0 = \lim_{k\to\infty}\|v_k-w_k\| = \lim_{k\to\infty}\alpha_k\|d_k\|$. □

Theorem 4.5. *Suppose that Hypotheses* H1 *and* H2 *hold. If the sequence* $\{\nu_k\}$ *is generated by the Algorithm* 3.1, *then the following result holds:*

$$\lim_{k\to\infty} \inf \|E(\nu_k)\| = 0.$$

Proof. We prove this result by contradiction. Suppose, on the contrary, that the conclusion does not hold. Then there exists a positive constant M_4 such that $\|E(\nu_k)\| \geqslant M_4$ for all k. Under this assumption, we revisit the sufficient descent condition (2.6), which yields $\|d_k\| \geqslant c_1\|E(\nu_k)\| \geqslant c_1 M_4$ for all k. Now, from Lemma 4.4, we know that $\lim\limits_{k\to\infty} \alpha_k = 0$. Combining this with the boundedness results from Lemma 4.3, we may extract a convergent subsequence such that:

$$\lim_{i\to\infty, i\in\mathcal{K}} d_{k_i} = \tilde{d}, \quad \lim_{i\to\infty, i\in\mathcal{K}} \nu_{k_i} = \tilde{\nu},$$

where $\mathcal{K}$ is an infinite index set. Taking limits in (2.6) along this subsequences yields:

$$-E(\tilde{\nu})^T \tilde{d} \geqslant c_1\|E(\tilde{\nu})\|^2 \geqslant c_1 M_4 > 0. \tag{4.4}$$

On the other hand, since $\rho^{-1}\alpha_k$ fails to satisfy the line search approach (3.1), we have the inequality:

$$-E(\nu_k + \rho^{-1}\alpha_k d_k)^T d_k < \sigma\rho^{-1}\alpha_k \mathcal{M}(i_{k-1})\|d_k\|^2.$$

Taking limits along the same subsequence again, and using the continuity of E, we deduce $-E(\tilde{\nu})^T\tilde{d} \leqslant 0$, which contradicts the earlier strict inequality in (4.4). Therefore, the conclusion of Theorem 4.5 holds. □

Theorem 4.6. *Suppose that Hypotheses* H1 *and* H2 *hold. If the sequences* $\{x_k\}$ *and* $\{\nu_k\}$ *are generated by the Algorithm* 3.1, *then both sequences converge to the same solution of problem* (1.1).

Proof. To establish this result, we begin by analyzing the relationship between x_k and ν_k. From the definition of t_k and Remark 3.2, we have

$$\|x_k - \nu_k\| = \|x_k - (x_k + t_k(x_k - x_{k-1}))\| = t_k\|x_k - x_{k-1}\|.$$

Taking the limit on the above equality, it immediately follows that $\lim_{k\to\infty}\|x_k - \nu_k\| = 0$. Without loss of generality, we assume that

$$\lim_{i\to\infty} \nu_{k_i} = \nu_*, \quad \lim_{i\to\infty} x_{k_i} = x_*, \quad \lim_{i\to\infty} \|E(\nu_{k_i})\| = \|E(\nu_*)\| = 0.$$

Moreover, we also deduce $\lim\limits_{i\to\infty}\|E(x_{k_i})\| = \|E(\nu_*)\| = 0$. Since $\mathcal{C}$ is a closed set and $x_{k_i} \in \mathcal{C}$, we have $\nu_* \in \mathcal{C}$, which further implies that $\nu_* \in \mathcal{C}_*$. Now, by setting $x_* := \nu_* \in \mathcal{C}_*$ and invoking Lemma 4.2, we obtain

$$\lim_{k\to\infty} \|x_k - x_*\| = \lim_{i\to\infty} \|x_{k_i} - x_*\| = \lim_{i\to\infty} \|x_{k_i} - \nu_*\| = 0.$$

Therefore, both $\{x_k\}$ and $\{\nu_k\}$ converge to the same solution $x_* \in \mathcal{C}_*$. □

5. Numerical results and performance evaluation

To illustrate the effectiveness and competitiveness of the proposed algorithm, we apply it to solve large-scale constrained nonlinear equations and sparse signal reconstruction problems. All experiments are conducted on a Lenovo PC with a 2.10 GHz CPU and 16 GB RAM, running Windows 11, to ensure a consistent computational environment for performance comparison.

5.1. Experiments on constrained nonlinear equations

The goal of this experiment is to demonstrate the effectiveness and competitiveness of the proposed algorithm for solving large-scale constrained nonlinear equations. To this end, we compare the performance of the Algorithm 3.1 against two established algorithm: HZCGP algorithm [3] and CGAIS algorithm [11]. The algorithmic parameters for HZCGP and CGAIS algorithms are taken directly from their original references. For the Algorithm 3.1, the algorithmic parameters are set as:

$$\epsilon = 10^{-6}, \quad t = 0.35, \quad \mu = 0.51, \quad \sigma = 0.0001, \quad \rho = 0.32, \quad \eta_1 = 0.001, \quad \eta_2 = 0.8, \quad \varpi = 1.9.$$

For each algorithm and test problem, the computation is terminated when one of the following conditions is met:

(i) $\|E(x_k)\| \leqslant \epsilon$;
(ii) $\|E(\nu_k)\| \leqslant \epsilon$;
(iii) $\|E(w_k)\| \leqslant \epsilon$; or
(iv) NumI > 3000.

Let $E(x) = (E_1(x), E_2(x), \dots, E_n(x))^T$ denote the nonlinear equations to be solved with $x = (x_1, x_2, \dots, x_n)^T$. We consider the following test problems as follows.

Problem 5.1. Set $E_i(x) = e^{x_i} - 1$, for $i = 1, 2, \dots, n$, and $\mathcal{C} = R^n_+$.

Problem 5.2. Set $E_i(x) = \frac{i}{n} e^{x_i} - 1$, for $i = 1, 2, \dots, n$, and $\mathcal{C} = R^n_+$.

Problem 5.3. Set

$$\begin{aligned} E_1(x) &= 2x_1 + 0.5h^2(x_1 + h)^3 x_2, \\ E_i(x) &= 2x_i - x_{i-1} + x_{i+1} + 0.5h^2(x_i + ih)^3, \quad \text{for} \ \ i = 2, 3, \dots, n-1, \\ E_n(x) &= 2x_n - x_{n-1} + 0.5h^2(x_n + nh)^3, \end{aligned}$$

and $\mathcal{C} = R^n_+$, $h = \frac{1}{n+1}$.

Problem 5.4. Set $E_i(x) = 2x_i - \sin(x_i)$, for $i = 1, 2, \dots, n$, and $\mathcal{C} = [-2, +\infty)$.

Problem 5.5. Set $E_i(x) = (e^{x_i})^2 + 3\sin(x_i)\cos(x_i) - 1$, for $i = 1, 2, \dots, n$, and $\mathcal{C} = R^n_+$.

Problem 5.6. Set

$$\begin{aligned} E_1(x) &= x_1 - e^{\cos(\frac{x_1 + x_2}{2})}, \\ E_i(x) &= x_i - e^{\cos(\frac{x_{i-1} + x_i + x_{i+1}}{i})}, \quad \text{for} \ \ i = 2, 3, \dots, n-1, \\ E_n(x) &= x_n - e^{\cos(\frac{x_{n-1} + x_n}{n})}, \end{aligned}$$

and $\mathcal{C} = R^n_+$.

Problem 5.7. Set $E_i(x) = 2c(x_i - 1) + 4(\bar{x} - 0.25)x_i$, for $i = 1, 2, \dots, n$, and $\mathcal{C} = R^n_+$, $\bar{x} = \sum_{i=1}^{n} x_i^2$, $c = 10^{-5}$.

Problem 5.8. Set

$$\begin{aligned} E_1(x) &= x_1 + \sin(x_1) - 1, \\ E_i(x) &= -x_{i-1} + 2x_i + \sin(x_i) - 1, \quad \text{for} \ \ i = 2, 3, \dots, n-1, \\ E_n(x) &= x_n + \sin(x_n) - 1, \end{aligned}$$

and $\mathcal{C} = \{x \in R^n : x \geqslant -3\}$.

Problem 5.9. Set $E_i(x) = \ln(|x_i| + 1) - \frac{x_i}{n}$, for $i = 1, 2, \dots, n$, and $\mathcal{C} = R^n_+$.

For all test problems, we consider the problem dimension $n \in \{5000, 10000, 50000, 100000, 150000\}$. To examine the robustness of each algorithm with respect to different initial points, we employ the following types of initializations, while the initial points are generated as follows:

$$x_1 = \left(\frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^n}\right), \qquad x_2 = \left(1, \frac{1}{2}, \dots, \frac{1}{n}\right), \qquad x_3 = \left(0, \frac{1}{n}, \dots, \frac{n-1}{n}\right),$$
$$x_4 = \left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right), \qquad x_5 = \left(\frac{n-1}{n}, \frac{n-2}{n}, \dots, \frac{n-n}{n}\right), \qquad x_6 = \left(\frac{1}{3}, \frac{1}{3^2}, \dots, \frac{1}{3^n}\right),$$
$$x_7 = (1, 1, \dots, 1), \qquad x_8 = \mathrm{randn}(n, 1).$$

The numerical results are summarized in Tables 1-9, where "IP" represents the initial point, "Dim" represents the dimension of test problems multiplied by 1000, "RunT" represents the running time in seconds, "NumF" represents the number of function evaluations, "NumI" represents the number of iterations, "E_*" represents the approximate final value of $E(x_k)$. To further visualize the performance across all test cases, we adopt the performance profiles proposed by Dolan et al. [2], as illustrated in Figures 1-3. From the profiles, we draw the following observations.

- The Algorithm 3.1 consistently achieves the highest performance profiles across all metrics, indicating superior overall competitiveness compared to HZCGP and CGAIS algorithms.
- The Algorithm 3.1 solves approximately 68.06%, of test problems with the fewest iterations, 85.00% with the fewest function evaluations, and 74.72% with the least running time in seconds, demonstrating its efficiency and competitiveness in large-scale settings.

