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Numerical analysis for heat and moisture transfer in fibrous insulation

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Abstract

The study of heat and moisture transfer in fibrous insulation materials is crucial for optimizing thermal performance and ensuring durability in various applications. This paper presents a comprehensive analysis of the linearized Crank-Nicolson finite element method (FEM) as applied to the coupled heat and moisture transfer equations in fibrous insulation. The Finite Element Method (FEM) is particularly well-suited for solving complex, nonlinear problems like heat and moisture transfer in fibrous insulation materials. FEM allows for the accurate modeling of complex geometries and nonlinear material properties, which is crucial for simulating the behavior of fibrous insulation materials. Compared to other finite methods, FEM provides a flexible framework for handling coupled heat and moisture transfer equations, allowing for the accurate prediction of temperature and moisture distributions within the material. The semi-discrete formulation is presented using FEM and optimal error estimates for L^2 and energy norms are established. For the fully discrete scheme, the stability and optimal error estimates for the numerical solution are also derived for the same norms, highlighting the method's convergence properties and accuracy. The analysis considers various factors influencing the transfer processes, including fiber orientation, density, and environmental conditions.

Keywords: Heat transfer, moisture transfer, finite element method, Crank-Nicolson method, error estimates, fibrous insulation.

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1. Introduction

Mathematical and numerical modeling of coupled heat and moisture transfer in porous materials, such as fibrous insulations, is an increasingly crucial research area in engineering and materials science. This process is particularly relevant in various sectors, including the food industry, building materials, and more recently in the textile industry [7]. A typical example of this type of modeling is that of clothing assemblies composed of a thick porous fiber mat of 10 mm thickness, sandwiched between a thin inner fabric layer (0.1 mm) in contact with human skin and another fabric layer (0.1 mm) in contact with a colder environment. This configuration is schematically represented in [7]. The fibrous mat, being highly porous, presents a large temperature difference between the skin and the external environment, making radiative heat transfer within the fibrous mat particularly significant.

Over the years, various mathematical models have been developed to capture the complex processes of moisture absorption and heat transfer in these systems. Fan et al. [8] introduced one of the first

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dynamic models integrating both moisture absorption and heat transfer in garment assemblies. They studied the effects of condensation and evaporation on the thermal behavior of textile materials. Later, in [16], an improvement of this model was proposed, integrating mobile condensates and phase changes, with extensive experimental validation.

Numerical methods and simulations for heat and moisture transport in porous textile materials have been studied by many authors [4, 6, 12, 17] and references theirin. The Crank-Nicolson finite element method has been the framework used by many researchers [2, 5, 9, 13]. Rashidinia and Barzegar in [14] presented the Galerkin spectral method to solve the one-dimensional viscoelastic wave equations. Hashim et al. in [10] investigated the numerical analysis of the damped wave equation, focusing on the properties of numerical solutions achieved via semi-discrete approximation and fully-discrete approximation FEMs. Cheng and Wang [4] proposed an approach of the Galerkin finite element method for a one-dimensional model, including the vapor diffusion and heat conduction processes. They proved the optimal-order error estimates for the energy norm using the Euler method in time. Sun et al. [15] analyzed the errors of a semi-implicit finite difference method for the temperature and vapor system in one-dimensional space. Hou et al in [11] proposed a splitting Galerkin method with semi-implicit Euler scheme in time direction for heat and sweat transport in textile materials. In that paper, a linearized scheme is applied for the approximation to Darcy's velocity simultaneously in the mass and energy equations, which leads to physical conservation of the method in the flow convection.

In this paper, we propose a linearized numerical scheme based on finite element method in space and Crank-Niclson scheme in time direction of a process model of moisture heat transfer in porous clothing assemblies, which takes into account heat transfer and water vapor sorption in the fibers of the medium. The difficulty lies on the strong nonlinearity. We prove that the full scheme is unconditionally stable. The optimal-order error estimates is obtained in L^2 and energy norms by first analyzing the spatial discrete scheme and without any restriction on spatial and time steps.

The remainder of this paper is organized as follows. In Section 2 we present a mathematical model to describe fibrous insulation, review some results, and discuss some regularities that are used to prove the optimal error estimates. Spatial errors are discussed in Section 3. The fully discrete scheme is presented in Section 4, where we analyze the stability and the global error.

2. Mathematical model and preliminaries

The following nonlinear system of energy balance equation and mass conservation equation of water vapor is considered:

$$G(x)\partial_t T - \partial_x(a(x)\partial_x T) = \Theta(T), \quad x \in (0, L), \quad t \in (0, t_f], \tag{2.1}$$

$$[\Phi(x) + E(T, C)]\partial_t C - \partial_x \left(d(x)\Phi(x)\partial_x C\right) = H(T, C)\partial_t T, \quad x \in (0, L), \quad t \in (0, t_f], \tag{2.2}$$

$$a(x)\partial_x T(0, t) = \mu_1 \left(T(0, t) - T_0\right), \quad a(x)\partial_x T(L, t) = \mu_2 \left(T_L - T(L, t)\right), \quad t \in (0, t_f], \tag{2.3}$$

$$d(x)\Phi(x)\partial_x C(0, t) = \gamma_1 \left(C(0, t) - C_0\right), \quad d(x)\Phi(x)\partial_x C(L, t) = \gamma_2 \left(C_L - C(L, t)\right), \tag{2.4}$$

$$T(x, 0) = T^0(x), \quad C(x, 0) = C^0(x), \quad x \in (0, L), \tag{2.5}$$

where $T = T(x,t)$ is the temperature, $C = C(x,t)$ is the concentration of water vapor, $a(x) \geqslant a_*$ is the effective thermal conductivity, and $G(x) \geqslant G_*$ is the effective volumetric heat capacity with a_* and G_* positive constants. $\Phi(x) \geqslant \Phi_*$ and $d(x) \geqslant d_*$ represent the porosity of the medium and the molecular diffusion coefficient of water vapor, respectively. The functions $E(T, C)$ and $-H(T, C)$ represent the rate of water vapor accumulation contributed by the change in C and T, respectively. C_0, C_L, T_0, and T_L are constants. The function Θ is given by

$$\begin{aligned}\Theta(T) &= \alpha_1 T^4(x,t) + \alpha_2 e^{\beta x}\int_0^x e^{-\beta s}T^4(s,t)ds + \alpha_3 e^{-\beta x}\int_0^x e^{\beta s}T^4(s,t)ds \\ &\quad + \alpha_4\left(e^{\beta x} + (1-\zeta_1)e^{-\beta x}\right)\int_0^x e^{\beta s}T^4(s,t)ds + \alpha_5\left(e^{\beta x} + (1-\zeta_1)e^{-\beta x}\right)\int_0^x e^{-\beta s}T^4(s,t)ds\end{aligned}$$

$$+\alpha_6 e^{\beta x}T^4(0,t)+\alpha_7 e^{\beta x}T^4(L,t)+\alpha_8 e^{-\beta x}T^4(0,t)+\alpha_9 e^{-\beta x}T^4(L,t),$$

with the constants α_i defined by:

$$\alpha_1 = 2\beta\sigma, \qquad \alpha_2 = \beta^2\sigma, \qquad \alpha_3 = -\beta^2\sigma,$$
$$\alpha_4 = \frac{\beta^2\sigma(1-\zeta_2)}{(1-\zeta_2)(1-\zeta_1)-e^{2\beta L}}, \qquad \alpha_5 = \frac{\beta^2\sigma}{(1-\zeta_2)(1-\zeta_1)e^{-2\beta L}-1}, \qquad \alpha_6 = \frac{\beta\sigma\zeta_1}{(1-\zeta_1)-(1-\zeta_2)e^{2\beta L}},$$
$$\alpha_7 = \frac{\beta\sigma\zeta_2}{(1-\zeta_1)(1-\zeta_2)e^{-\beta l}-e^{\beta L}}, \qquad \alpha_8 = -\beta\sigma\zeta_1\left[1-\frac{1-\zeta_1}{(1-\zeta_2)e^{2\beta L}}\right]^{-1}, \qquad \alpha_9 = \frac{\beta\sigma\zeta_2}{(1-\zeta_2)e^{-\beta L}-(1-\zeta_1)e^{\beta L}}.$$

β, σ, ζ_1, and ζ_2 are positive constants. The definitions and notations of Sobolev spaces will be borrowed from [3, 9]. The norm of the Sobolev space $W^{m,p}(\Omega)$ will be denoted by $\|\cdot\|_{m,p}$. When $p=2$, $W^{m,p}(\Omega) = H^m(\Omega)$ and its norm will be denoted by $\|\cdot\|_m$. The Lebesgue space is denoted as usual by $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, with norms $\|\cdot\|_{L^p}$ (except the $L^2(\Omega)$-norm which is denoted by $\|\cdot\|$). The Bochner spaces, such as $L^q(0,T,X)$ with norm denoted by $\|\cdot\|_{L^q(X)}$, where X is an Hilbert space are also employed.

Based on the boundedness of the functions $a(\cdot)$, $d(\cdot)$, $\Phi(\cdot)$, and the parameters μ_1, μ_2, γ_1, and γ_2, there exists a positive constant μ such that

$$\int_0^L a(x)(u_x)^2dx + \mu_1 u(0)^2 + \mu_2 u(L)^2 \geqslant \mu\|u\|_1^2, \quad \forall u \in H^1(0,L), \tag{2.6}$$
$$\int_0^L d(x)\Phi(x)(u_x)^2dx + \gamma_1 u(0)^2 + \gamma_2 u(L)^2 \geqslant \mu\|u\|_1^2, \quad \forall u \in H^1(0,L). \tag{2.7}$$

Throughout this paper, we set $T_0 = T_L = C_0 = C_L = 0$ and we make the following regularity assumptions regarding the exact solution (T,C) of (2.1)-(2.5):

$$E(\cdot,\cdot),\ H(\cdot,\cdot) \text{ are positive and Lipschitz continuous functions,} \tag{2.8}$$
$$T^0,\ C^0 \in H^2(0,L), \tag{2.9}$$
$$T,\ C \in L^\infty(0,t_f,W^1_\infty(0,L)) \cap L^\infty(0,t_f,H^{r+1}(0,L)), \tag{2.10}$$
$$T_t,\ C_t \in L^\infty(0,t_f,H^{r+1}(0,L)), \tag{2.11}$$
$$T_{tt},\ C_{tt} \in L^\infty(0,t_f,H^{-1}(0,L)). \tag{2.12}$$

Lemma 2.1. *Let a_k, b_k, c_k, and γ_k, for integers $k \geqslant 0$, be the positive numbers such that*

$$a_n + \tau\sum_{k=0}^n b_k \leqslant \tau\sum_{k=0}^n \gamma_k a_k + \tau\sum_{k=0}^n c_k + B, \quad \textit{for} \quad n \geqslant 0. \tag{2.13}$$

Suppose that $\tau\gamma_k < 1$, for all k, and set $\sigma_k = \frac{1}{(1-\tau\gamma_k)}$. Then

$$a_n + \tau\sum_{k=0}^n b_k \leqslant \exp\left(\tau\sum_{k=0}^n \gamma_k\sigma_k\right)\left(\tau\sum_{k=0}^n c_k + B\right), \quad \textit{for} \quad n \geqslant 0. \tag{2.14}$$

Remark 2.2. If the first sum on the right hand side of (2.13) extends only up to $n-1$, then estimate (2.14) holds for all $k > 0$ with $\sigma_k = 1$. The weak form of the system (2.1)-(2.5) is given by: find for all t, $(T(t), C(t)) \in H^1(0,L) \times H^1(0,L)$ such that:

$$(G(x)\partial_t T, v) + (a(x)\partial_x T, \partial_x v) + \mu_1 T(0,t)v(0) + \mu_2 T(L,t)v(L) = (\Theta(T), v), \tag{2.15}$$
$$([\Phi(x)+E(T,C)]\partial_t C, z) + (d(x)\Phi(x)\partial_x C, \partial_x z) + \gamma_1 C(0,t)z(0) + \gamma_2 C(L,t)z(L) = (H(T,C)\partial_t T, z), \tag{2.16}$$

$\forall v,\ z \in H^1(0,L)$.

3. Semi-discrete scheme in space

Let $V_h \subset H^1(0,L)$ be a finite element space on a quasi-uniform partition of $(0,L)$ which consists of continuous piecewise polynomials of degree $r \geqslant 1$ with a maximum interval of length h. The spatial discret problem based on (2.15)-(2.16) is written as follows. Given $T_h(\cdot,0) = R_h T^0$ and $C_h(\cdot,0) = R_h C^0$, find $(T_h(t), C_h(t)) \in V_h \times V_h$ such that

$$(G(x)\partial_t T_h, v_h) + (a(x)\partial_x T_h, \partial_x v_h) + \mu_1 T_h(0,t)v_h(0) + \mu_2 T_h(L,t)v_h(L) = (\Theta(T_h), v_h), \tag{3.1}$$

$$\begin{aligned}&([\Phi(x) + E(T_h, C_h)]\partial_t C_h, z_h) + (d(x)\Phi(x)\partial_x C_h, \partial_x z_h) + \gamma_1 C_h(0,t)z_h(0) + \gamma_2 C_h(L,t)z_h(L)\\ &\quad = (H(T_h, C_h)\partial_t T_h, z_h), \ \forall v_h,\ z_h \in V_h.\end{aligned} \tag{3.2}$$

Let us decompose $T - T_h$ and $C - C_h$, respectively, as $T - T_h = T - R_h T + R_h T - T_h = E_h^T + e_h^T$ and $C - C_h = C - R_h C + R_h C - C_h = E_h^C + e_h^C$, where $R_h T : (0, t_f] \to V_h$ and $R_h C : (0, t_f] \to V_h$ are the elliptic projections defined as follows:

$$\int_0^l a(x)\partial_x E_h^T \partial_x v dx + \mu_1 E_h^T(0)v_h(0) + \mu_2 E_h^T(L)v_h(L) = 0,$$

$$\int_0^l d(x)\Phi(x)\partial_x E_h^C \partial_x v dx + \gamma_1 E_h^C(0)v_h(0) + \gamma_2 E_h^C(L)v_h(L) = 0, \quad v_h \in V_h.$$

From [4], one has the following error estimates.

Lemma 3.1. *For the elliptic projections defined above, the following error estimates hold:*

$$\begin{aligned}\|T - R_h T\|_{L^2} + h\|T - R_h T\|_{H^1} &\leqslant M\|T\|_{L^\infty(H^{r+1})} h^{r+1},\\ \|C - R_h C\|_{L^2} + h\|C - R_h C\|_{H^1} &\leqslant M\|C\|_{L^\infty(H^{r+1})} h^{r+1},\\ \|\partial_t(T - R_h T)\|_{L^2} + h\|\partial_t(T - R_h T)\|_{H^1} &\leqslant M\|T\|_{W^1_\infty(H^{r+1})} h^{r+1},\\ \|\partial_t(C - R_h C)\|_{L^2} + h\|\partial_t(C - R_h C)\|_{H^1} &\leqslant M\|C\|_{W^1_\infty(H^{r+1})} h^{r+1},\\ \|T - R_h T\|_{L^\infty(W^1_\infty)} &\leqslant M\|T\|_{L^\infty(W^{r+1}_\infty)} h^r,\\ \|C - R_h C\|_{L^\infty(W^1_\infty)} &\leqslant M\|C\|_{L^\infty(W^{r+1}_\infty)} h^r,\end{aligned}$$

for all $r \geqslant 1$ and M a generic positive constant.

One of the main result of this paper is as follows.

Theorem 3.2. *Let (T_h, C_h) be the solution to the semi-discretized problem (3.1)-(3.2). Suppose that the exact solution (T, C) of the system (2.1)-(2.5) satisfies the regularity conditions (2.8)-(2.12). Then, there exists a positive constant M, independent of Δt and h, such that:*

$$\|e_h^T\|^2_{L^\infty(L^2(0,L))} + \|e_h^C\|^2_{L^\infty(L^2(0,L))} + \|e_h^T\|^2_{L^2(H^1(0,L))} + \|e_h^C\|^2_{L^2(H^1(0,L))} \leqslant M h^{2r+2}. \tag{3.3}$$

Proof. e_h^T satisfies the following equation:

$$\begin{aligned}&(G(x)\partial_t e_h^T, v_h) + (a(x)\partial_x e_h^T, \partial_x v_h) + \mu_1 e_h^T(0,t)v_h(0)\\ &\quad + \mu_2 e_h^T(L,t)v_h(L) = (G(x)\partial_t E_h^T, v_h) + (\Theta(T) - \Theta(T_h), v_h).\end{aligned}$$

Taking $v_h = e_h^T$ and using the properties of $G(.)$ and (2.6), we have:

$$(G(x)\partial_t e_h^T, e_h^T) + (a(x)\partial_x e_h^T, \partial_x e_h^T) + \mu_1[e_h^T(0,t)]^2 + \mu_2[e_h^T(L,t)]^2 \geqslant \frac{G_*}{2}\frac{d}{dt}\|e_h^T\|^2 + \mu\|e_h^T\|_1^2.$$

For the nonlinear terms $(\Theta(T)-\Theta(T_h), e_h^T)$, we obtain the following estimates:

$$\begin{aligned}(\Theta(T)-\Theta(T_h), e_h^T) &= (\alpha_1[T(x,t)]^4-[T_h(x,t)]^4, e_h^T) + (\alpha_2 e^{\beta x}\int_0^x e^{-\beta s}[T(s,t)]^4-[T_h(s_t)]^4 ds, e_h^T)\\ &+ (\alpha_3 e^{-\beta x}\int_0^x e^{\beta s}[T(s,t)]^4-[T_h(s,t)]^4 ds, e_h^T)\\ &+ (\alpha_4(e^{\beta x}+(1-\zeta_1)e^{-\beta x})\int_0^x e^{\beta s}[T(s,t)]^4-[T_h(s,t)]^4 ds, e_h^T)\\ &+ (\alpha_5(e^{\beta x}+(1-\zeta_1)e^{-\beta x})\int_0^x e^{-\beta s}[T(s,t)]^4-[T_h(s,t)]^4 ds, e_h^T)\\ &+ (\alpha_6 e^{\beta x}[T(0,t)]^4-[T_h(0,t)]^4, e_h^T) + (\alpha_7 e^{\beta x}[T(L,t)]^4-[T_h(L,t)]^4, e_h^T)\\ &+ (\alpha_8 e^{-\beta x}[T(0,t)]^4-[T_h(0,t)]^4, e_h^T) + (\alpha_9 e^{-\beta x}[T(L,t)]^4-[T_h(L,t)]^4, e_h^T) = \sum_{i=1}^{9} w_i,\end{aligned}$$

$$\begin{aligned}w_1 &= (\alpha_1[T(x,t)]^4-[T_h(x,t)]^4, e_h^T)\\ &= (\alpha_1(T(x)-T_h(x))(T(x)+[T_h(x))([T(x)]^2+[T_h(x)]^2), e_h^T)\\ &\leqslant M(\|E_h^T\|^2+\|e_h^T\|^2)+\frac{\mu}{20}\|e_h^T\|_1^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}|w_2+w_3+w_4+w_5| &\leqslant M(\|E_h^T\|^2+\|e_h^T\|^2)+\frac{\mu}{5}\|e_h^T\|_1^2,\\ |w_6+w_7+w_8+w_9| &\leqslant M([E_h^T(0,t)]^2+[e_h^T(0,t)]^2+[E_h^T(L,t)]^2+[e_h^T(L,t)]^2)+\frac{\mu}{5}\|e_h^T\|_1^2.\end{aligned}$$

Then

$$\begin{aligned}(\Theta(T)-\Theta(T_h), e_h^T) &\leqslant M(\|E_h^T\|^2+\|e_h^T\|^2+[E_h^T(0,t)]^2+[e_h^T(0,t)]^2\\ &\quad +[E_h^T(L,t)]^2+[e_h^T(L,t)]^2)+\frac{9\mu}{20}\|e_h^T\|_1^2 \leqslant Mh^{2r+2}+M\|e_h^T\|^2+\frac{\mu}{2}\|e_h^T\|_1^2.\end{aligned}$$

Adding all these estimates together, one obtains

$$\frac{d}{dt}\|e_h^T\|^2+\frac{\mu}{2}\|e_h^T\|_1^2 \leqslant Mh^{2r+2}+M\|e_h^T\|^2. \tag{3.4}$$

To deal with e_h^C, setting $z_h = e_h^C$ in (2.2) and taking into account the hypotheses on $\Phi(.)$, $E(.,.)$, $H(.,.)$, $G(.)$, and $a(.)$, one has

$$\begin{aligned}([\Phi(x)+E(T_h,C_h)]\partial_t e_h^C, e_h^C) &\geqslant \frac{\Phi_*}{2}\frac{d}{dt}\|e_h^C\|^2,\\ (d(x)\Phi(x)\partial_x e_h^C, \partial_x e_h^C)+\gamma_1[e_h^C(0,t)]^2+\gamma_2[e_h^C(L,t)]^2 &\geqslant \mu\|e_h^C\|_1^2,\\ ([\Phi(x)+E(T_h,C_h)]\partial_t E_h^C, e_h^C) &\leqslant M\|\partial_t E_h^C\|^2+\frac{\mu}{8}\|e_h^C\|_1^2,\\ ([E(T,C)-E(T_h,C_h)]\partial_t C, e_h^C) &\leqslant M(\|T-T_h\|^2+\|C-C_h\|^2)+\frac{\mu}{8}\|e_h^C\|_1^2,\\ &\leqslant M(\|E_h^T\|^2+\|e_h^T\|^2+\|E_h^C\|^2+\|e_h^C\|^2)+\frac{\mu}{8}\|e_h^C\|_1^2,\\ (H(T_h,C_h)[\partial_t E_h^T+\partial_t e_h^T], e_h^C) &\leqslant M(\|\partial_t E_h^T\|^2+\|\partial_t e_h^T\|^2)+\frac{\mu}{8}\|e_h^C\|_1^2,\\ ([H(T,C)-H(T_h,C_h)]\partial_t T, e_h^C) &\leqslant M(\|T-T_h\|^2+\|C-C_h\|^2)+\frac{\mu}{8}\|e_h^C\|_1^2,\end{aligned}$$

$$\leqslant M(\|E_h^T\|^2 + \|e_h^T\|^2 + \|E_h^C\|^2 + \|e_h^C\|^2) + \frac{\mu}{8}\|e_h^C\|_1^2.$$

Combining all these estimates, we obtain

$$\frac{d}{dt}\|e_h^C\|^2 + \frac{\mu}{4}\|e_h^C\|_1^2 \leqslant M(h^{2r+2} + \|e_h^T\|^2 + \|E_h^C\|^2 + \|\partial_t e_h^T\|^2). \tag{3.5}$$