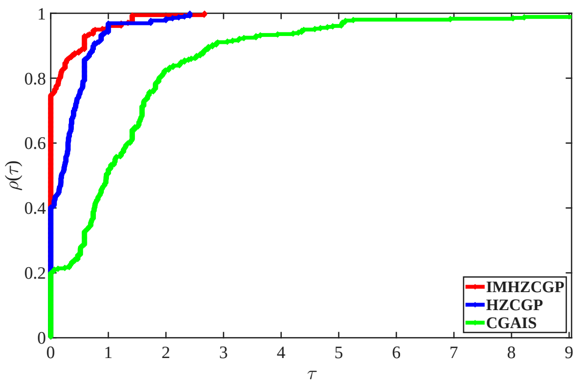

Figure 1: Performance profiles on NumI.

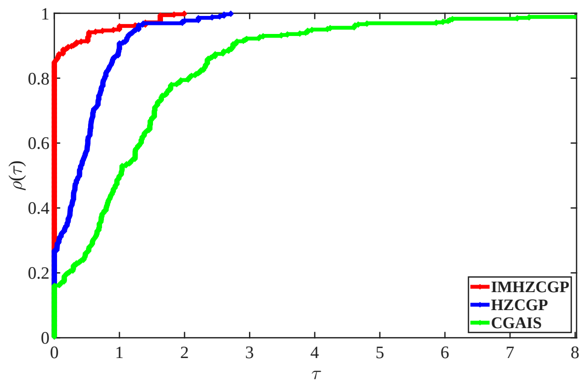

Figure 2: Performance profiles on NumF.

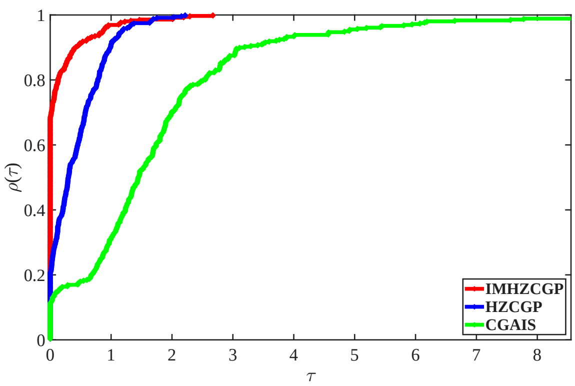

Figure 3: Performance profiles on RunT.

Table 1: Numerical results for Problem 5.1.

IP(Dim)	IMHZCGP	HZCGP	CGAIS
	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*
$x_1(5)$	5.00e-03/5/1/0.00e+00	1.09e-03/5/1/0.00e+00	1.02e-03/5/1/0.00e+00
$x_2(5)$	2.07e-03/13/3/0.00e+00	1.35e-03/9/2/0.00e+00	2.13e-03/10/2/0.00e+00
$x_3(5)$	4.32e-03/13/3/0.00e+00	3.33e-03/17/4/0.00e+00	3.30e-03/21/2/0.00e+00
$x_4(5)$	1.04e-03/22/5/0.00e+00	2.40e-03/27/6/0.00e+00	6.37e-03/56/13/0.00e+00
$x_5(5)$	1.42e-03/9/2/0.00e+00	1.49e-03/9/2/0.00e+00	2.73e-03/25/3/0.00e+00
$x_6(5)$	7.64e-04/22/5/0.00e+00	7.31e-04/27/6/0.00e+00	1.46e-03/40/12/0.00e+00
$x_7(5)$	4.57e-04/22/5/0.00e+00	5.40e-04/27/6/0.00e+00	1.77e-03/56/13/0.00e+00
$x_8(5)$	4.85e-04/22/5/0.00e+00	4.97e-04/23/5/0.00e+00	1.05e-03/34/10/0.00e+00
$x_1(10)$	2.52e-04/5/1/0.00e+00	1.74e-04/5/1/0.00e+00	2.12e-04/5/1/0.00e+00
$x_2(10)$	5.22e-04/13/3/0.00e+00	2.98e-04/9/2/0.00e+00	7.02e-04/10/2/0.00e+00
$x_3(10)$	5.74e-04/13/3/0.00e+00	7.08e-04/17/4/0.00e+00	6.99e-04/21/2/0.00e+00
$x_4(10)$	1.01e-03/22/5/0.00e+00	1.12e-03/27/6/0.00e+00	5.43e-04/11/3/0.00e+00
$x_5(10)$	4.58e-04/9/2/0.00e+00	4.28e-04/9/2/0.00e+00	1.02e-03/25/3/0.00e+00
$x_6(10)$	7.25e-04/22/5/0.00e+00	1.18e-03/27/6/0.00e+00	5.24e-04/11/3/0.00e+00
$x_7(10)$	8.47e-04/22/5/0.00e+00	9.14e-04/27/6/0.00e+00	6.61e-04/11/3/0.00e+00
$x_8(10)$	8.61e-04/22/5/0.00e+00	2.71e-03/64/15/8.93e-07	8.05e-04/11/3/0.00e+00
$x_1(50)$	9.88e-04/5/1/0.00e+00	6.30e-04/5/1/0.00e+00	5.44e-04/5/1/0.00e+00
$x_2(50)$	1.69e-03/13/3/0.00e+00	1.40e-03/9/2/0.00e+00	1.59e-03/10/2/0.00e+00
$x_3(50)$	1.17e-03/13/3/0.00e+00	1.88e-03/17/4/0.00e+00	2.00e-03/21/2/0.00e+00
$x_4(50)$	3.48e-03/29/7/0.00e+00	6.41e-03/67/16/0.00e+00	1.57e-03/11/3/0.00e+00
$x_5(50)$	7.48e-04/9/2/0.00e+00	7.42e-04/9/2/0.00e+00	2.59e-03/25/3/0.00e+00
$x_6(50)$	2.69e-03/29/7/0.00e+00	6.41e-03/72/16/8.85e-07	1.44e-03/11/3/0.00e+00
$x_7(50)$	2.69e-03/29/7/0.00e+00	6.46e-03/67/16/0.00e+00	1.39e-03/11/3/0.00e+00
$x_8(50)$	2.51e-03/26/6/0.00e+00	2.45e-03/27/6/0.00e+00	1.44e-03/11/3/0.00e+00
$x_1(100)$	8.63e-04/5/1/0.00e+00	5.82e-04/5/1/0.00e+00	5.48e-04/5/1/0.00e+00
$x_2(100)$	1.68e-03/13/3/0.00e+00	9.19e-04/9/2/0.00e+00	1.67e-03/10/2/0.00e+00
$x_3(100)$	1.54e-03/13/3/0.00e+00	1.98e-03/17/4/0.00e+00	2.55e-03/21/2/0.00e+00
$x_4(100)$	4.53e-03/34/8/0.00e+00	6.51e-03/51/12/5.57e-07	2.08e-03/11/3/0.00e+00
$x_5(100)$	1.07e-03/9/2/0.00e+00	1.28e-03/9/2/0.00e+00	3.79e-03/25/3/0.00e+00
$x_6(100)$	5.05e-03/34/8/0.00e+00	6.93e-03/51/13/8.73e-07	2.16e-03/11/3/0.00e+00
$x_7(100)$	4.53e-03/34/8/0.00e+00	5.85e-03/51/12/5.57e-07	1.81e-03/11/3/0.00e+00
$x_8(100)$	4.46e-03/34/8/0.00e+00	3.51e-03/27/6/0.00e+00	2.00e-03/11/3/0.00e+00
$x_1(150)$	2.62e-03/5/1/0.00e+00	2.43e-03/5/1/0.00e+00	2.23e-03/5/1/0.00e+00
$x_2(150)$	6.91e-03/13/3/0.00e+00	4.58e-03/9/2/0.00e+00	7.70e-03/10/2/0.00e+00
$x_3(150)$	6.58e-03/13/3/0.00e+00	8.89e-03/17/4/0.00e+00	1.24e-02/21/2/0.00e+00
$x_4(150)$	2.19e-02/34/8/0.00e+00	2.78e-02/43/10/0.00e+00	1.13e-02/11/3/0.00e+00
$x_5(150)$	5.29e-03/9/2/0.00e+00	4.33e-03/9/2/0.00e+00	1.64e-02/25/3/0.00e+00
$x_6(150)$	2.15e-02/34/8/0.00e+00	3.63e-02/59/14/1.16e-08	1.17e-02/11/3/0.00e+00
$x_7(150)$	1.80e-02/34/8/0.00e+00	2.55e-02/43/10/0.00e+00	1.03e-02/11/3/0.00e+00
$x_8(150)$	1.90e-02/34/8/0.00e+00	2.52e-02/44/10/0.00e+00	1.05e-02/11/3/0.00e+00

Table 2: Numerical results for Problem 5.2.

IP(Dim)	IMHZCGP	HZCGP	CGAIS
	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*
$x_1(5)$	3.29e-03/83/20/6.80e-07	3.33e-03/125/28/6.86e-07	4.51e-03/120/29/4.57e-07
$x_2(5)$	2.10e-03/74/18/4.70e-07	3.06e-03/116/26/9.16e-07	5.98e-02/1395/462/9.87e-07
$x_3(5)$	1.93e-03/71/18/4.29e-07	3.58e-03/140/29/8.69e-07	8.25e-03/209/62/9.77e-07
$x_4(5)$	2.25e-03/87/21/9.19e-07	3.17e-03/124/26/6.40e-07	5.78e-03/121/27/7.89e-07
$x_5(5)$	2.32e-03/80/20/5.87e-07	2.77e-03/96/20/6.37e-07	1.77e-02/404/128/5.62e-07
$x_6(5)$	2.66e-03/87/21/9.22e-07	3.61e-03/130/27/4.80e-07	4.54e-03/121/27/8.16e-07
$x_7(5)$	2.55e-03/94/23/4.31e-07	3.09e-03/109/24/7.52e-07	4.06e-03/111/24/5.82e-07
$x_8(5)$	2.58e-03/87/21/8.71e-07	3.16e-03/114/25/5.40e-07	7.45e-03/155/42/3.06e-07
$x_1(10)$	4.29e-03/87/21/5.29e-07	6.49e-03/115/26/7.59e-07	9.82e-03/137/31/5.64e-07
$x_2(10)$	4.26e-03/90/22/8.05e-07	6.74e-03/151/30/3.97e-07	6.29e-02/803/260/9.90e-07
$x_3(10)$	5.73e-03/90/22/6.88e-07	6.79e-03/112/25/3.56e-07	6.64e-02/794/260/9.94e-07
$x_4(10)$	5.26e-03/87/21/6.56e-07	7.21e-03/117/26/6.99e-07	8.51e-02/1187/388/9.94e-07
$x_5(10)$	4.57e-03/74/18/9.97e-07	4.35e-03/98/22/7.53e-07	1.59e-01/2064/686/1.00e-06
$x_6(10)$	3.94e-03/87/21/6.56e-07	6.09e-03/103/23/1.94e-07	3.02e-02/320/99/1.98e-07
$x_7(10)$	5.21e-03/87/22/3.80e-07	5.94e-03/105/24/7.37e-07	5.46e-03/97/20/6.10e-07
$x_8(10)$	3.55e-03/78/19/8.99e-07	5.08e-03/115/25/9.89e-07	2.95e-02/419/126/9.87e-07
$x_1(50)$	1.36e-02/92/22/4.52e-07	2.02e-02/137/30/4.51e-07	2.79e-02/121/28/4.05e-07
$x_2(50)$	1.40e-02/87/22/9.70e-07	1.65e-02/122/27/7.60e-07	3.36e-02/166/43/5.60e-07
$x_3(50)$	1.60e-02/99/24/6.18e-07	1.68e-02/117/24/8.62e-07	1.10e-01/490/154/9.95e-07
$x_4(50)$	1.31e-02/95/24/1.82e-07	2.28e-02/157/32/5.15e-07	1.09e-01/475/145/9.85e-07
$x_5(50)$	1.31e-02/87/21/7.81e-07	1.56e-02/131/28/8.02e-07	2.10e-01/975/322/9.89e-07
$x_6(50)$	1.43e-02/95/24/1.82e-07	1.80e-02/136/29/8.67e-07	1.04e-01/475/145/9.95e-07
$x_7(50)$	1.39e-02/79/20/9.31e-07	1.70e-02/112/26/5.47e-07	3.30e-01/1447/480/9.96e-07
$x_8(50)$	1.33e-02/98/24/8.71e-07	1.52e-02/117/23/3.33e-07	3.13e-02/161/39/3.24e-07
$x_1(100)$	2.14e-02/89/22/8.86e-07	2.60e-02/115/26/5.37e-07	2.43e-01/666/213/9.89e-07
$x_2(100)$	2.22e-02/90/22/4.41e-07	2.40e-02/112/26/8.77e-07	5.26e-01/1397/462/9.86e-07
$x_3(100)$	1.89e-02/90/22/4.60e-07	2.46e-02/111/24/3.81e-07	7.18e-02/207/52/9.81e-07
$x_4(100)$	2.47e-02/96/23/4.25e-07	3.07e-02/142/31/6.59e-07	7.23e-02/213/58/8.21e-07
$x_5(100)$	2.00e-02/90/22/9.81e-07	2.22e-02/103/22/2.81e-07	4.74e-01/1308/433/9.92e-07
$x_6(100)$	2.20e-02/96/23/4.25e-07	2.55e-02/114/27/7.08e-07	2.66e-01/744/235/9.88e-07
$x_7(100)$	2.04e-02/87/22/7.47e-07	2.53e-02/115/25/7.86e-07	3.18e-02/100/24/7.90e-07
$x_8(100)$	2.08e-02/91/22/5.31e-07	3.98e-02/181/33/9.90e-07	2.98e-01/840/274/9.99e-07
$x_1(150)$	6.45e-02/92/22/7.52e-07	8.56e-02/123/28/5.96e-07	6.85e-01/458/138/9.82e-07
$x_2(150)$	6.17e-02/90/22/5.35e-07	7.10e-02/95/22/9.04e-07	6.05e-01/416/125/9.82e-07
$x_3(150)$	6.02e-02/86/22/4.04e-07	7.49e-02/102/23/5.86e-07	8.18e-01/518/164/9.89e-07
$x_4(150)$	6.97e-02/96/23/7.14e-07	8.15e-02/121/28/2.82e-07	2.30e+00/1424/471/9.92e-07
$x_5(150)$	7.03e-02/94/23/5.69e-07	8.23e-02/108/23/9.24e-07	1.79e-01/130/33/5.76e-07
$x_6(150)$	7.67e-02/96/23/7.14e-07	8.43e-02/116/27/1.72e-07	2.31e+00/1424/471/9.93e-07
$x_7(150)$	6.26e-02/87/21/8.07e-07	8.67e-02/120/26/9.82e-07	2.01e-01/152/38/6.18e-07
$x_8(150)$	7.55e-02/100/25/9.79e-07	1.06e-01/148/30/5.89e-07	1.11e+00/699/223/9.94e-07

Table 3: Numerical results for Problem 5.3.

IP(Dim)	IMHZCGP RunT/NumF/NumI/E_*	HZCGP RunT/NumF/NumI/E_*	CGAIS RunT/NumF/NumI/E_*
$x_1(5)$	4.14e-02/101/24/8.86e-07	4.98e-02/113/24/9.56e-07	1.92e-01/387/94/9.10e-07
$x_2(5)$	3.08e-02/78/19/9.19e-07	5.77e-02/149/28/9.68e-07	1.28e-01/262/66/9.96e-07
$x_3(5)$	3.76e-02/95/23/8.68e-07	5.44e-02/141/26/9.93e-07	1.34e-01/273/68/8.79e-07
$x_4(5)$	1.45e-02/37/9/7.90e-07	2.85e-02/73/15/8.04e-07	9.98e-02/205/52/9.19e-07
$x_5(5)$	3.08e-02/79/19/8.82e-07	3.02e-02/76/16/8.28e-07	9.72e-02/198/48/8.63e-07
$x_6(5)$	1.44e-02/37/9/7.90e-07	2.82e-02/73/15/8.04e-07	1.00e-01/205/52/9.19e-07
$x_7(5)$	4.20e-02/109/26/7.52e-07	4.32e-02/112/23/7.94e-07	1.66e-01/339/85/9.63e-07
$x_8(5)$	4.78e-02/122/27/8.14e-07	6.50e-02/168/30/9.55e-07	1.55e-01/318/80/9.71e-07
$x_1(10)$	7.90e-02/101/24/7.39e-07	1.03e-01/132/27/8.42e-07	3.51e-01/351/85/9.67e-07
$x_2(10)$	6.19e-02/78/19/7.72e-07	1.05e-01/135/25/9.07e-07	2.70e-01/269/67/8.90e-07
$x_3(10)$	7.42e-02/95/23/7.07e-07	1.11e-01/140/26/9.29e-07	2.76e-01/277/69/9.40e-07
$x_4(10)$	2.63e-02/33/8/5.39e-07	4.02e-02/50/11/9.55e-07	2.00e-01/200/50/8.84e-07
$x_5(10)$	6.20e-02/79/19/7.31e-07	5.67e-02/72/16/8.69e-07	1.98e-01/199/49/9.66e-07
$x_6(10)$	2.64e-02/33/8/5.39e-07	3.89e-02/50/11/9.54e-07	1.96e-01/200/50/8.84e-07
$x_7(10)$	7.90e-02/101/24/7.48e-07	8.07e-02/104/22/8.79e-07	3.85e-01/384/96/9.22e-07
$x_8(10)$	9.96e-02/126/28/6.81e-07	1.58e-01/199/34/7.03e-07	3.24e-01/321/80/9.93e-07
$x_1(50)$	4.19e-01/101/24/6.77e-07	4.81e-01/117/24/7.63e-07	2.51e+00/480/120/9.61e-07
$x_2(50)$	3.22e-01/78/19/7.49e-07	5.64e-01/135/25/8.74e-07	1.39e+00/269/67/9.47e-07
$x_3(50)$	3.90e-01/95/23/6.80e-07	5.80e-01/140/26/8.69e-07	1.44e+00/277/69/9.99e-07
$x_4(50)$	1.08e-01/25/6/9.42e-07	2.05e-01/50/11/9.92e-07	8.91e-01/173/44/9.89e-07
$x_5(50)$	3.22e-01/79/19/7.07e-07	2.91e-01/72/16/8.36e-07	1.03e+00/199/49/9.98e-07
$x_6(50)$	1.10e-01/25/6/9.42e-07	2.07e-01/50/11/9.95e-07	8.98e-01/173/44/9.89e-07
$x_7(50)$	4.25e-01/101/24/6.93e-07	4.93e-01/119/24/6.56e-07	1.97e+00/388/94/7.58e-07
$x_8(50)$	6.39e-01/157/35/5.91e-07	6.84e-01/166/29/8.76e-07	1.77e+00/337/84/9.45e-07
$x_1(100)$	9.83e-01/101/24/6.75e-07	1.37e+00/136/27/8.72e-07	6.43e+00/520/130/9.11e-07
$x_2(100)$	7.74e-01/78/19/7.49e-07	1.35e+00/135/25/8.74e-07	3.37e+00/269/67/9.47e-07
$x_3(100)$	9.55e-01/95/23/6.80e-07	1.41e+00/140/26/8.69e-07	3.47e+00/277/69/9.99e-07
$x_4(100)$	2.49e-01/25/6/7.49e-07	5.02e-01/50/11/6.89e-07	2.06e+00/165/42/9.68e-07
$x_5(100)$	8.05e-01/79/19/7.07e-07	7.27e-01/72/16/8.35e-07	2.48e+00/199/49/9.98e-07
$x_6(100)$	2.52e-01/25/6/7.49e-07	4.99e-01/50/11/6.88e-07	2.07e+00/165/42/9.68e-07
$x_7(100)$	1.02e+00/101/24/6.80e-07	1.45e+00/145/28/8.53e-07	5.67e+00/452/113/9.12e-07
$x_8(100)$	1.61e+00/160/35/7.40e-07	1.65e+00/165/30/8.14e-07	4.32e+00/345/86/9.19e-07
$x_1(150)$	1.52e+00/101/24/6.74e-07	1.86e+00/123/25/6.28e-07	8.14e+00/457/115/9.88e-07
$x_2(150)$	9.47e-01/78/19/7.49e-07	1.63e+00/135/25/8.74e-07	4.15e+00/269/67/9.47e-07
$x_3(150)$	1.15e+00/95/23/6.80e-07	1.69e+00/140/26/8.69e-07	4.27e+00/277/69/1.00e-06
$x_4(150)$	3.01e-01/25/6/6.46e-07	6.00e-01/50/11/7.49e-07	2.47e+00/160/40/9.75e-07
$x_5(150)$	9.51e-01/79/19/7.07e-07	8.69e-01/72/16/8.35e-07	3.03e+00/199/49/9.98e-07
$x_6(150)$	3.01e-01/25/6/6.46e-07	6.02e-01/50/11/7.49e-07	2.46e+00/160/40/9.75e-07
$x_7(150)$	1.21e+00/101/24/6.77e-07	1.41e+00/117/24/7.62e-07	7.40e+00/480/120/9.62e-07
$x_8(150)$	1.78e+00/148/33/9.03e-07	2.89e+00/240/41/8.90e-07	5.37e+00/349/87/9.36e-07