To bound $\|\partial_t e_h^T\|^2$, let $v_h = \partial_t e_h^T$ in (2.1), we have

$$\begin{aligned}(G(x)\partial_t e_h^T, \partial_t e_h^T) + (a(x)\partial_x e_h^T, \partial_x\partial_t e_h^T) + \mu_1 e_h^T(0,t)\partial_t e_h^T(0,t)\\ + \mu_2 e_h^T(L,t)\partial_t e_h^T(L,t) = (G(x)\partial_t E_t^\partial e_h^T, \partial_t e_h^T) + (\Theta(T) - \Theta(T_h), \partial_t e_h^T).\end{aligned}$$

Similarly as what we did above leads to

$$\begin{aligned}&(G(x)\partial_t e_h^T, \partial_t e_h^T) \geqslant G_*\|\partial_t e_h^T\|^2,\\ &(a(x)\partial_x e_h^T, \partial_x\partial_t e_h^T) + \mu_1 e_h^T(0,t)\partial_t e_h^T(0,t) + \mu_2 e_h^T(L,t)\partial_t e_h^T(L,t)\\ &\qquad\qquad \geqslant \frac{1}{2}\frac{d}{dt}(a_*\|\partial_x e_h^T\|^2 + \mu_1[e_h^T(0,t)]^2 + [\mu_2 e_h^T(L,t)]^2).\end{aligned}$$

For the nonlinear term, we obtain

$$\begin{aligned}(\Theta(T) - \Theta(T_h), \partial_t e_h^T) &\leqslant M(\|E_h^T\|^2 + \|e_h^T\|^2 + [E_h^T(0,t)]^2 + [e_h^T(0,t)]^2\\ &\quad + [E_h^T(L,t)]^2 + [e_h^T(L,t)]^2) + \frac{G_*}{4}\|\partial_t e_h^T\|^2 \leqslant Mh^{2r+2} + M\|e_h^T\|^2 + \frac{G_*}{4}\|\partial_t e_h^T\|^2.\end{aligned}$$

Again $a_*\|\partial_x e_h^T\|^2 + \mu_1[e_h^T(0,t)]^2 + \mu_2[e_h^T(L,t)]^2 \geqslant \mu\|e_h^T\|_1^2$. Therefore

$$\|\partial_t e_h^T\|^2 + \frac{d}{dt}\|e_h^T\|_1^2 \leqslant Mh^{2r+2} + M\|e_h^T\|^2. \tag{3.6}$$

Combining (3.4), (3.5), and (3.6) and integrating the obtained relation over $(0, t_f)$ and using the Gronwall inequality, one ends the proof with the generic consatnt $M = M(\|T\|_\infty, \|T_h\|_\infty, \|C\|_\infty, \|C_h\|_\infty)$. □

4. Fully discrete problem

Let $\{t_n | t_n = n\Delta t; 0 \leqslant n \leqslant N\}$ be a uniform partition of the time interval $[0, t_f]$, where $\Delta t = t_f/N$ represents the time step. At each time t_n, w^n denotes an approximation of the function $w(x, t_n)$. For a sequence of functions $\{w^n\}_{n=0}^N$, the following definitions are introduced:

$$D_t w^n = \frac{w^n - w^{n-1}}{\Delta t}, \quad \bar{w}^n = \frac{1}{2}(w^n + w^{n-1}),\ n = 1, 2, \ldots, N, \quad \hat{w}^n = \frac{1}{2}(3w^{n-1} - w^{n-2}),\ n = 2, \ldots, N.$$

The following linearized scheme for the system (2.1)-(2.5) is proposed. Taking $T_h^0 = R_h T^0$ and $C_h^0 = R_h C^0$, for $n = 1, \ldots, N$, we seek $(T_h^n, C_h^n) \in V_h \times V_h$ such that

$$(G(x)D_t T_h^n, v_h) + (a(x)\partial_x \bar{T}_h^n, \partial_x v_h) + \mu_1 \bar{T}_h^n(0)v_h(0) + \mu_2 \bar{T}_h^n(L)v_h(L) = (\Theta(\hat{T}_h^n), v_h), \tag{4.1}$$

$$\begin{aligned}&([\Phi(x) + E(\hat{T}_h^n, \hat{C}_h^n)]D_t C_h^n, z_h) + (d(x)\Phi(x)\partial_x \bar{C}_h^n, \partial_x z_h) + \gamma_1 \bar{C}_h^n(0)z_h(0) + \gamma_2 \bar{C}_h^n(L)z_h(L)\\ &\quad = (H(\hat{T}_h^n, \hat{C}_h^n)D_t T_h^n, z_h), \quad \forall v_h, z_h \in V_h,\end{aligned} \tag{4.2}$$

where $(\hat{T}_h^1, \hat{C}_h^1)$ is the solution of the following system of linear equations:

$$(G(x)\frac{\hat{T}_h^1 - T_h^0}{\Delta t/2}, v_h) + (a(x)\partial_x \hat{T}_h^1, \partial_x v_h) + \mu_1 \hat{T}_h^1(0)v_h(0) + \mu_2 \hat{T}_h^1(L)v_h(L) = (\Theta(T_h^0), v_h), \tag{4.3}$$

$$
\begin{aligned}
&([\Phi(x)+E(T_h^0,C_h^0)]\frac{\hat{C}_h^1-C_h^0}{\Delta t/2},z_h)+(d(x)\Phi(x)\partial_x\hat{C}_h^1,\partial_x z_h)+\gamma_1\hat{C}_h^1(0)z_h(0)+\gamma_2\hat{C}_h^1(L)z_h(L)\\
&=(H(T_h^0,C_h^0)\frac{\hat{T}_h^1-T_h^0}{\Delta t/2},z_h),\quad \forall v_h,z_h\in V_h.
\end{aligned}
\tag{4.4}
$$

Thanks to the assumptions on the functions $G(\cdot)$, $E(\cdot,\cdot)$ and $H(\cdot,\cdot)$, it is easy that the matrices resulting from the linear systems (4.1)-(4.2) and (4.3)-(4.4), respectively, are symmetric and positive definite. This ensures the existence and uniqueness of the numerical solutions (T_h^n, C_h^n). The following error estimates come from the definition of R_h:

$$
\|T^0-T_h^0\|+\|C^0-C_h^0\|+h(\|\partial_x(T^0-T_h^0)\|+\|\partial_x(C^0-C_h^0)\|)\leqslant Mh^{r+1},\quad \|T_h^0\|_\infty+\|C_h^0\|_\infty\leqslant M.
$$

4.1. *Stability result*

We now present a stability result that is essential for convergence analysis.

Lemma 4.1. *Assume the exact solution* (T,C) *satisfies the regularity assumptions* (2.8)-(2.12)*. Then, there exists a positive constant* M *such that*

$$
\begin{aligned}
\|\hat{T}_h^1\|^2+\|\hat{C}_h^1\|^2+\Delta t(\|\hat{T}_h^1\|_1^2+\|\hat{C}_h^1\|_1^2)&\leqslant M(\|T^0\|^2+\|C^0\|^2+\|T^0\|^6+[T^0(0)]^6+[T^0(L)]^6), \quad (4.5)\\
\|\hat{T}_h^1\|_\infty+\|\hat{C}_h^1\|_\infty&\leqslant M.
\end{aligned}
$$

Proof. Taking $v_h=\Delta t\hat{T}_h^1$ in (4.3), one has

$$
(2G(x)(\hat{T}_h^1-T_h^0),\hat{T}_h^1)+\Delta t(a(x)\partial_x\hat{T}_h^1,\partial_x\hat{T}_h^1)+\mu_1\Delta t[\hat{T}_h^1(0)]^2+\mu_2\Delta t[\hat{T}_h^1(L)]^2=\Delta t(\Theta(T_h^0),\hat{T}_h^1). \tag{4.6}
$$

For the first two terms on the left hand side, since $G(.)$ and $a(.)$ are bounded by positive constants, one obtains

$$
(2G(x)\hat{T}_h^1,\hat{T}_h^1)\geqslant 2G_*\|\hat{T}_h^1\|^2, \tag{4.7}
$$
$$
\Delta t(a(x)\partial_x\hat{T}_h^1,\partial_x\hat{T}_h^1)+\mu_1\Delta t[\hat{T}_h^1(0)]^2+\mu_2\Delta t[\hat{T}_h^1(L)]^2\geqslant \mu\Delta t\|\hat{T}_h^1\|_1^2, \tag{4.8}
$$
$$
(2G(x)T_h^0,\hat{T}_h^1)\leqslant M\|T^0\|^2+G_*\|\hat{T}_h^1\|^2. \tag{4.9}
$$

For the right hand side of (4.6), we have

$$
\begin{aligned}
\Delta t(\Theta(T_h^0),\hat{T}_h^1)&=\Delta t(\alpha_1(T_h^0)^4(x,t)+\alpha_2e^{\beta x}\int_0^x e^{-\beta s}(T_h^0)^4(s,t)ds\\
&\quad+\alpha_3e^{-\beta x}\int_0^x e^{\beta s}(T_h^0)^4(s,t)ds+\alpha_4(e^{\beta x}+(1-\zeta_1)e^{-\beta x})\int_0^x e^{\beta s}(T_h^0)^4(s,t)ds\\
&\quad+\alpha_5(e^{\beta x}+(1-\zeta_1)e^{-\beta x})\int_0^x e^{-\beta s}(T_h^0)^4(s,t)ds\\
&\quad+\alpha_6e^{\beta x}(T_h^0)^4(0,t)+\alpha_7e^{\beta x}(T_h^0)^4(L,t)+\alpha_8e^{-\beta x}(T_h^0)^4(0,t)\\
&\quad+\alpha_9e^{-\beta x}(T_h^0)^4(L,t),\hat{T}_h^1)=\sum_{i=1}^{9}J_i,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1&=\Delta t(\alpha_1(T_h^0)^4(x,t),\hat{T}_h^1)\leqslant M\Delta t\|T^0\|^6+\frac{1}{20}\mu\Delta t\|\hat{T}_h^1\|_1^2,\\
J_2&=\Delta t(\alpha_2e^{\beta x}\int_0^x e^{-\beta s}(T_h^0)^4(s,t)ds,\hat{T}_h^1)\leqslant M\Delta t\|T^0\|^6+\frac{1}{20}\mu\Delta t\|\hat{T}_h^1\|_1^2,
\end{aligned}
$$

$$J_6 = \Delta t(\alpha_6 e^{\beta x}(T_h^0)^4(0,t), \hat{T}_h^1) \leqslant M\Delta t[T_h^0(0)]^4\|\hat{T}_h^1\| \leqslant M\Delta t[T^0(0)]^6 + \frac{1}{20}\mu\Delta t\|\hat{T}_h^1\|_1^2.$$

Similarly,

$$\begin{aligned}
|J_3 + J_4 + J_5| &\leqslant M\Delta t\|T^0\|^6 + \frac{3}{20}\mu\Delta t\|\hat{T}_h^1\|_1^2,\\
|J_7 + J_8 + J_9| &\leqslant M\Delta t([T^0(0)]^6 + [T^0(L)]^6) + \frac{3}{20}\mu\Delta t\|\hat{T}_h^1\|_1^2.
\end{aligned} \tag{4.10}$$

Combining these estimates with (4.7)-(4.10), we obtain

$$\|\hat{T}_h^1\|^2 + \frac{\mu}{2}\Delta t\|\hat{T}_h^1\|_1^2 \leqslant M\|T^0\|^2 + Mt_f(\|T^0\|^6 + [T^0(0)]^6 + [T^0(L)]^6). \tag{4.11}$$