Table 4: Numerical results for Problem 5.4.

IP(Dim)	IMHZCGP	HZCGP	CGAIS
	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*
$x_1(5)$	2.24e-03/65/16/4.84e-07	1.60e-03/81/20/5.40e-07	6.38e-03/218/59/1.55e-07
$x_2(5)$	7.29e-04/38/10/2.33e-07	1.12e-03/61/15/9.18e-07	2.02e-03/74/15/9.59e-07
$x_3(5)$	1.02e-03/53/13/4.91e-07	1.20e-03/65/16/5.91e-07	3.03e-03/109/18/2.41e-07
$x_4(5)$	1.49e-03/72/18/6.49e-07	1.68e-03/85/19/8.33e-07	5.44e-03/175/48/7.48e-07
$x_5(5)$	1.34e-03/53/13/4.42e-07	1.68e-03/76/17/9.75e-07	4.05e-02/1296/431/9.94e-07
$x_6(5)$	1.41e-03/72/18/6.51e-07	1.93e-03/100/21/2.99e-07	4.62e-03/162/48/1.19e-07
$x_7(5)$	1.33e-03/72/18/6.49e-07	1.49e-03/85/19/8.33e-07	4.79e-03/175/48/7.48e-07
$x_8(5)$	1.23e-03/64/16/8.74e-07	1.42e-03/80/19/7.07e-07	5.04e-03/190/49/3.16e-07
$x_1(10)$	2.03e-03/65/16/5.10e-07	2.86e-03/81/20/5.55e-07	1.19e-02/211/61/5.23e-07
$x_2(10)$	1.12e-03/38/10/2.33e-07	1.73e-03/61/15/9.18e-07	3.32e-03/74/15/9.59e-07
$x_3(10)$	1.59e-03/53/13/4.91e-07	1.94e-03/65/16/5.91e-07	4.03e-03/109/18/2.41e-07
$x_4(10)$	2.22e-03/69/18/5.97e-07	3.06e-03/104/23/5.95e-07	8.85e-03/174/52/2.05e-07
$x_5(10)$	1.65e-03/53/13/4.43e-07	3.21e-03/110/22/9.23e-07	7.20e-02/1347/448/9.98e-07
$x_6(10)$	2.17e-03/69/18/6.24e-07	2.90e-03/98/23/2.88e-07	9.09e-03/174/52/1.41e-07
$x_7(10)$	2.26e-03/69/18/5.97e-07	3.02e-03/104/23/5.97e-07	8.72e-03/174/52/2.05e-07
$x_8(10)$	2.63e-03/72/18/6.63e-07	3.92e-03/97/22/5.82e-07	9.80e-03/186/53/9.26e-07
$x_1(50)$	5.62e-03/62/16/6.74e-07	6.79e-03/81/20/8.57e-07	3.20e-02/231/67/3.63e-08
$x_2(50)$	3.13e-03/38/10/2.33e-07	4.59e-03/61/15/9.18e-07	8.28e-03/74/15/9.59e-07
$x_3(50)$	4.50e-03/53/13/4.91e-07	5.44e-03/65/16/5.91e-07	1.23e-02/109/18/2.41e-07
$x_4(50)$	6.65e-03/76/19/7.35e-07	9.33e-03/121/26/3.04e-07	3.03e-02/228/62/2.60e-07
$x_5(50)$	4.45e-03/53/13/4.43e-07	4.92e-03/67/16/3.78e-07	1.91e-01/1302/433/9.94e-07
$x_6(50)$	6.72e-03/76/19/7.80e-07	1.01e-02/129/27/8.36e-07	2.92e-02/228/62/2.67e-07
$x_7(50)$	6.18e-03/76/19/7.35e-07	9.32e-03/121/26/7.71e-07	3.01e-02/228/62/2.60e-07
$x_8(50)$	6.08e-03/75/19/7.50e-07	9.55e-03/113/25/7.95e-07	2.85e-02/210/61/8.36e-07
$x_1(100)$	7.94e-03/58/15/1.40e-07	1.08e-02/85/21/4.61e-07	5.21e-02/246/72/5.44e-07
$x_2(100)$	4.68e-03/38/10/2.33e-07	6.88e-03/61/15/9.18e-07	1.27e-02/74/15/9.59e-07
$x_3(100)$	6.39e-03/53/13/4.91e-07	7.88e-03/65/16/5.91e-07	1.64e-02/109/18/2.41e-07
$x_4(100)$	1.29e-02/96/24/4.09e-07	1.32e-02/110/25/8.05e-07	4.69e-02/227/66/2.92e-08
$x_5(100)$	6.45e-03/53/13/4.43e-07	7.87e-03/67/16/9.96e-07	2.85e-02/142/41/8.71e-07
$x_6(100)$	1.16e-02/96/24/4.10e-07	1.38e-02/114/25/7.70e-07	4.82e-02/227/66/2.71e-08
$x_7(100)$	1.25e-02/96/24/4.09e-07	1.36e-02/110/25/8.05e-07	4.86e-02/227/66/2.92e-08
$x_8(100)$	1.11e-02/80/20/5.05e-07	1.39e-02/111/25/6.37e-07	4.38e-02/219/64/2.28e-07
$x_1(150)$	4.46e-02/69/17/6.78e-07	4.84e-02/85/21/4.72e-07	3.08e-01/237/74/9.24e-07
$x_2(150)$	2.33e-02/38/10/2.33e-07	3.46e-02/61/15/9.18e-07	7.11e-02/74/15/9.59e-07
$x_3(150)$	3.11e-02/53/13/4.91e-07	3.39e-02/65/16/5.91e-07	1.05e-01/109/18/2.41e-07
$x_4(150)$	4.28e-02/73/18/8.58e-07	7.35e-02/129/27/8.84e-07	2.97e-01/228/66/7.20e-07
$x_5(150)$	3.25e-02/53/13/4.43e-07	4.07e-02/70/16/4.41e-07	1.82e+00/1290/429/9.97e-07
$x_6(150)$	4.41e-02/73/18/8.52e-07	7.12e-02/121/27/6.42e-07	2.85e-01/228/66/7.34e-07
$x_7(150)$	4.15e-02/73/18/8.58e-07	7.27e-02/125/28/5.98e-07	2.99e-01/228/66/7.20e-07
$x_8(150)$	4.81e-02/85/21/7.37e-07	8.32e-02/145/30/9.17e-07	2.85e-01/223/65/7.34e-07

Table 5: Numerical results for Problem 5.5.