Next, let us take $z_h = \Delta t\hat{C}_h^1$ in (4.4), we obtain

$$\begin{aligned}
&2((\Phi(x) + E(T_h^0, C_h^0))(\hat{C}_h^1 - C_h^0), \hat{C}_h^1) + \Delta t(d(x)\Phi(x)\partial_x\hat{C}_h^1, \partial_x\hat{C}_h^1)\\
&\quad + \gamma_1\Delta t[\hat{C}_h^1(0)]^2 + \gamma_2\Delta t[\hat{C}_h^1(L)]^2 = 2(H(T_h^0, C_h^0)(\hat{T}_h^1 - T_h^0), \hat{C}_h^1).
\end{aligned}$$

Taking into account the assumptions on $d(.)$, $\Phi(.)$, $E(.,.)$, and $H(.,.)$ we have the following estimates

$$\begin{aligned}
2((\Phi(x) + E(T_h^0, C_h^0))\hat{C}_h^1, \hat{C}_h^1) &\geqslant 2\Phi_*\|\hat{C}_h^1\|^2,\\
\Delta t(d(x)\Phi(x)\partial_x\hat{C}_h^1, \partial_x\hat{C}_h^1) + \gamma_1\Delta t[\hat{C}_h^1(0)]^2 + \gamma_2\Delta t[\hat{C}_h^1(L)]^2 &\geqslant \mu\Delta t\|\hat{C}_h^1\|_1^2,\\
2((\Phi(x) + E(T_h^0, C_h^0))C_h^0, \hat{C}_h^1) &\leqslant M\|C^0\|^2 + \Phi_*\|\hat{C}_h^1\|^2,\\
2\Delta t(H(T_h^0, C_h^0)(\hat{T}_h^1 - T_h^0), \hat{C}_h^1) &\leqslant M\Delta t(\|\hat{T}_h^1\|^2 + \|T_h^0\|^2) + \frac{\mu}{4}\Delta t\|\hat{C}_h^1\|_1^2.
\end{aligned}$$

Combining these estimates with (4.11) we get

$$\|\hat{C}_h^1\|^2 + \Delta t\|\hat{C}_h^1\|_1^2 \leqslant M\|C^0\|^2 + Mt_f(\|\hat{T}_h^1\|^2 + \|T^0\|^2). \tag{4.12}$$

Adding (4.11) and (4.12) we will have the inequality (4.5). Furthermore, by the Sobolev embedding theorem [1], we have $H^1(0,L) \hookrightarrow C^0(0,L)$, then

$$\|\hat{T}_h^1\|_\infty \leqslant K\|\hat{T}_h^1\|_1 \leqslant M \quad \text{and } \|\hat{C}_h^1\|_\infty \leqslant K\|\hat{C}_h^1\|_1 \leqslant M.$$

□

The following main results present the unconditional stability and error estimation for the full discrete scheme.

Lemma 4.2. *If the exact solution* (T, C) *satisfies the regularity assumptions* (2.8)-(2.12)*, then for all* $n = 1, \ldots, N$*, the problem* (4.1)-(4.2) *has a unique solution* $(T_h^n, C_h^n) \in V_h \times V_h$*, which satisfies the following inequality*

$$\|T_h^n\|^2 + \|C_h^n\|^2 + \Delta t\sum_{k=1}^{n}(\|\bar{T}_h^k\|_1^2 + \|\bar{C}_h^k\|_1^2) \leqslant M, \tag{4.13}$$

$$\|T_h^n\|_\infty + \|C_h^n\|_\infty \leqslant M, \tag{4.14}$$

where M *is a constant* $M = M(\|T^0\|_\infty, \|C^0\|_\infty, \|T^0\|^2, \|C^0\|^2, \|T^0\|^6, [T^0(0)]^6, [T^0(L)]^6)$.

Proof. The result of (4.14) comes from (4.13) and $H^1(0,L) \hookrightarrow C^0(0,L)$. So we just need to prove (4.13). The proof is done by mathematical induction. For $n = 1$, the pair $(T_h^1, C_h^1) \in V_h \times V_h$ satisfies

$$(G(x)D_tT_h^1, v_h) + (a(x)\partial_x\bar{T}_h^1, \partial_x v_h) + \mu_1\bar{T}_h^1(0)v_h(0) + \mu_2\bar{T}_h^1(L)v_h(L) = (\Theta(\hat{T}_h^1), v_h), \tag{4.15}$$

$$\begin{aligned}&([\Phi(x)+E(\hat{T}_h^1,\hat{C}_h^1)]D_tC_h^1,z_h)+(d(x)\Phi(x)\partial_x\bar{C}_h^1,\partial_x z_h)+\gamma_1\bar{C}_h^1(0)z_h(0)+\gamma_2\bar{C}_h^1(L)z_h(L)\\&\quad=(H(\hat{T}_h^1,\hat{C}_h^1)D_tT_h^1,z_h).\end{aligned}\tag{4.16}$$

Set $v_h=\Delta t\bar{T}_h^1$ in (4.15) to have

$$\Delta t(G(x)D_tT_h^1,\bar{T}_h^1)+\Delta t(a(x)\partial_x\bar{T}_h^1,\partial_x\bar{T}_h^1)+\Delta t\mu_1[\bar{T}_h^1(0)]^2+\mu_2[\bar{T}_h^1(L)]^2)=\Delta t(\Theta(\hat{T}_h^1),\bar{T}_h^1).$$

Similarly to what we do above, we have

$$\begin{aligned}&\Delta t(G(x)D_tT_h^1,\bar{T}_h^1)\geqslant\frac{G_*}{2}(\|T_h^1\|^2-\|T_h^0\|^2),\\&\Delta t(a(x)\partial_x\bar{T}_h^1,\partial_x\bar{T}_h^1)+\Delta t\mu_1[\bar{T}_h^1(0)]^2+\mu_2[\bar{T}_h^1(L)]^2)\geqslant\mu\Delta t\|\bar{T}_h^1\|_1^2,\\&\Delta t(\Theta(\hat{T}_h^1),\bar{T}_h^1)=\Delta t(\alpha_1[\hat{T}_h^1(x)]^4+\alpha_2e^{\beta x}\int_0^x e^{-\beta s}[\hat{T}_h^1(s)]^4ds\\&\qquad+\alpha_3e^{-\beta x}\int_0^x e^{\beta s}[\hat{T}_h^1(s)]^4ds+\alpha_4(e^{\beta x}+(1-\zeta_1)e^{-\beta x})\int_0^x e^{\beta s}[\hat{T}_h^1(s)]^4ds\\&\qquad+\alpha_5(e^{\beta x}+(1-\zeta_1)e^{-\beta x})\int_0^x e^{-\beta s}[\hat{T}_h^1(s)]^4ds+\alpha_6e^{\beta x}[\hat{T}_h^1(0)]^4\\&\qquad+\alpha_7e^{\beta x}[\hat{T}_h^1(L)]^4+\alpha_8e^{-\beta x}[\hat{T}_h^1(0)]^4+\alpha_9e^{-\beta x}[\hat{T}_h^1(L)]^4,\bar{T}_h^1)=\sum_{i=1}^{9}K_i.\end{aligned}\tag{4.17}$$

We can bound each K_i as follows.

$$\begin{aligned}|K_1+K_2+K_3+K_4+K_5|&\leqslant M\Delta t\|\hat{T}_h^1\|^6+\frac{5}{18}\mu\Delta t\|\bar{T}_h^1\|_1^2,\\|K_6+K_7+K_8+K_9|&\leqslant M\Delta t([\hat{T}_h^1(0))]^6+[\hat{T}_h^1(L))]^6)+\frac{2}{9}\mu\Delta t\|\bar{T}_h^1\|_1^2.\end{aligned}\tag{4.18}$$

By combining these estimates with (4.17)-(4.18) we obtain

$$\|T_h^1\|^2+\Delta t\|\bar{T}_h^1\|_1^2\leqslant M\|T_h^0\|^2+Mt_f(\|\hat{T}_h^1\|^6+[\hat{T}_h^1(0))]^6+[\hat{T}_h^1(L))]^6).\tag{4.19}$$

To bound C_h^1, take $z_h=\Delta t\bar{C}_h^1$ in (4.16) to have

$$\begin{aligned}&\Delta t([\Phi(x)+E(\hat{T}_h^1,\hat{C}_h^1)]D_tC_h^1,\bar{C}_h^1)+\Delta t((d(x)\Phi(x)\partial_x\bar{C}_h^1,\partial_x\bar{C}_h^1)\\&\quad+\gamma_1[\bar{C}_h^1(0)]^2+\gamma_2[\bar{C}_h^1(L)]^2)=\Delta t(H(\hat{T}_h^1,\hat{C}_h^1)D_tT_h^1,\bar{C}_h^1).\end{aligned}$$

Each term is bounded as follows:

$$\begin{aligned}&\Delta t([\Phi(x)+E(\hat{T}_h^1,\hat{C}_h^1)]D_tC_h^1,\bar{C}_h^1)\geqslant\frac{\Phi_*}{2}(\|C_h^1\|^2-\|C_h^0\|^2),\\&\Delta t((d(x)\Phi(x)\partial_x\bar{C}_h^1,\partial_x\bar{C}_h^1)+\gamma_1[\bar{C}_h^1(0)]^2+\gamma_2[\bar{C}_h^1(L)]^2)\geqslant\mu\Delta t\|\bar{C}_h^1\|_1^2,\\&\Delta t(H(\hat{T}_h^1,\hat{C}_h^1)D_tT_h^1,\bar{C}_h^1)\leqslant M(\|T_h^1\|^2+\|T_h^0\|^2+\|C_h^0\|^2)+\frac{\Phi_*}{4}\|C_h^1\|^2.\end{aligned}$$

Therefore,

$$\|C_h^1\|^2+\Delta t\|\bar{C}_h^1\|_1^2\leqslant M(\|T_h^1\|^2+\|T_h^0\|^2+\|C_h^0\|^2).\tag{4.20}$$

Adding (4.19) and (4.20), we obtain the stability result (4.13) for $n=1$. We now assume that (4.13) holds for $n\leqslant m$ with $m\leqslant N-1$ and prove (4.13) subsequently at rank $n=m+1$. Let us set $v_h=\Delta t\bar{T}_h^n$ in (4.1), we obtain

$$\Delta t(G(x)D_tT_h^n,\bar{T}_h^n)+\Delta t((a(x)\partial_x\bar{T}_h^n,\partial_x\bar{T}_h^n)+\mu_1[\bar{T}_h^n(0)]^2+\mu_2[\bar{T}_h^n(L)]^2)=\Delta t(\Theta(\hat{T}_h^n),\bar{T}_h^n).\tag{4.21}$$

The left hand side of (4.21) is bounded as follows.

$$\Delta t(GD_t T_h^n, \bar{T}_h^n) \geqslant \frac{G_*}{2}(\|T_h^n\|^2 - \|T_h^{n-1}\|^2), \quad \Delta t((a(x)\partial_x \bar{T}_h^n, \partial_x \bar{T}_h^n) + \mu_1[\bar{T}_h^n(0)]^2 + \mu_2[\bar{T}_h^n(L)]^2) \geqslant \mu\Delta t\|\bar{T}_h^n\|_1^2.$$