IP(Dim)	IMHZCGP	HZCGP	CGAIS
	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*
$x_1(5)$	9.26e-04/6/1/0.00e+00	2.30e-04/6/1/0.00e+00	2.36e-04/6/1/0.00e+00
$x_2(5)$	1.32e-04/6/1/0.00e+00	2.64e-04/6/1/0.00e+00	1.31e-04/6/1/0.00e+00
$x_3(5)$	9.04e-05/4/1/0.00e+00	9.24e-05/4/1/0.00e+00	7.42e-05/4/1/0.00e+00
$x_4(5)$	1.15e-03/44/8/0.00e+00	1.20e-03/49/8/0.00e+00	1.97e-03/66/10/0.00e+00
$x_5(5)$	2.65e-04/11/2/0.00e+00	2.37e-04/11/2/0.00e+00	6.83e-04/26/3/0.00e+00
$x_6(5)$	1.08e-03/44/8/0.00e+00	1.06e-03/43/7/0.00e+00	1.95e-03/66/10/1.44e-08
$x_7(5)$	1.17e-03/44/8/0.00e+00	1.26e-03/49/8/0.00e+00	1.98e-03/66/10/0.00e+00
$x_8(5)$	9.90e-04/38/7/0.00e+00	1.38e-03/54/9/0.00e+00	1.94e-03/66/10/1.15e-08
$x_1(10)$	3.54e-04/6/1/0.00e+00	3.19e-04/6/1/0.00e+00	2.93e-04/6/1/0.00e+00
$x_2(10)$	2.16e-04/6/1/0.00e+00	1.96e-04/6/1/0.00e+00	2.01e-04/6/1/0.00e+00
$x_3(10)$	1.41e-04/4/1/0.00e+00	1.39e-04/4/1/0.00e+00	1.26e-04/4/1/0.00e+00
$x_4(10)$	1.60e-03/40/7/0.00e+00	1.70e-03/43/7/0.00e+00	2.80e-03/49/10/0.00e+00
$x_5(10)$	5.56e-04/11/2/0.00e+00	4.66e-04/11/2/0.00e+00	1.30e-03/26/3/0.00e+00
$x_6(10)$	1.75e-03/40/7/0.00e+00	1.73e-03/42/7/0.00e+00	2.82e-03/49/10/0.00e+00
$x_7(10)$	1.74e-03/40/7/0.00e+00	1.83e-03/43/7/0.00e+00	2.87e-03/49/10/0.00e+00
$x_8(10)$	1.75e-03/40/7/0.00e+00	1.66e-03/42/7/0.00e+00	2.86e-03/49/10/0.00e+00
$x_1(50)$	1.38e-03/6/1/0.00e+00	1.03e-03/6/1/0.00e+00	1.03e-03/6/1/0.00e+00
$x_2(50)$	6.07e-04/6/1/0.00e+00	5.36e-04/6/1/0.00e+00	5.17e-04/6/1/0.00e+00
$x_3(50)$	3.86e-04/4/1/0.00e+00	3.64e-04/4/1/0.00e+00	3.81e-04/4/1/0.00e+00
$x_4(50)$	4.94e-03/39/7/0.00e+00	7.03e-03/57/9/0.00e+00	1.36e-02/73/21/3.75e-12
$x_5(50)$	1.34e-03/11/2/0.00e+00	1.27e-03/11/2/0.00e+00	3.42e-03/26/3/0.00e+00
$x_6(50)$	4.96e-03/39/7/0.00e+00	7.26e-03/57/9/0.00e+00	4.96e-01/2582/858/1.00e-06
$x_7(50)$	5.28e-03/39/7/0.00e+00	7.10e-03/57/9/0.00e+00	1.24e-02/73/21/3.75e-12
$x_8(50)$	4.73e-03/34/6/0.00e+00	7.99e-03/63/10/0.00e+00	8.92e-01/4700/1564/9.97e-07
$x_1(100)$	1.93e-03/6/1/0.00e+00	2.13e-03/6/1/0.00e+00	2.03e-03/6/1/0.00e+00
$x_2(100)$	9.95e-04/6/1/0.00e+00	1.01e-03/6/1/0.00e+00	9.94e-04/6/1/0.00e+00
$x_3(100)$	6.94e-04/4/1/0.00e+00	6.70e-04/4/1/0.00e+00	6.52e-04/4/1/0.00e+00
$x_4(100)$	7.68e-03/34/6/0.00e+00	1.30e-02/57/9/0.00e+00	1.45e-02/49/13/8.88e-16
$x_5(100)$	2.12e-03/11/2/0.00e+00	2.03e-03/11/2/0.00e+00	5.20e-03/26/3/0.00e+00
$x_6(100)$	7.30e-03/34/6/0.00e+00	1.23e-02/56/9/0.00e+00	1.54e-02/49/13/0.00e+00
$x_7(100)$	7.56e-03/34/6/0.00e+00	1.41e-02/57/9/0.00e+00	1.48e-02/49/13/8.88e-16
$x_8(100)$	8.67e-03/40/7/0.00e+00	1.30e-02/60/10/0.00e+00	1.90e+00/6271/2088/9.98e-07
$x_1(150)$	4.54e-03/6/1/0.00e+00	4.23e-03/6/1/0.00e+00	4.60e-03/6/1/0.00e+00
$x_2(150)$	3.16e-03/6/1/0.00e+00	2.90e-03/6/1/0.00e+00	2.79e-03/6/1/0.00e+00
$x_3(150)$	1.88e-03/4/1/0.00e+00	1.79e-03/4/1/0.00e+00	1.73e-03/4/1/0.00e+00
$x_4(150)$	2.22e-02/34/6/0.00e+00	3.82e-02/57/9/0.00e+00	5.37e-02/43/11/0.00e+00
$x_5(150)$	6.11e-03/11/2/0.00e+00	6.33e-03/11/2/0.00e+00	1.95e-02/26/3/0.00e+00
$x_6(150)$	2.25e-02/34/6/0.00e+00	3.58e-02/56/9/0.00e+00	9.75e-02/82/15/8.17e-07
$x_7(150)$	2.40e-02/34/6/0.00e+00	3.70e-02/57/9/0.00e+00	5.31e-02/43/11/0.00e+00
$x_8(150)$	2.05e-02/34/6/0.00e+00	3.50e-02/56/9/0.00e+00	6.13e-02/56/11/0.00e+00

Table 6: Numerical results for Problem 5.6.

IP(Dim)	IMHZCGP	HZCGP	CGAIS
	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*
$x_1(5)$	7.87e-03/58/14/3.24e-07	1.68e-02/138/27/4.04e-07	2.17e-02/145/30/9.74e-07
$x_2(5)$	9.33e-03/75/18/8.84e-07	1.41e-02/114/23/5.17e-07	2.54e-02/171/35/7.29e-07
$x_3(5)$	9.48e-03/79/20/4.77e-07	1.71e-02/141/27/5.13e-07	2.31e-02/147/34/9.56e-07
$x_4(5)$	9.44e-03/76/18/4.64e-07	1.84e-02/151/29/6.73e-07	1.98e-02/125/32/5.53e-07
$x_5(5)$	8.74e-03/70/17/6.36e-07	1.34e-02/108/23/7.78e-07	2.15e-02/139/33/9.91e-07
$x_6(5)$	9.57e-03/76/18/4.64e-07	1.36e-02/110/23/8.05e-07	1.92e-02/122/31/7.53e-07
$x_7(5)$	8.60e-03/70/17/5.67e-07	1.51e-02/120/25/8.59e-07	2.38e-02/158/33/8.74e-07
$x_8(5)$	7.75e-03/63/15/6.11e-07	1.27e-02/103/22/8.75e-07	2.51e-02/159/40/9.66e-07
$x_1(10)$	1.72e-02/70/17/2.65e-07	3.02e-02/122/25/8.31e-07	4.65e-02/134/35/7.67e-07
$x_2(10)$	1.79e-02/72/17/9.29e-07	3.29e-02/135/27/4.71e-07	4.70e-02/146/37/4.43e-07
$x_3(10)$	1.76e-02/72/17/9.17e-07	3.15e-02/129/27/6.67e-07	4.31e-02/129/33/3.77e-07
$x_4(10)$	1.80e-02/71/17/6.26e-07	2.86e-02/113/23/7.90e-07	3.88e-02/117/29/6.43e-07
$x_5(10)$	1.76e-02/70/17/4.98e-07	4.46e-02/179/32/5.97e-07	5.39e-02/173/42/7.37e-07
$x_6(10)$	1.75e-02/71/17/6.26e-07	3.81e-02/155/28/7.34e-07	3.90e-02/123/31/7.60e-07
$x_7(10)$	1.66e-02/67/16/7.65e-07	2.99e-02/123/25/3.68e-07	4.05e-02/128/33/8.67e-07
$x_8(10)$	1.64e-02/67/16/8.72e-07	2.92e-02/119/24/9.37e-07	4.00e-02/129/30/8.50e-07
$x_1(50)$	8.82e-02/67/16/2.31e-07	1.84e-01/144/27/6.28e-07	1.98e-01/122/32/9.07e-07
$x_2(50)$	1.01e-01/80/19/6.46e-07	2.21e-01/176/34/6.01e-07	2.38e-01/148/37/8.38e-07
$x_3(50)$	1.00e-01/80/19/5.23e-07	1.81e-01/144/28/2.79e-07	1.90e-01/120/31/9.30e-07
$x_4(50)$	1.07e-01/84/20/9.79e-07	1.99e-01/156/30/6.34e-07	2.67e-01/167/39/9.29e-07
$x_5(50)$	1.01e-01/80/19/9.32e-07	1.47e-01/117/24/9.33e-07	4.00e-01/241/68/5.24e-07
$x_6(50)$	1.11e-01/88/21/4.02e-07	1.82e-01/143/27/5.71e-07	2.18e-01/134/34/9.30e-07
$x_7(50)$	1.06e-01/83/20/7.98e-07	1.90e-01/150/30/8.50e-07	2.29e-01/146/31/6.25e-07
$x_8(50)$	1.02e-01/80/19/2.56e-07	1.86e-01/146/29/9.77e-07	2.07e-01/126/33/3.60e-07
$x_1(100)$	1.72e-01/67/16/2.17e-07	3.46e-01/133/27/9.89e-07	3.70e-01/111/27/9.97e-07
$x_2(100)$	2.48e-01/97/23/8.30e-07	4.15e-01/158/31/3.06e-07	9.22e-01/268/79/7.77e-07
$x_3(100)$	2.64e-01/101/24/7.76e-07	4.89e-01/188/34/5.57e-07	9.21e-01/268/79/7.81e-07
$x_4(100)$	2.32e-01/89/21/1.52e-07	3.94e-01/150/28/2.86e-07	8.66e-01/251/74/9.61e-07
$x_5(100)$	1.83e-01/71/17/1.66e-07	3.94e-01/153/28/6.33e-07	4.05e-01/126/29/5.58e-07
$x_6(100)$	2.29e-01/89/21/1.52e-07	5.11e-01/198/35/4.16e-07	8.40e-01/251/74/9.61e-07
$x_7(100)$	1.81e-01/71/17/8.25e-07	3.22e-01/124/25/6.01e-07	4.53e-01/142/35/9.22e-07
$x_8(100)$	2.16e-01/85/20/5.87e-07	4.31e-01/168/30/4.17e-07	8.83e-01/261/77/4.19e-07
$x_1(150)$	2.77e-01/67/16/3.10e-07	6.04e-01/147/28/7.20e-07	6.47e-01/118/29/9.89e-07
$x_2(150)$	2.95e-01/73/17/5.53e-07	7.65e-01/188/32/5.22e-07	6.68e-01/124/30/9.34e-07
$x_3(150)$	3.26e-01/80/19/6.49e-07	5.98e-01/147/27/1.35e-07	7.82e-01/148/32/3.07e-07
$x_4(150)$	4.67e-01/113/27/3.76e-07	7.40e-01/180/33/9.42e-07	1.56e+00/268/79/7.80e-07
$x_5(150)$	3.48e-01/85/20/3.19e-07	6.92e-01/170/33/6.88e-07	7.16e-01/132/32/8.21e-07
$x_6(150)$	4.66e-01/113/27/3.77e-07	7.59e-01/185/34/9.00e-07	1.56e+00/268/79/7.80e-07
$x_7(150)$	2.76e-01/66/16/1.10e-07	6.30e-01/154/29/7.43e-07	1.63e+00/279/84/3.93e-07
$x_8(150)$	3.24e-01/79/19/3.52e-07	4.51e-01/110/22/6.07e-07	1.45e+00/249/73/6.44e-07

Table 7: Numerical results for Problem 5.7.