The right hand side of (4.21) is treated as

$$\begin{aligned}\Delta t(\Theta(\hat{T}_h^n), \bar{T}_h^n) = \Delta t(&\alpha_1[\hat{T}_h^n(x)]^4 + \alpha_2 e^{\beta x}\int_0^x e^{-\beta s}[\hat{T}_h^n(s)]^4 ds \\ &+ \alpha_3 e^{-\beta x}\int_0^x e^{\beta s}[\hat{T}_h^n(s)]^4 ds + \alpha_4(e^{\beta x} + (1-\zeta_1)e^{-\beta x})\int_0^x e^{\beta s}[\hat{T}_h^n(s)]^4 ds \\ &+ \alpha_5(e^{\beta x} + (1-\zeta_1)e^{-\beta x})\int_0^x e^{-\beta s}[\hat{T}_h^n(s)]^4 ds \\ &+ \alpha_6 e^{\beta x}[\hat{T}_h^n(0)]^4 + \alpha_7 e^{\beta x}[\hat{T}_h^n(L)]^4 + \alpha_8 e^{-\beta x}[\hat{T}_h^n(0)]^4 + \alpha_9 e^{-\beta x}[\hat{T}_h^n(L)]^4, \bar{T}_h^n) = \sum_{i=1}^{9} S_i,\end{aligned}$$

where

$$\begin{aligned}|S_1 + S_2 + S_3 + S_4 + S_5| &\leqslant M\Delta t\|\hat{T}_h^n\|^6 + \frac{\mu}{5}\Delta t\|\bar{T}_h^n\|_1^2, \\ |S_6 + S_7 + S_8 + S_9| &\leqslant M\Delta t([\hat{T}_h^n(0)]^6 + [\hat{T}_h^n(L)]^6) + \frac{\mu}{5}\Delta t\|\bar{T}_h^n\|_1^2, \\ \Delta t(\Theta(\hat{T}_h^n), \bar{T}_h^n) &\leqslant M\Delta t(\|\hat{T}_h^n)\|^6 + \hat{T}_h^n(0)]^6 + [\hat{T}_h^n(L)]^6) + \frac{\mu}{5}\Delta t\|\bar{T}_h^n\|_1^2.\end{aligned}$$

Then

$$\begin{aligned}\|T_h^n\|^2 + \Delta t\|\bar{T}_h^n\|_1^2 &\leqslant \|T_h^{n-1}\|^2 + M\Delta t(\|\hat{T}_h^n\|^6 + \hat{T}_h^n(0)]^6 + [\hat{T}_h^n(L)]^6) \\ &\leqslant \|T_h^{n-1}\|^2 + M\Delta t\|\hat{T}_h^n\|_\infty^4\|\hat{T}_h^n\|_1^2, \quad (\text{using } H^1(0,L) \hookrightarrow C^0(0,L) \hookrightarrow L^2(0,L)).\end{aligned}$$

By the defintion of $\hat{T}_h^n$ and the induction assumption, $\|\hat{T}_h^n\|_\infty \leqslant M$, therefore

$$\|T_h^n\|^2 + \Delta t\|\bar{T}_h^n\|_1^2 \leqslant \|T_h^{n-1}\|^2 + M\Delta t\|\hat{T}_h^n\|_1^2.$$

Summing up the above inequality, one has

$$\|T_h^n\|^2 + \Delta t\sum_{k=1}^{n}\|\bar{T}_h^k\|_1^2 \leqslant \|T_h^0\|^2 + M\Delta t\sum_{k=1}^{n}\|\hat{T}_h^k\|_1^2$$

and we obtain require result for T_h^n with the help of the induction assuption. To get the bound of C_h^n, we take $z_h = \Delta t\bar{C}_h^n$ in (4.2) to obtain

$$\begin{aligned}\Delta t([\Phi(x) + E(\hat{T}_h^n, \hat{C}_h^n)]D_t C_h^n, \bar{C}_h^n) + \Delta t((d(x)\Phi(x)\partial_x \bar{C}_h^n, \partial_x \bar{C}_h^n) + \gamma_1[\bar{C}_h^n(0)]^2 + \gamma_2[\bar{C}_h^n(L)]^2) \\ = \Delta t(H(\hat{T}_h^n, \hat{C}_h^n)D_t T_h^n, \bar{C}_h^n),\end{aligned}$$

$$\Delta t([\Phi(x) + E(\hat{T}_h^n, \hat{C}_h^n)]D_t C_h^n, \bar{C}_h^n) \geqslant \frac{\Phi_*}{2}(\|C_h^n\|^2 - \|C_h^{n-1}\|^2),$$

$$\Delta t((d(x)\Phi(x)\partial_x \bar{C}_h^n, \partial_x \bar{C}_h^n) + \gamma_1\bar{C}_h^n(0)]^2 + \gamma_2\bar{C}_h^n(L)]^2) \geqslant \mu\Delta t\|\bar{C}_h^n\|_1^2,$$

$$\Delta t(H(\hat{T}_h^n, \hat{C}_h^n)D_t T_h^n, \bar{C}_h^n) \leqslant M\Delta t\|D_t T_h^n\|^2 + \frac{\mu}{2}\Delta t\|\bar{C}_h^n\|_1^2.$$

We obtain

$$\|C_h^n\|^2 + \Delta t\|\bar{C}_h^n\|_1^2 \leqslant M\Delta t\|D_t T_h^n\|^2 + \|C_h^{n-1}\|^2. \tag{4.22}$$

By summing inequality (4.22) for $k = 1$ to n, we obtain

$$\|C_h^n\|^2 + \Delta t\sum_{k=1}^{n}\|\bar{C}_h^k\|_1^2 \leqslant M(\|T_h^0\|^2 + \|C_h^0\|^2). \qquad \square$$

4.2. Error estimates for the fully discrete system

This subsection focuses on the optimal error estimation for the fully discrete solutions. Let decompose the errors as follow, for $V \in \{T,\ C\}$, we have

$$\hat{V}^n - \hat{V}_h^n = \underbrace{\hat{V}^n - R_h\hat{V}^n}_{\hat{E}_{vh}^n} + \underbrace{R_h\hat{V}^n - \hat{V}_h^n}_{\hat{e}_{vh}^n}, \qquad V^n - V_h^n = \underbrace{V^n - R_hV^n}_{E_{vh}} + \underbrace{R_hV^n - V_h^n}_{e_{vh}^n}.$$

To derive the stated error estimates, we have from the weak form (2.15)-(2.16), the error equations given by

$$\begin{aligned}&(G(x)D_tT^n, v) + (a(x)\partial_x T^{n-1/2}, \partial_x v) + \mu_1 T^{n-1/2}(0)v(0) + \mu_2 T^{n-1/2}(L)v(L)\\&\quad= (G(x)[D_tT^n - \partial_t T(t^{n-1/2})], v) + (\Theta(T^{n-1/2}), v),\end{aligned} \tag{4.23}$$

$$\begin{aligned}&([\Phi(x) + E(\hat{T}^n, \hat{C}^n)]D_tC^n, z) + (d(x)\Phi(x)\partial_x C^{n-1/2}, \partial_x z) + \gamma_1 C^{n-1/2}(0)z(0) + \gamma_2 C^{n-1/2}(L)z(L)\\&\quad= (\Phi(x)[D_tC^n - \partial_t C(t^{n-1/2})], z) + (H(\hat{T}^n, \hat{C}^n)D_tT^n, z)\\&\qquad+ (E(\hat{T}^n, \hat{C}^n)D_tC^n - E(T^{n-1/2}, C^{n-1/2})\partial_t C(t^{n-1/2}), z)\\&\qquad+ (H(T^{n-1/2}, C^{n-1/2})\partial_t T(t^{n-1/2}) - H(\hat{T}^n, \hat{C}^n)D_tT^n, z),\end{aligned} \tag{4.24}$$

where $(\hat{T}^1, \hat{C}^1)$ satisfies

$$\begin{aligned}&(G(x)\frac{\hat{T}^1 - T^0}{\Delta t/2}, v) + (a(x)\partial_x \hat{T}^1, \partial_x v) + \mu_1 \hat{T}^1(0)v(0) + \mu_2 \hat{T}^1(L)v(L)\\&\quad= (G(x)[\frac{\hat{T}^1 - T^0}{\Delta t/2} - \partial_t T^{1/2}], v) + (\Theta(T^{1/2}), v),\end{aligned} \tag{4.25}$$

$$\begin{aligned}&([\Phi(x) + E(T^0, C^0)]\frac{\hat{C}^1 - C^0}{\Delta t/2}, z) + (d(x)\Phi(x)\partial_x \hat{C}^1, \partial_x z) + \gamma_1 \hat{C}^1(0)z(0) + \gamma_2 \hat{C}^1(L)z(L)\\&\quad= (H(T^{1/2}, C^{1/2})\partial_t T^{1/2}, z) + ([\Phi(x) + E(T^0, C^0)]\frac{\hat{C}^1 - C^0}{\Delta t/2} - [\Phi(x) + E(T^{1/2}, C^{1/2})]\partial_t C^{1/2}, z).\end{aligned} \tag{4.26}$$

Lemma 4.3. *Suppose that the exact solution* $(T,\ C)$ *of the continuous equations* (2.1)-(2.5) *satisfies the regularity hypothesis* (2.8)-(2.12)*. Then, there exists a positive constant M such that:*

$$\|\hat{e}_{th}^1\|^2 + \|\hat{e}_{ch}^1\|^2 + \Delta t\|\hat{e}_{th}^1\|_1^2 + \Delta t\|\hat{e}_{ch}^1\|_1^2 \leqslant M(\Delta t^4 + h^{2r+2}).$$

Proof. Subtracting (4.3) from (4.25), taking $v = v_h = \hat{e}_{th}^1$, we get

$$\begin{aligned}&2(G(x)\hat{e}_{th}^1, \hat{e}_{th}^1) + \Delta t(a(x)\partial_x \hat{e}_{th}^1, \partial_x \hat{e}_{th}^1) + \mu_1\Delta t[\hat{e}_{th}^1(0)]^2 + \mu_2\Delta t[\hat{e}_{th}^1(L)]^2\\&\quad= \Delta t(G(x)[\frac{\hat{T}^1 - T^0}{\Delta t/2} - \partial_t T^{1/2}], \hat{e}_{th}^1) - \Delta t(G(x)\frac{\hat{E}_{th}^1 - E_{th}^0}{\Delta t/2}, \hat{e}_{th}^1) + \Delta t(\Theta(T^{1/2}) - \Theta(T_h^0), \hat{e}_{th}^1).\end{aligned} \tag{4.27}$$

The left hand side of (4.27) is bounded as follows:

$$2(G(x)\hat{e}_{th}^1, \hat{e}_{th}^1) \geqslant 2G_*\|\hat{e}_{th}^1\|^2, \tag{4.28}$$

$$\Delta t(a(x)\partial_x \hat{e}_{th}^1, \partial_x \hat{e}_{th}^1) + \mu_1\Delta t[\hat{e}_{th}^1(0)]^2 + \mu_2\Delta t[\hat{e}_{th}^1(L)]^2 \geqslant \mu\Delta t\|\hat{e}_{th}^1\|_1^2. \tag{4.29}$$

For the right hand side of (4.27), we have

$$
\begin{aligned}
&\Delta t(G(x)[\frac{\hat{T}^1-T^0}{\Delta t/2}-\partial_t T^{1/2}],\hat{e}^1_{th}) \leqslant M\Delta t^4+\frac{\mu\Delta t}{12}\|\hat{e}^1_{th}\|_1^2,\\
&\Delta t(G(x)\frac{\hat{E}^1_{th}-E^0_{th}}{\Delta t/2},\hat{e}^1_{th}) \leqslant Mh^{2r+2}+\frac{\mu\Delta t}{12}\|\hat{e}^1_{th}\|_1^2,\\
&\Delta t(\Theta(T^{1/2})-\Theta(T^0_h),\hat{e}^1_{th})\\
&\quad=\Delta t(\alpha_1([T^{1/2}(x)]^4-[T^0_h(x)]^4),\hat{e}^1_{th})\\
&\qquad+\Delta t(\alpha_2 e^{\beta x}\int_0^x e^{-\beta s}([T^{1/2}(s)]^4-[T^0_h(s)]^4)ds,\hat{e}^1_{th})\\
&\qquad+(\alpha_3 e^{-\beta x}\int_0^x e^{\beta s}([T^{1/2}(s)]^4-[T^0_h(s)]^4)ds,\hat{e}^1_{th})\\
&\qquad+\Delta t(\alpha_4(e^{\beta x}+(1-\zeta_1)e^{-\beta x})\int_0^x e^{\beta s}([T^{1/2}(s)]^4-[T^0_h(s)]^4)ds,\hat{e}^1_{th})\\
&\qquad+\Delta t(\alpha_5(e^{\beta x}+(1-\zeta_1)e^{-\beta x})\int_0^x e^{-\beta s}([T^{1/2}(s)]^4-[T^0_h(s)]^4)ds,\hat{e}^1_{th})\\
&\qquad+\Delta t(\alpha_6 e^{\beta x}([T^{1/2}(0)]^4-[T^0_h(0)]^4),\hat{e}^1_{th})+\Delta t(\alpha_7 e^{\beta x}([T^{1/2}(L)]^4-[T^0_h(L)]^4),\hat{e}^1_{th})\\
&\qquad+\Delta t(\alpha_8 e^{-\beta x}([T^{1/2}(0)]^4-[T^0_h(0)]^4),\hat{e}^1_{th})+\Delta t(\alpha_9 e^{-\beta x}([T^{1/2}(L)]^4-[T^0_h(L)]^4),\hat{e}^1_{th})=\sum_{i=1}^{9}V_i,
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
V_1&=\alpha_1\Delta t([T^{1/2}(x)]^4-[T^0_h(x)]^4,\hat{e}^1_{th})\\
&=\alpha_1\Delta t([T^{1/2}(x)]^4-[T^0(x)]^4+[T^0(x)]^4-[T^0_h(x)]^4,\hat{e}^1_{th})\\
&=\alpha_1\Delta t(([T^{1/2}(x)]^2+[T^0(x)]^2)(T^{1/2}(x)-T^0(x))(T^{1/2}(x)+T^0(x)),\hat{e}^1_{th})\\
&\quad+\alpha_1\Delta t(([T^0(x)]^2+[T^0(x)]^2)(T^0(x)-T^0_h(x))(T^0(x)+T^0_h(x)),\hat{e}^1_{th})\\
&\leqslant M(\|T^0\|_\infty,\|T^0_h\|_\infty,\|T^{1/2}\|_\infty)\Delta t(\|T^{1/2}-T^0\|^2+\|T^0-T^0_h\|^2)+\frac{\mu\Delta t}{12}\|\hat{e}^1_{th}\|_1^2\\
&\leqslant M(\Delta t^4+h^{2r+2})+\frac{\mu\Delta t}{12}\|\hat{e}^1_{th}\|_1^2.
\end{aligned}
$$

The bounds of V_2, V_3, V_4, and V_5 are similar to V_1. That is

$$
\begin{aligned}
|V_2+V_3+V_4+V_5|&\leqslant M(\Delta t^4+h^{2r+2})+\frac{4\mu\Delta t}{12}\|\hat{e}^1_{th}\|_1^2,\\
V_6&=\Delta t(\alpha_6 e^{\beta x}([T^{1/2}(0)]^4-[T^0_h(0)]^4),\hat{e}^1_{th})\\
&=\Delta t(\alpha_6 e^{\beta x}([T^{1/2}(0)]^4-[T^0(0)]^4+[T^0(0)]^4-[T^0_h(0)]^4),\hat{e}^1_{th})\\
&=\Delta t(\alpha_6 e^{\beta x}([T^{1/2}(0)]^2+[T^0(0)]^2)(T^{1/2}(0)-T^0(0))(T^{1/2}(0)+T^0(0)),\hat{e}^1_{th})\\
&\quad+\Delta t(\alpha_6 e^{\beta x}([T^0(0)]^2+[T^0(0)]^2)(T^0(0)-T^0_h(0))(T^0(0)+T^0_h(0)),\hat{e}^1_{th})\\
&\leqslant M(\|T^0\|_\infty,\|T^0_h\|_\infty,\|T^{1/2}\|_\infty)\Delta t([T^{1/2}(0)-T^0(0)]^2+[T^0(0)-T^0_h(0)]^2)\\
&\quad+\frac{\mu\Delta t}{12}\|\hat{e}^1_{th}\|_1^2\leqslant M(\Delta t^4+h^{2r+2})+\frac{\mu\Delta t}{12}\|\hat{e}^1_{th}\|_1^2.
\end{aligned}
\tag{4.31}
$$

Also the bounds of V_7,V_8, and V_9 are similar to V_6,

$$
\begin{aligned}
|V_7+V_8+V_9|&\leqslant M([T^{1/2}(0)-T^0(0)]^2+[T^0(0)-T^0_h(0)]^2+[T^{1/2}(l)-T^0(L)]^2\\
&\quad+[T^0(L)-T^0_h(L)]^2)+\frac{\mu\Delta t}{12}\|\hat{e}^1_{th}\|_1^2\leqslant M_3(\Delta t^4+h^{2r+2})+\frac{\mu\Delta t}{12}\|\hat{e}^1_{th}\|_1^2.
\end{aligned}
\tag{4.32}
$$

with $M_3 = M_3(\|T^0\|_\infty, \|T_h^0\|_\infty, \|T^{1/2}\|_\infty, \|T^0\|_\infty, \|T_h^0\|_\infty, \|T^{1/2}\|_\infty)$. By combining (4.28)-(4.32) together with (4.27), we obtain

$$\|\hat{e}_{th}^1\|^2 + \Delta t\|\hat{e}_{th}^1\|_1^2 \leqslant M(\Delta t^4 + h^{2r+2}). \tag{4.33}$$

We now give the error estimates for $\hat{e}_{ch}^1$. For that, subtracting (4.4) from (4.26) and taking $z = z_h = \hat{e}_{ch}^1$, we get

$$\begin{aligned}
&([\Phi(x) + E(T_h^0, C_h^0)]\hat{e}_{ch}^1, \hat{e}_{ch}^1) + \Delta t(d(x)\Phi(x)\partial_x\hat{e}_{ch}^1, \partial_x\hat{e}_{ch}^1) + \gamma_1\Delta t[\hat{e}_{ch}^1(0)]^2 + \gamma_2\Delta t[\hat{e}_{ch}^1(L)]^2\\
&= -\Delta t((\Phi(x) + E(T_h^0, C_h^0))(\hat{E}_{ch}^1 - E_{ch}^0), \hat{e}_{ch}^1) - \Delta t((E(T^0, C^0) - E(T_h^0, C_h^0))\frac{\hat{C}^1 - C^0}{\Delta t/2}, \hat{e}_{ch}^1)\\
&\quad + \Delta t(H(T^{1/2}, C^{1/2})\partial_t T^{1/2} - H(\hat{T}_h^1, \hat{C}_h^1)\frac{\hat{T}_h^1 - T_h^0}{\Delta t/2}, \hat{e}_{ch}^1)\\
&\quad + \Delta t\left([\Phi(x) + E(T^0, C^0)]\frac{\hat{C}^1 - C^0}{\Delta t/2} - [\Phi(x) + E(T^{1/2}, C^{1/2})]\partial_t C^{1/2}, \hat{e}_{ch}^1\right) = \sum_{i=1}^{4} U_i.
\end{aligned} \tag{4.34}$$

The left hand side of (4.34) is bounded as follows:

$$([\Phi(x) + E(T_h^0, C_h^0)]\hat{e}_{ch}^1, \hat{e}_{ch}^1) \geqslant \Phi_*\|\hat{e}_{ch}^1\|^2, \tag{4.35}$$

$$\Delta t(d(x)\Phi(x)\partial_x\hat{e}_{ch}^1, \partial_x\hat{e}_{ch}^1) + \gamma_1\Delta t[\hat{e}_{ch}^1(0)]^2 + \gamma_2\Delta t[\hat{e}_{ch}^1(L)]^2 \geqslant \mu\Delta t\|\hat{e}_{ch}^1\|_1^2. \tag{4.36}$$

For the right hand side of (4.34), we have

$$\begin{aligned}
U_1 &= \Delta t((\Phi(x) + E(T_h^0, C_h^0))(\hat{E}_{ch}^1 - E_{ch}^0), \hat{e}_{ch}^1)\\
&\leqslant M\Delta t(\|\hat{E}_{ch}^1\|^2 + \|E_{ch}^0\|^2) + \frac{\mu\Delta t}{5}\|\hat{e}_{ch}^1\|_1^2 \leqslant Mh^{2r+2} + \frac{\mu\Delta t}{5}\|\hat{e}_{ch}^1\|_1^2,\\
U_2 &= \Delta t((E(T^0, C^0) - E(T_h^0, C_h^0))\frac{\hat{C}^1 - C^0}{\Delta t/2}, \hat{e}_{ch}^1)\\
&\leqslant M(\|T^0 - T_h^0\| + \|C^0 - C_h^0\|)\|\hat{C}^1 - C^0\|\|\hat{e}_{ch}^1\| \leqslant Mh^{2r+2} + \frac{\Phi_*}{7}\|\hat{e}_{ch}^1\|^2,\\
U_3 &= \Delta t\left(H(T^{1/2}, C^{1/2})\partial_t T^{1/2} - H(\hat{T}_h^1, \hat{C}_h^1)\frac{\hat{T}_h^1 - T_h^0}{\Delta t/2}, \hat{e}_{ch}^1\right)\\
&= \Delta t(H(T^{1/2}, C^{1/2})(\partial_t T^{1/2} - \frac{\hat{T}^1 - T^0}{\Delta t/2}, \hat{e}_{ch}^1) + ((H(T^{1/2}, C^{1/2}) - H(\hat{T}_h^1, \hat{C}_h^1))(\hat{T}^1 - T^0), \hat{e}_{ch}^1)\\
&\quad + (H(\hat{T}_h^1, \hat{C}_h^1)((\hat{T}^1 - \hat{T}_h^1) - (T^0 - T_h^0), \hat{e}_{ch}^1)\\
&\leqslant M\Delta t^4 + \frac{\Phi_*}{7}\|\hat{e}_{ch}^1\|^2 + M\|\hat{e}_{th}^1\|^2 + \frac{\Phi_*}{7}\|\hat{e}_{ch}^1\|^2 + M(h^{2r+2} + \|\hat{e}_{th}^1\|^2) + \frac{\Phi_*}{7}\|\hat{e}_{ch}^1\|^2\\
&\leqslant M(\Delta t^4 + h^{2r+2} + \|\hat{e}_{th}^1\|^2) + \frac{3\Phi_*}{7}\|\hat{e}_{ch}^1\|^2,\\
U_4 &= \Delta t\left([\Phi(x) + E(T^0, C^0)]\frac{\hat{C}^1 - C^0}{\Delta t/2} - [\Phi(x) + E(T^{1/2}, C^{1/2})]\partial_t C^{1/2}, \hat{e}_{ch}^1\right)\\
&= \Delta t\left([\Phi(x) + E(T^{1/2}, C^{1/2})](\frac{\hat{C}^1 - C^0}{\Delta t/2} - \partial_t C^{1/2}, \hat{e}_{ch}^1\right)\\
&\quad + \left((E(T^0, C^0 - E(T^{1/2}, C^{1/2})(\hat{C}^1 - C^0), \hat{e}_{ch}^1\right) \leqslant M\Delta t^4 + \frac{2\Phi_*}{7}\|\hat{e}_{ch}^1\|^2.
\end{aligned} \tag{4.37}$$