IP(Dim)	IMHZCGP	HZCGP	CGAIS
	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*
$x_1(5)$	2.46e-03/124/26/5.99e-07	8.91e-04/80/13/2.57e-07	2.24e-03/130/26/9.98e-07
$x_2(5)$	9.20e-03/683/195/9.44e-07	1.61e-02/1468/277/8.80e-07	1.28e-02/678/189/9.67e-07
$x_3(5)$	8.25e-03/669/192/9.21e-07	1.09e-02/953/199/7.50e-07	5.65e-02/3138/795/9.98e-07
$x_4(5)$	7.09e-03/597/164/9.79e-07	9.54e-03/908/182/8.25e-07	3.92e-02/2487/396/9.37e-07
$x_5(5)$	8.09e-03/658/187/9.96e-07	1.26e-02/1235/210/8.48e-07	1.47e-02/694/186/8.50e-07
$x_6(5)$	1.45e-02/634/174/9.85e-07	9.54e-03/942/192/9.38e-07	1.25e-01/7582/1663/1.00e-06
$x_7(5)$	7.09e-03/597/164/9.79e-07	1.28e-02/1293/242/9.41e-07	5.16e-02/3309/614/9.94e-07
$x_8(5)$	7.53e-03/639/174/9.61e-07	7.16e-03/671/120/9.73e-07	4.20e-02/2752/477/7.19e-07
$x_1(10)$	3.34e-03/112/22/9.45e-07	3.10e-03/99/17/2.53e-07	4.71e-03/122/28/9.60e-08
$x_2(10)$	1.64e-02/493/141/9.10e-07	2.18e-02/948/166/9.78e-07	2.60e-02/640/167/9.21e-07
$x_3(10)$	1.26e-02/487/140/9.23e-07	2.87e-02/1101/197/9.14e-07	2.98e-02/528/144/9.44e-07
$x_4(10)$	1.78e-02/557/147/8.98e-07	2.09e-02/888/181/9.81e-07	2.60e-02/651/153/7.68e-07
$x_5(10)$	1.32e-02/476/135/9.53e-07	1.37e-02/577/123/9.57e-07	2.64e-02/673/164/8.72e-07
$x_6(10)$	2.14e-02/583/157/8.72e-07	1.71e-02/743/124/7.76e-07	1.96e-02/507/121/5.86e-07
$x_7(10)$	1.42e-02/557/147/8.98e-07	2.95e-02/1289/234/8.84e-07	2.37e-02/593/149/9.91e-07
$x_8(10)$	2.08e-02/578/160/8.82e-07	1.59e-02/652/132/9.90e-07	1.77e-01/4407/1106/9.98e-07
$x_1(50)$	1.57e-02/180/38/8.50e-07	8.35e-03/96/15/4.71e-07	1.48e-02/117/29/7.27e-07
$x_2(50)$	2.18e-02/254/72/9.36e-07	4.15e-02/539/92/9.34e-07	5.51e-02/429/109/6.21e-07
$x_3(50)$	1.83e-02/213/62/8.68e-07	2.63e-02/340/69/8.36e-07	5.90e-02/424/101/6.19e-07
$x_4(50)$	3.19e-02/365/94/9.11e-07	3.69e-02/394/75/4.78e-07	2.26e+00/22876/2506/9.38e-07
$x_5(50)$	1.82e-02/212/60/9.09e-07	3.37e-02/392/80/8.87e-07	3.22e-02/258/55/5.46e-07
$x_6(50)$	3.45e-02/339/86/9.49e-07	4.66e-02/571/98/6.86e-07	2.32e+00/23487/2560/6.28e-07
$x_7(50)$	3.38e-02/365/94/9.11e-07	3.57e-02/427/83/8.71e-07	2.08e+00/21294/2317/7.69e-07
$x_8(50)$	3.02e-02/335/80/9.69e-07	3.83e-02/457/71/7.05e-07	2.20e+00/22612/2462/7.27e-07
$x_1(100)$	3.88e-02/311/70/9.16e-07	8.49e-03/78/11/3.23e-07	2.16e-02/122/27/5.83e-08
$x_2(100)$	3.66e-02/292/83/9.73e-07	4.45e-02/400/74/3.27e-07	5.97e-02/313/74/9.80e-07
$x_3(100)$	3.82e-02/305/87/9.97e-07	2.88e-02/245/51/8.93e-07	6.84e-02/355/90/8.85e-07
$x_4(100)$	4.17e-02/355/93/9.27e-07	2.91e-02/267/50/7.95e-07	1.76e-01/1110/195/9.05e-07
$x_5(100)$	1.91e-02/151/42/9.65e-07	2.81e-02/223/45/2.43e-07	7.18e-02/393/90/2.72e-07
$x_6(100)$	5.23e-02/403/105/9.69e-07	3.74e-02/348/66/8.56e-07	1.71e-01/1097/197/8.44e-07
$x_7(100)$	4.50e-02/355/93/9.27e-07	3.84e-02/340/65/7.46e-07	1.83e-01/1129/200/8.65e-07
$x_8(100)$	2.40e-02/196/48/6.81e-07	2.64e-02/248/45/2.36e-07	1.67e-01/1035/186/5.08e-07
$x_1(150)$	2.17e-01/261/57/8.78e-07	5.33e-02/73/9/8.35e-07	1.78e-01/132/29/1.12e-08
$x_2(150)$	1.44e-01/175/48/9.19e-07	3.15e-01/402/67/9.40e-07	5.03e-01/358/80/6.33e-07
$x_3(150)$	1.40e-01/168/47/8.67e-07	2.09e-01/262/53/4.90e-07	4.48e-01/298/76/2.54e-07
$x_4(150)$	1.46e-01/172/40/8.72e-07	1.94e-01/243/43/6.76e-07	NaN/NaN/NaN/NaN
$x_5(150)$	1.88e-01/232/57/9.36e-07	1.13e-01/139/29/6.42e-07	4.88e-01/345/77/8.43e-07
$x_6(150)$	2.20e-01/269/67/8.74e-07	1.85e-01/245/43/9.03e-07	NaN/NaN/NaN/NaN
$x_7(150)$	1.41e-01/172/40/8.72e-07	1.81e-01/221/39/8.32e-07	NaN/NaN/NaN/NaN
$x_8(150)$	1.65e-01/198/47/8.30e-07	1.78e-01/229/39/8.90e-07	NaN/NaN/NaN/NaN

Table 8: Numerical results for Problem 5.8.