Taking (4.35)-(4.37) into (4.34) and combine with (4.33), we obtain the desired result. □

Theorem 4.4. *Let* (T_h^n, C_h^n) *be the solution to the discretized problem* (4.1)-(4.4)*. Assume that the exact solution* (T, C) *of the continuous equations* (2.1)-(2.5) *satisfies the regularity conditions* (2.8)-(2.12)*. Then, there exists a positive constant* M*, independent of* Δt *and* h*, such that:*

$$\|e_{th}^n\|^2 + \|e_{ch}^n\|^2 + \Delta t \sum_{k=1}^{n} (\|\bar{e}_{th}^k\|_1^2 + \|\bar{e}_{ch}^k\|_1^2) \leqslant M(\Delta t^4 + h^{2r+2}). \tag{4.38}$$

Proof. For $n = 1$, subtracting (4.1) from (4.23) and taking $v = v_h = \Delta t \bar{e}_{th}^1$, one has

$$\begin{aligned}
&\Delta t(G(x)D_t e_{th}^1, \bar{e}_{th}^1) + \Delta t(a(x)\partial_x \bar{e}_{th}^1, \partial_x \bar{e}_{th}^1) + \mu_1 \Delta t[\bar{e}_{th}^1(0)]^2 + \mu_2 \Delta t[\bar{e}_{th}^1(L)]^2 \\
&= \Delta t(G(x)[D_t T^1 - \partial_t T^{1/2}], \bar{e}_{th}^1) - \Delta t(G(x)D_t E_{th}^1, \bar{e}_{th}^1) - \Delta t(a(x)(\partial_x(T^{1/2} - \bar{T}^1), \partial_x \bar{e}_{th}^1) \\
&\quad - \mu_1 \Delta t(T^{1/2}(0) - \bar{T}^1(0))\bar{e}_{th}^1(0) - \mu_2 \Delta t(T^{1/2}(L) - \bar{T}^1(L))\bar{e}_{th}^1(L) + \Delta t(\Theta(T^{1/2}) - \Theta(\hat{T}_h^1), \bar{e}_{th}^1) \\
&= \sum_{i=1}^{5} V_i.
\end{aligned} \tag{4.39}$$

The bound of the left hand side of (4.39) is given by

$$\Delta t(G(x)D_t e_{th}^1, \bar{e}_{th}^1) + \Delta t(a(x)\partial_x \bar{e}_{th}^1, \partial_x \bar{e}_{th}^1) + \mu_1 \Delta t[\bar{e}_{th}^1(0)]^2 + \mu_2 \Delta t[\bar{e}_{th}^1(L)]^2) \geqslant G_* \|e_{th}^1\|^2 + \mu \Delta t \|\bar{e}_{th}^1\|_1^2. \tag{4.40}$$

For the right hand side of (4.39), we have

$$\begin{aligned}
V_1 &= \Delta t(G(x)[D_t T^1 - \partial_t T^{1/2}, \bar{e}_{th}^1) \leqslant M\Delta t^4 + \frac{\mu \Delta t}{8}\|\bar{e}_{th}^1\|_1^2, \\
V_2 &= \Delta t(G(x)D_t E_{th}^1, \bar{e}_t^1) \leqslant M h^{2r+2} + \frac{\mu \Delta t}{8}\|\bar{e}_{th}^1\|_1^2, \\
V_3 &= \Delta t(a(x)(\partial_x(T^{1/2} - \bar{T}^1), \partial_x \bar{e}_{th}^1) \leqslant M\Delta t^4 + \frac{\mu \Delta t}{8}\|\bar{e}_{th}^1\|_1^2, \\
V_4 &= \mu_1 \Delta t(T^{1/2}(0) - \bar{T}^1(0))\bar{e}_{th}^1(0) + \mu_2 \Delta t(T^{1/2}(L) - \bar{T}^1(L))\bar{e}_{th}^1(L) \\
&\leqslant M\Delta t^4 + \frac{\mu \Delta t}{8}\|\bar{e}_{th}^1\|_1^2 \ (\text{using } H^1(0,L) \hookrightarrow C^0(0,L)), \\
V_5 &= \Delta t(\Theta(T^{1/2}) - \Theta(\hat{T}_h^1), \bar{e}_{th}^1) = \Delta t[(\alpha_1[T^{1/2}(x)]^4 - [\hat{T}_h^1(x)]^4, \bar{e}_{th}^1) \\
&\quad + (\alpha_2 e^{\beta x}\int_0^x e^{-\beta s}[T^{1/2}(s)]^4 - [\hat{T}_h^1(s)]^4 ds, \bar{e}_{th}^1) \\
&\quad + (\alpha_3 e^{-\beta x}\int_0^x e^{\beta s}[T^{1/2}(s)]^4 - [\hat{T}_h^1(s)]^4 ds, \bar{e}_{th}^1) \\
&\quad + (\alpha_4(e^{\beta x} + (1-\zeta_1)e^{-\beta x})\int_0^x e^{\beta s}[T^{1/2}(s)]^4 - [\hat{T}_h^1(s)]^4 ds, \bar{e}_{th}^1) \\
&\quad + (\alpha_5(e^{\beta x} + (1-\zeta_1)e^{-\beta x})\int_0^x e^{-\beta s}[T^{1/2}(s)]^4 - [\hat{T}_h^1(s)]^4 ds, \bar{e}_{th}^1) \\
&\quad + (\alpha_6 e^{\beta x}[T^{1/2}(0)]^4 - [\hat{T}_h^1(0)]^4, \bar{e}_{th}^1) + (\alpha_7 e^{\beta x}[T^{1/2}(L)]^4 - [\hat{T}_h^1(L)]^4, \bar{e}_{th}^1) \\
&\quad + (\alpha_8 e^{-\beta x}[T^{1/2}(0)]^4 - [\hat{T}_h^1(0)]^4, \bar{e}_{th}^1) + (\alpha_9 e^{-\beta x}[T^{1/2}(L)]^4 - [\hat{T}_h^1(L)]^4, \bar{e}_{th}^1)] = \sum_{i=1}^{9} w_i,
\end{aligned} \tag{4.41}$$

where

$$\begin{aligned}
w_1 &= \Delta t(\alpha_1[T^{1/2}(x)]^4 - [\hat{T}_h^1(x)]^4, \bar{e}_{th}^1) \\
&= \Delta t(\alpha_1([T^{1/2}(x)] - [\hat{T}_h^1(x)])([T^{1/2}(x)] + [\hat{T}_h^1(x)])([T^{1/2}(x)]^2 + [\hat{T}_h^1(x)]^2), \bar{e}_{th}^1)
\end{aligned}$$

$$\leqslant M(h^{2r+2} + \|\hat{e}^1_{th}\|^2) + \frac{\mu\Delta t}{8}\|\bar{e}^1_{th}\|_1^2.$$

Similarly to w_1, we obtain

$$|w_2 + w_3 + w_4 + w_5| \leqslant M(h^{2r+2} + \|\hat{e}^1_{th}\|^2) + \frac{\mu\Delta t}{8}\|\bar{e}^1_{th}\|_1^2,$$
$$|w_6 + w_7 + w_8 + w_9| \leqslant M(h^{2r+2} + [\hat{e}^1_{th}(0)]^2 + [\hat{e}^1_{th}(L)]^2) + \frac{\mu\Delta t}{8}\|\bar{e}^1_{th}\|_1^2.$$

Thus, combining (4.40)-(4.41) with (4.39), we end up with

$$\|e^1_{th}\|^2 + \Delta t\|\bar{e}^1_{th}\|_1^2 \leqslant M(\Delta t^4 + h^{2r+2}).$$

To give the error estimate for e^n_{ch} for $n = 1$, we subtract (4.2) from (4.24) and take $z = z_h = \Delta t\bar{e}^1_{ch}$ to get

$$\begin{aligned}
&\Delta t([\Phi(x) + E(\hat{T}^1_h, \hat{C}^1_h)]D_t e^1_{ch}, \bar{e}^1_{ch}) + \Delta t(d(x)\Phi(x)(\partial_x \bar{e}^1_{ch}, \partial_x \bar{e}^1_{ch}) + \Delta t\gamma_1[\bar{e}^1_{ch}(0)]^2 + \Delta t\gamma_2[\bar{e}^1_{ch}(L)]^2\\
&\quad = -\Delta t(E(\hat{T}^1, \hat{C}^1) - E(\hat{T}^1_h, \hat{C}^1_h))D_t C^1, \bar{e}^1_{ch}) - \Delta t([\Phi(x) + E(\hat{T}^1_h, \hat{C}^1_h)]D_t E^1_{ch}, \bar{e}^1_{ch})\\
&\qquad + \Delta t\Phi[D_t C^1 - \partial_t C^{1/2}], \bar{e}^1_{ch}) + \Delta t(E(\hat{T}^1, \hat{C}^1)D_t C^1 - E(T^{1/2}, C^{1/2})\partial_t C^{1/2}, \bar{e}^1_{ch})\\
&\qquad - \Delta t(d(x)\Phi(x)(\partial_x(C^{1/2} - \bar{C}^1), \partial_x \bar{e}^1_{ch}) - \Delta t\gamma_1(C^{1/2}(0) - \bar{C}^1(0))\bar{e}^1_{ch}(0) - \Delta t\gamma_2(C^{1/2}(L) - \bar{C}^1(L))\bar{e}^1_{ch}(L)\\
&\qquad + \Delta t(H(T^{1/2}, C^{1/2})\partial_t T^{1/2} - H(\hat{T}^1, \hat{C}^1)D_t T^1, \bar{e}^1_{ch}) + \Delta t(H(\hat{T}^1, \hat{C}^1)D_t T^1 - H(\hat{T}^1_h, \hat{C}^1_h)D_t T^1_h, \bar{e}^1_{ch}) = \sum_{i=1}^{8} P_i.
\end{aligned} \tag{4.42}$$

The bound of the left hand side of (4.42) is similar to (4.39) and is given by

$$\begin{aligned}
&\Delta t([\Phi(x) + E(\hat{T}^1_h, \hat{C}^1_h)]D_t e^1_{ch}, \bar{e}^1_{ch}) + \Delta t(d(x)\Phi(x)(\partial_x \bar{e}^1_{ch}, \partial_x \bar{e}^1_{ch})\\
&\quad + \Delta t\gamma_1[\bar{e}^1_{ch}(0)]^2 + \Delta t\gamma_2[\bar{e}^1_{ch}(L)]^2 \geqslant \phi_*\Delta t D_t\|e^1_{ch}\|^2 + \mu\Delta t\|\bar{e}^1_{ch}\|_1^2.
\end{aligned} \tag{4.43}$$