IP(Dim)	IMHZCGP	HZCGP	CGAIS
	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*
$x_1(5)$	4.69e-03/68/16/5.33e-07	2.92e-03/66/14/2.26e-07	8.92e-03/106/20/9.22e-07
$x_2(5)$	6.79e-03/99/22/6.00e-07	8.01e-03/115/21/9.77e-07	1.38e-02/135/34/9.03e-07
$x_3(5)$	9.32e-03/103/22/3.90e-07	1.06e-02/151/28/9.82e-07	1.62e-02/169/43/9.21e-07
$x_4(5)$	6.68e-03/88/19/5.45e-07	7.32e-03/104/20/3.15e-07	1.41e-02/161/35/8.15e-07
$x_5(5)$	7.27e-03/102/23/6.31e-07	7.92e-03/115/21/3.51e-07	1.31e-02/145/36/8.68e-07
$x_6(5)$	6.16e-03/88/19/5.13e-07	7.54e-03/107/21/8.25e-07	1.21e-02/133/33/9.49e-07
$x_7(5)$	6.26e-03/89/19/6.19e-07	7.79e-03/114/20/8.52e-07	1.21e-02/134/34/9.27e-07
$x_8(5)$	8.27e-03/118/25/4.17e-07	1.04e-02/147/25/7.20e-07	1.50e-02/162/42/7.09e-07
$x_1(10)$	9.36e-03/64/15/7.33e-07	9.21e-03/66/14/4.10e-07	2.11e-02/111/22/1.12e-07
$x_2(10)$	1.35e-02/98/22/9.66e-07	1.58e-02/115/21/9.20e-07	2.80e-02/145/37/9.75e-07
$x_3(10)$	1.55e-02/97/21/9.57e-07	1.82e-02/132/24/4.83e-07	2.46e-02/135/34/9.99e-07
$x_4(10)$	1.34e-02/94/21/8.55e-07	1.36e-02/94/17/7.48e-07	2.86e-02/148/35/8.66e-07
$x_5(10)$	1.57e-02/99/21/4.07e-07	1.91e-02/134/25/7.94e-07	2.99e-02/159/39/8.24e-07
$x_6(10)$	1.34e-02/94/21/8.87e-07	1.49e-02/105/19/7.90e-07	2.49e-02/138/32/8.97e-07
$x_7(10)$	1.24e-02/88/19/6.78e-07	1.51e-02/110/21/8.71e-07	2.84e-02/159/38/9.88e-07
$x_8(10)$	1.75e-02/124/26/8.27e-07	2.15e-02/150/26/7.24e-07	3.21e-02/172/41/4.88e-07
$x_1(50)$	4.91e-02/69/16/6.31e-07	4.94e-02/71/15/9.37e-08	9.41e-02/117/20/6.82e-07
$x_2(50)$	7.02e-02/103/22/3.94e-07	9.92e-02/138/25/6.83e-07	1.57e-01/171/41/6.96e-07
$x_3(50)$	8.13e-02/108/23/5.70e-07	8.77e-02/128/23/7.58e-07	1.59e-01/180/44/7.26e-07
$x_4(50)$	6.62e-02/95/21/4.39e-07	5.28e-02/75/15/4.62e-07	1.33e-01/153/36/8.75e-07
$x_5(50)$	7.11e-02/103/23/9.71e-07	1.00e-01/146/27/5.85e-07	1.61e-01/180/44/9.18e-07
$x_6(50)$	6.68e-02/95/21/4.45e-07	5.13e-02/75/15/4.63e-07	1.34e-01/153/36/8.28e-07
$x_7(50)$	6.39e-02/90/20/7.53e-07	5.11e-02/75/15/4.56e-07	1.34e-01/153/37/9.66e-07
$x_8(50)$	1.00e-01/138/29/9.06e-07	1.44e-01/193/32/5.72e-07	1.37e-01/153/38/9.60e-07
$x_1(100)$	1.00e-01/69/16/6.63e-07	9.96e-02/71/15/5.51e-07	1.82e-01/109/19/2.61e-07
$x_2(100)$	1.34e-01/97/20/9.63e-07	1.73e-01/124/23/5.35e-07	2.86e-01/161/39/8.97e-07
$x_3(100)$	1.86e-01/133/30/6.86e-07	2.29e-01/165/30/6.21e-07	3.15e-01/177/43/6.03e-07
$x_4(100)$	1.32e-01/94/21/9.50e-07	1.36e-01/97/20/9.48e-07	2.32e-01/129/34/9.75e-07
$x_5(100)$	1.47e-01/105/23/8.88e-07	1.60e-01/114/21/9.03e-07	3.12e-01/173/43/9.50e-07
$x_6(100)$	1.33e-01/94/21/9.50e-07	1.42e-01/101/19/7.53e-07	2.52e-01/140/36/5.70e-07
$x_7(100)$	1.16e-01/83/19/9.14e-07	1.13e-01/80/16/8.67e-07	2.50e-01/139/35/3.57e-07
$x_8(100)$	2.11e-01/149/32/7.51e-07	2.15e-01/147/25/9.99e-07	2.97e-01/166/41/6.76e-07
$x_1(150)$	1.76e-01/73/17/3.62e-07	1.79e-01/75/15/3.59e-07	6.12e-01/174/48/1.89e-08
$x_2(150)$	2.27e-01/96/20/8.70e-07	3.22e-01/136/24/7.80e-07	6.13e-01/181/44/9.54e-07
$x_3(150)$	2.27e-01/96/20/7.26e-07	3.31e-01/140/26/3.93e-07	6.29e-01/188/43/9.75e-07
$x_4(150)$	2.00e-01/84/18/4.01e-07	1.91e-01/80/16/9.20e-07	4.82e-01/141/37/9.94e-07
$x_5(150)$	2.94e-01/122/28/8.12e-07	3.19e-01/134/23/2.75e-07	6.39e-01/191/45/9.32e-07
$x_6(150)$	2.03e-01/84/18/4.01e-07	1.89e-01/80/16/9.19e-07	5.13e-01/150/39/9.82e-07
$x_7(150)$	2.31e-01/92/20/3.29e-07	1.80e-01/75/15/7.40e-07	5.10e-01/154/35/5.10e-07
$x_8(150)$	3.40e-01/142/30/2.29e-07	3.74e-01/150/26/7.33e-07	5.45e-01/161/40/9.69e-07

Table 9: Numerical results for Problem 5.9.

IP(Dim)	IMHZCGP	HZCGP	CGAIS
	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*	RunT/NumF/NumI/E_*
$x_1(5)$	7.52e-04/4/1/0.00e+00	3.74e-04/4/1/0.00e+00	1.18e-04/4/1/0.00e+00
$x_2(5)$	1.07e-04/4/1/0.00e+00	2.67e-04/4/1/0.00e+00	1.08e-04/4/1/0.00e+00
$x_3(5)$	9.33e-05/4/1/0.00e+00	8.57e-05/4/1/0.00e+00	9.57e-05/4/1/0.00e+00
$x_4(5)$	3.53e-04/10/3/0.00e+00	3.41e-04/10/3/0.00e+00	1.08e-03/29/5/0.00e+00
$x_5(5)$	3.94e-04/10/3/0.00e+00	3.60e-04/10/3/0.00e+00	2.89e-04/7/2/0.00e+00
$x_6(5)$	3.52e-04/10/3/0.00e+00	3.24e-04/10/3/0.00e+00	1.04e-03/29/5/0.00e+00
$x_7(5)$	3.52e-04/10/3/0.00e+00	3.43e-04/10/3/0.00e+00	1.02e-03/29/5/0.00e+00
$x_8(5)$	3.54e-04/10/3/0.00e+00	3.41e-04/10/3/0.00e+00	1.06e-03/29/5/0.00e+00
$x_1(10)$	2.15e-04/4/1/0.00e+00	1.72e-04/4/1/0.00e+00	1.69e-04/4/1/0.00e+00
$x_2(10)$	1.63e-04/4/1/0.00e+00	1.49e-04/4/1/0.00e+00	1.58e-04/4/1/0.00e+00
$x_3(10)$	1.53e-04/4/1/0.00e+00	1.51e-04/4/1/0.00e+00	1.61e-04/4/1/0.00e+00
$x_4(10)$	6.27e-04/10/3/0.00e+00	5.56e-04/10/3/0.00e+00	2.05e-03/33/7/8.20e-07
$x_5(10)$	6.93e-04/10/3/0.00e+00	7.60e-04/10/3/0.00e+00	6.59e-04/7/2/0.00e+00
$x_6(10)$	7.60e-04/10/3/0.00e+00	7.27e-04/10/3/0.00e+00	2.35e-03/29/5/0.00e+00
$x_7(10)$	8.61e-04/10/3/0.00e+00	7.21e-04/10/3/0.00e+00	2.64e-03/33/7/8.20e-07
$x_8(10)$	7.33e-04/10/3/0.00e+00	7.41e-04/10/3/0.00e+00	2.45e-03/29/5/0.00e+00
$x_1(50)$	7.93e-04/4/1/0.00e+00	8.56e-04/4/1/0.00e+00	6.24e-04/4/1/0.00e+00
$x_2(50)$	5.72e-04/4/1/0.00e+00	5.23e-04/4/1/0.00e+00	4.84e-04/4/1/0.00e+00
$x_3(50)$	4.58e-04/4/1/0.00e+00	4.48e-04/4/1/0.00e+00	4.93e-04/4/1/0.00e+00
$x_4(50)$	1.92e-03/10/3/0.00e+00	1.83e-03/10/3/0.00e+00	9.98e-03/51/8/3.57e-25
$x_5(50)$	2.09e-03/10/3/0.00e+00	2.07e-03/10/3/0.00e+00	1.67e-03/7/2/0.00e+00
$x_6(50)$	2.03e-03/10/3/0.00e+00	2.72e-03/10/3/0.00e+00	5.89e-03/29/5/9.75e-07
$x_7(50)$	2.08e-03/10/3/0.00e+00	2.04e-03/10/3/0.00e+00	9.58e-03/51/8/3.57e-25
$x_8(50)$	1.88e-03/10/3/0.00e+00	1.81e-03/10/3/0.00e+00	8.94e-03/51/8/0.00e+00
$x_1(100)$	9.28e-04/4/1/0.00e+00	9.83e-04/4/1/0.00e+00	9.51e-04/4/1/0.00e+00
$x_2(100)$	7.11e-04/4/1/0.00e+00	6.94e-04/4/1/0.00e+00	7.09e-04/4/1/0.00e+00
$x_3(100)$	8.05e-04/4/1/0.00e+00	7.03e-04/4/1/0.00e+00	7.12e-04/4/1/0.00e+00
$x_4(100)$	3.11e-03/10/3/0.00e+00	3.15e-03/10/3/0.00e+00	1.35e-02/49/8/8.31e-07
$x_5(100)$	3.43e-03/10/3/0.00e+00	3.35e-03/10/3/0.00e+00	2.80e-03/7/2/0.00e+00
$x_6(100)$	3.33e-03/10/3/0.00e+00	3.29e-03/10/3/0.00e+00	1.67e-02/67/9/2.43e-26
$x_7(100)$	3.17e-03/10/3/0.00e+00	3.09e-03/10/3/0.00e+00	1.30e-02/49/8/8.31e-07
$x_8(100)$	3.08e-03/10/3/0.00e+00	3.18e-03/10/3/0.00e+00	1.43e-02/51/8/7.89e-27
$x_1(150)$	2.85e-03/4/1/0.00e+00	2.97e-03/4/1/0.00e+00	3.15e-03/4/1/0.00e+00
$x_2(150)$	3.44e-03/4/1/0.00e+00	3.43e-03/4/1/0.00e+00	2.86e-03/4/1/0.00e+00
$x_3(150)$	2.85e-03/4/1/0.00e+00	2.79e-03/4/1/0.00e+00	2.89e-03/4/1/0.00e+00
$x_4(150)$	9.56e-03/10/3/0.00e+00	9.74e-03/10/3/0.00e+00	8.24e-02/67/9/9.98e-24
$x_5(150)$	9.69e-03/10/3/0.00e+00	9.62e-03/10/3/0.00e+00	8.84e-03/7/2/0.00e+00
$x_6(150)$	1.00e-02/10/3/0.00e+00	1.02e-02/10/3/0.00e+00	7.71e-02/67/9/9.76e-27
$x_7(150)$	9.44e-03/10/3/0.00e+00	8.49e-03/10/3/0.00e+00	7.77e-02/67/9/9.96e-24
$x_8(150)$	9.85e-03/10/3/0.00e+00	9.84e-03/10/3/0.00e+00	6.80e-02/51/8/3.77e-26

5.2. Experiments on sparse signal reconstruction

Sparse signal reconstruction involves solving the following regularized optimization problem:

$$\min_x \left\{ \frac{1}{2}\|Hx - b\|^2 + \lambda\|x\|_1 \right\}, \tag{5.1}$$

where $H \in R^{m\times n}$ is ill-conditioned matrix, $b \in R^n$ is the measurement vector, and $\lambda > 0$ is the regularization parameter. To facilitate computation, we introduce nonnegative vectors $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$, and rewrite $x = x^+ - x^-$. This leads to $\|x\|_1 = e_n^T x^+ + e_n^T x^-$, where $e_n = (1, 1, \dots, 1)^T \in R^n$. Substituting into the original objective, the problem becomes:

$$\min_{x^+, x^- \in R^+} \left\{ \frac{1}{2}\|H(x^+ - x^-) - b\|^2 + \lambda e_n^T x^+ + \lambda e_n^T x^- \right\},$$

which can be further reformulated as a standard quadratic optimization problem:

$$\min \left\{ \frac{1}{2} p^T D p + c^T p \right\},$$

where

$$p = \begin{pmatrix} x^+ \\ x^- \end{pmatrix}, \quad c = \begin{pmatrix} \lambda e_n - H^T b \\ \lambda e_n + H^T b \end{pmatrix}, \quad D = \begin{pmatrix} H^T H & -H^T H \\ -H^T H & H^T H \end{pmatrix}.$$

Here, D is a positive semi-definite. Therefore, problem (5.1) is equivalent to solving the following nonlinear equations $E(p) = 0$, where

$$E(p) = (E_1(p), E_2(p), E_3(p), \dots, E_{2n}(p))^T, \quad E_i(p) = \min\{p_i, \ (Dp + c)_i\}.$$

Note that the function $E : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is continuous and monotone from the work [16].

To validate the effectiveness of the Algorithm 3.1 in sparse signal reconstruction, we apply it to reconstruct a sparse signal of length $n = 6144$ from $m = 1536$ noisy observations. The original signal contains 192 randomly chosen nonzero entries. We evaluate the reconstruction performance using the mean squared error (MSE) defined as:

$$\text{MSE} = \frac{1}{n}\|\hat{p} - p\|^2,$$

where $\hat{p}$ is the ground-truth signal and p is the reconstructed signal. The Algorithm 3.1 is compared against the HZCGP and CGAIS algorithms. All algorithms start from the same initial point $p_0 = H^T b$, and the stopping condition is based on the relative change of the objective function:

$$\frac{|f(p_k) - f(p_{k-1})|}{|f(p_{k-1})|} < 10^{-6},$$

where $f(p_k) = \frac{1}{2}\|Hp_k - b\|^2 + \lambda\|p_k\|_1$.

Table 10 summarizes the reconstruction performance under various noise realizations. The Algorithm 3.1 demonstrates clear advantages over the HZCGP and CGAIS algorithm in terms of the number of iterations and running time in seconds. Figure 4 visually compares the reconstructed signals from all three algorithms. Additionally, Figure 4 presents a detailed performance analysis, including the mean square error (MSE), number of iterations (NumI), and running tim in seconds (RunT). As shown in these figures, Algorithm 3.1 achieves faster convergence and lower computational cost while maintaining reconstruction accuracy, further demonstrating its competitiveness in sparse signal reconstruction.

Table 10: Sparse signal reconstruction results for these three algorithms.

No.	IMHZCGP NumI/RunT/MSE	HZCGP NumI/RunT/MSE	CGAIS NumI/RunT/MSE
1	177/7.73/1.159e-05	194/12.53/1.159e-05	309/19.84/1.159e-05
2	182/7.20/1.453e-05	197/13.41/1.453e-05	308/19.78/1.453e-05
3	187/7.30/9.425e-06	196/13.42/9.425e-06	272/18.62/9.425e-06
4	169/7.05/1.428e-05	191/14.09/1.428e-05	269/17.98/1.428e-05
5	183/7.22/8.826e-06	203/13.28/8.825e-06	306/20.11/8.826e-06
6	172/8.17/1.168e-05	202/14.86/1.168e-05	283/18.80/1.168e-05
7	182/7.00/1.359e-05	164/10.34/1.359e-05	248/16.06/1.359e-05
8	196/7.88/8.247e-06	201/14.12/8.247e-06	326/22.05/8.247e-06
9	170/6.78/8.860e-06	176/12.67/8.860e-06	290/19.70/8.860e-06
10	194/7.86/1.154e-05	186/13.25/1.154e-05	306/19.17/1.154e-05
Average	181.20/7.42/1.126e-05	191.00/13.20/1.126e-05	291.70/19.21/1.126e-05

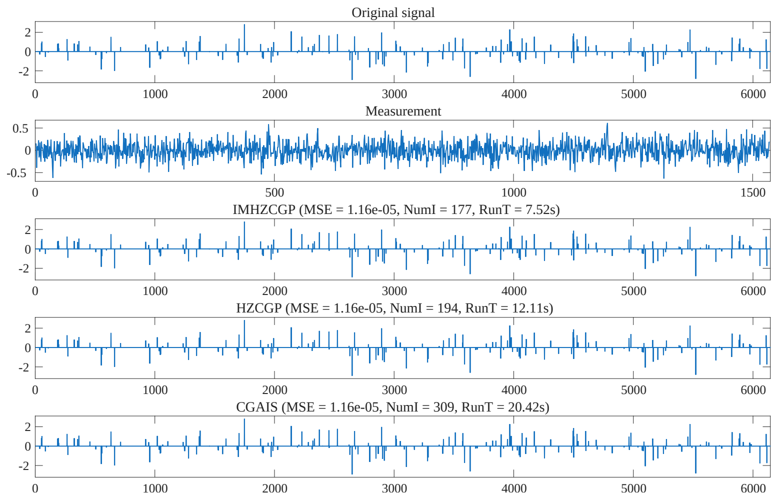

Figure 4: From top to bottom: original signal, measurement, and recovered signals by these three algorithms.

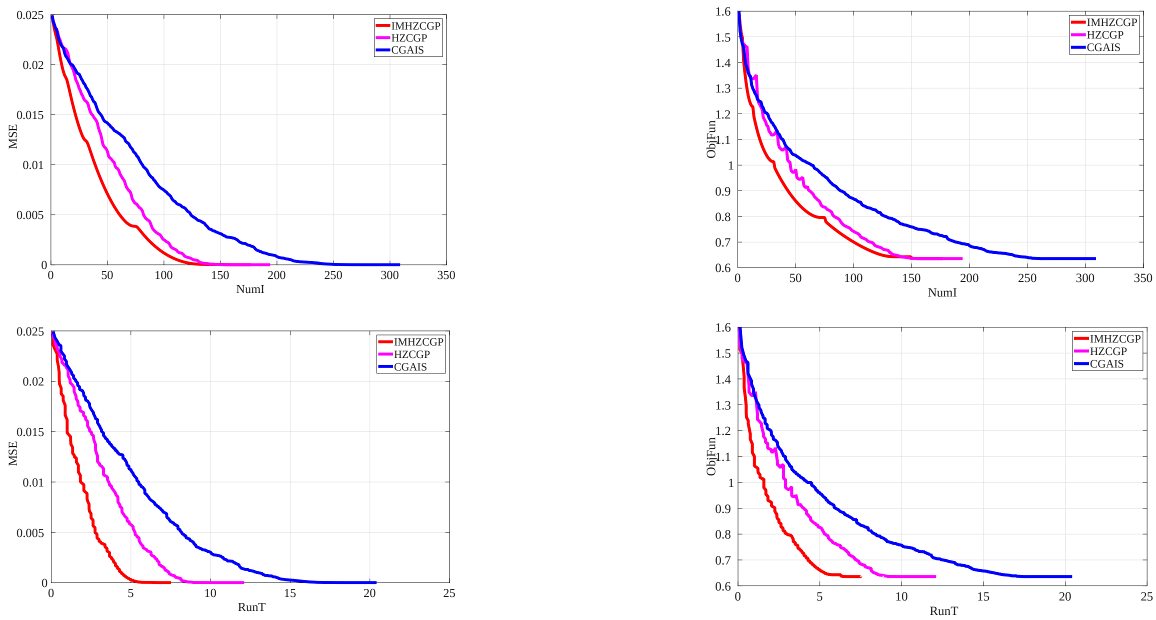

Figure 5: Comparison results of these three algorithms

6. Conclusion

In this paper, we propose a relaxed-inertial CGP algorithm with a modified Hager-Zhang-type conjugate parameter for solving constrained nonlinear equations. The proposed algorithm integrates the relaxed-inertial mechanism and modified Hager-Zhang-type conjugate parameter to construct a novel search direction that inherently satisfy the sufficient descent condition and trust region feature, without relying on any line search approaches. These characteristics make the proposed algorithm particularly attractive for large-scale equations, especially due to its derivative-free and low memory requirements. We establish the global convergence of the proposed algorithm under mild hypotheses, notably without the commonly required Lipschitz continuity condition. Comprehensive numerical experiments demonstrate the efficiency and competitiveness of the proposed algorithm on large-scale nonlinear equations and sparse signal reconstructions application when compared to two existing algorithms.

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