For the right hand side, we have

$$\begin{aligned}
P_1 &= \Delta t(E(\hat{T}^1, \hat{C}^1) - E(\hat{T}^1_h, \hat{C}^1_h))D_t C^1, \bar{e}^1_{ch})\\
&\leqslant \frac{\mu\Delta t}{18}\|\bar{e}^1_{ch}\|_1^2 + M\Delta t(\|\hat{T}^1 - \hat{T}^1_h\|^2 + \|\hat{C}^1 - \hat{C}^1_h\|^2)\|D_t C^1\|^2\\
&\leqslant \frac{\mu\Delta t}{18}\|\bar{e}^1_{ch}\|_1^2 + M(h^{2r+2} + \|\hat{e}^1_{th}\|^2 + \|\hat{e}^1_{ch}\|^2),\\
P_2 &= \Delta t([\Phi + E(\hat{T}^1_h, \hat{C}^1_h)]D_t E^1_{ch}, \bar{e}^1_{ch}) \leqslant Mh^{2r+2} + \frac{\mu\Delta t}{18}\|\bar{e}^1_{ch}\|_1^2,\\
P_3 &= \Delta t(\Phi[D_t C^1 - \partial_t C^{1/2}], \bar{e}^1_{ch}) \leqslant \frac{\mu\Delta t}{18}\|\bar{e}^1_{ch}\|_1^2 + M\Delta t^4,\\
P_4 &= \Delta t(E(T^{1/2}, C^{1/2})(D_t C^1 - \partial_t C^{1/2}), \bar{e}^1_{ch}) \leqslant \frac{\mu\Delta t}{18}\|\bar{e}^1_{ch}\|_1^2 + M\Delta t^4,\\
P_5 &= \Delta t(d(x)\Phi(x)(\partial_x(C^{1/2} - \bar{C}^1), \partial_x \bar{e}^1_{ch}) \leqslant M\Delta t^4 + \frac{\mu\Delta t}{18}\|\bar{e}^1_{ch}\|_1^2,\\
P_6 &= \Delta t\gamma_1(C^{1/2}(0) - \bar{C}^1(0))\bar{e}^1_{ch}(0) + \Delta t\gamma_2(C^{1/2}(L) - \bar{C}^1(L))\bar{e}^1_{ch}(L) \leqslant \frac{\mu\Delta t}{18}\|\bar{e}^1_{ch}\|_1^2 + M\Delta t^4,\\
P_7 &= \Delta t(H(T^{1/2}, C^{1/2})(\partial_t T^{1/2} - D_t T^1), \bar{e}^1_{ch}) \leqslant \frac{\mu\Delta t}{18}\|\bar{e}^1_{ch}\|_1^2 + M\Delta t^4,\\
P_8 &= \Delta t([H(\hat{T}^1, \hat{C}^1) - H(\hat{T}^1_h, \hat{C}^1_h)]D_t T^1, \bar{e}^1_c) + \Delta t(H(\hat{T}^1_h, \hat{C}^1_h)(D_t E^1_t), \bar{e}^1_c) + \Delta t(H(\hat{T}^1_h, \hat{C}^1_h)(D_t e^1_t), \bar{e}^1_c)\\
&\leqslant \frac{3\Phi_*}{5}(\|e^1_{ch}\|^2 + \|e^0_{ch}\|^2) + M(\|\hat{T}^1 - \hat{T}^1_h\|^2 + \|\hat{C}^1 - \hat{C}^1_h\|^2 + \|E^1_{th}\|^2 + \|e^1_{th}\|^2)\\
&\leqslant M(h^{2r+2} + \Delta t^4 + \|e^1_{th}\|^2) + \frac{3\Phi_*}{5}\|e^1_{ch}\|^2.
\end{aligned} \tag{4.44}$$

Putting (4.43)-(4.44) with (4.42), we obtain

$$\|e^1_{ch}\|^2 + \Delta t\|\bar{e}^1_{ch}\|^2_1 \leqslant M\Delta t(\Delta t^4 + h^{2r+2} + \|e^1_{th}\|^2).$$

Now let us assume that (4.38) holds for $n \leqslant m$ with $1 \leqslant m \leqslant N$, then we need to prove the inequality for $n = m+1$. For $n \geqslant 2$, subtracting (4.1) from (4.23) and taking $v = v_h = \Delta t\bar{e}^n_{th}$, we have

$$\begin{aligned}
&\Delta t(G(x)D_t e^n_{th}, \bar{e}^n_{th}) + \Delta t(a(x)\partial_x \bar{e}^n_{th}, \partial_x \bar{e}^n_{th}) + \mu_1\Delta t[\bar{e}^n_{th}(0)]^2 + \mu_2\Delta t[\bar{e}^n_{th}(L)]^2 \\
&\quad = \Delta t(G(x)[D_t T^n - \partial_t T^{n-1/2}], \bar{e}^n_{th}) - \Delta t(G(x)D_t E^n_{th}, \bar{e}^n_{th}) \\
&\qquad - \Delta t(a(x)\partial(T^{n-1/2} - \bar{T}^n), \partial_x \bar{e}^n_{th}) - \mu_1\Delta t(T^{n-1/2}(0) - \bar{T}^n(0))\bar{e}^n_{th}(0) \\
&\qquad - \mu_2\Delta t(T^{n-1/2}(L) - \bar{T}^n(L))\bar{e}^n_{th}(L) + \Delta t(\Theta(T^{n-1/2}) - \Theta(\hat{T}^n_h), \bar{e}^n_{th}) = \sum_{i=1}^{5} I_i.
\end{aligned} \tag{4.45}$$

The left hand side is treated as usual. That is

$$\begin{aligned}
&\Delta t(G(x)D_t e^n_{th}, \bar{e}^n_{th}) + \Delta t(a(x)\partial_x \bar{e}^n_{th}, \partial_x \bar{e}^n_{th}) \\
&\quad + \mu_1\Delta t[\bar{e}^n_{th}(0)]^2 + \mu_2\Delta t[\bar{e}^n_{th}(L)]^2 \geqslant \frac{G_*\Delta t}{2} D_t\|e^n_{th}\|^2 + \mu\Delta t\|\bar{e}^n_{th}\|^2_1.
\end{aligned} \tag{4.46}$$

For the right hand side, we have

$$\begin{aligned}
I_1 &= \Delta t(G(x)[D_t T^n - \partial_t T^{n-1/2}], \bar{e}^n_{th}) \leqslant \frac{\mu\Delta t}{8}\|\bar{e}^n_{ch}\|^2_1 + M\Delta t^5, \\
I_2 &= \Delta t(G(x)D_t E^n_t, \bar{e}^n_{th}) \leqslant M\Delta t h^{2r+2} + \frac{\mu\Delta t}{8}\|\bar{e}^n_{ch}\|^2_1, \\
I_3 &= \Delta t(a(x)\partial(T^{n-1/2} - \bar{T}^n), \partial_x \bar{e}^n_{th}) \leqslant \frac{\mu\Delta t}{8}\|\bar{e}^n_{ch}\|^2_1 + M\Delta t^5, \\
I_4 &= \Delta t[\mu_1(T^{n-1/2}(0) - \bar{T}^n(0))\bar{e}^n_{th}(0) + \mu_2(T^{n-1/2}(L) - \bar{T}^n(L))\bar{e}^n_{th}(L)] \leqslant \frac{\mu\Delta t}{8}\|\bar{e}^n_{ch}\|^2_1 + M\Delta t^5, \\
I_5 &= \Delta t(\Theta(T^{n-1/2}) - \Theta(\hat{T}^n_h), \bar{e}^n_{th}) \\
&= \Delta t[(\alpha_1[T^{n-1/2}(x)]^4 - [\hat{T}^n_h(x)]^4, \bar{e}^n_{th}) + (\alpha_2 e^{\beta x}\int_0^x e^{-\beta s}[T^{n-1/2}(s)]^4 - [\hat{T}^n_h(s)]^4 ds, \bar{e}^n_{th}) \\
&\quad + (\alpha_3 e^{-\beta x}\int_0^x e^{\beta s}[T^{n-1/2}(s)]^4 - [\hat{T}^n_h(s)]^4 ds, \bar{e}^n_{th}) \\
&\quad + (\alpha_4(e^{\beta x} + (1-\zeta_1)e^{-\beta x})\int_0^x e^{\beta s}[T^{n-1/2}(s)]^4 - [\hat{T}^n_h(s)]^4 ds, \bar{e}^n_{th}) \\
&\quad + (\alpha_5(e^{\beta x} + (1-\zeta_1)e^{-\beta x})\int_0^x e^{-\beta s}[T^{n-1/2}(s)]^4 - [\hat{T}^n_h(s)]^4 ds, \bar{e}^n_{th}) \\
&\quad + (\alpha_6 e^{\beta x}[T^{n-1/2}(0)]^4 - [\hat{T}^n_h(0)]^4, \bar{e}^n_{th}) + (\alpha_7 e^{\beta x}[T^{n-1/2}(L)]^4 - [\hat{T}^n_h(L)]^4, \bar{e}^n_{th}) \\
&\quad + (\alpha_8 e^{-\beta x}[T^{n-1/2}(0)]^4 - [\hat{T}^n_h(0)]^4, \bar{e}^n_{th}) + (\alpha_9 e^{-\beta x}[T^{n-1/2}(L)]^4 - [\hat{T}^n_h(L)]^4, \bar{e}^n_{th})] = \sum_{i=1}^{9} s_i,
\end{aligned}$$

where

$$\begin{aligned}
s_1 &= \Delta t(\alpha_1[T^{n-1/2}(x)]^4 - [\hat{T}^n_h(x)]^4, \bar{e}^n_{th}) \\
&= \Delta t(\alpha_1[T^{n-1/2}(x)]^4 - [\hat{T}^n(x)]^4, \bar{e}^n_{th}) + (\alpha_1[\hat{T}^n(x)]^4 - [\hat{T}^n_h(x)]^4, \bar{e}^n_{th}) \\
&= \Delta t(\alpha_1([T^{n-1/2}(x)] - [\hat{T}^n(x)])([T^{n-1/2}(x)] + [\hat{T}^n(x)])([T^{n-1/2}(x)]^2 + [\hat{T}^n(x)]^2), \bar{e}^n_{th}) \\
&\quad + \Delta t(\alpha_1([\hat{T}^n(x)] - [\hat{T}^n_h(x)])([\hat{T}^n(x)] + [\hat{T}^n_h(x)])([\hat{T}^n(x)]^2 + [\hat{T}^n_h(x)]^2), \bar{e}^n_{th}),
\end{aligned}$$

$$
\begin{aligned}
&\leqslant M\Delta t(\Delta t^4 + h^{2r+2} + \|\hat{e}^n_{th}\|^2) + \frac{\mu\Delta t}{12}\|\bar{e}^n_{th}\|_1^2\\
|s_2+s_3+s_4+s_5| &\leqslant M\Delta t(\Delta t^4 + h^{2r+2} + \|\hat{e}^n_{th}\|^2) + \frac{\mu\Delta t}{12}\|\bar{e}^n_{th}\|_1^2,\\
|s_6+s_7+s_8+s_9| &\leqslant M\Delta t(\Delta t^4 + h^{2r+2} + \|\hat{e}^n_{th}\|^2) + \frac{\mu\Delta t}{12}\|\bar{e}^n_{th}\|_1^2,
\end{aligned}
$$

which implies

$$
I_4 \leqslant M\Delta t(\Delta t^4 + h^{2r+2} + \|\hat{e}^n_{th}\|^2) + \frac{\mu\Delta t}{12}\|\bar{e}^n_{th}\|_1^2.
$$

Taking I_1–I_4 and (4.46) into (4.45), we have

$$
\|e^n_{th}\|^2 - \|e^{n-1}_{th}\|^2 + \Delta t\|\bar{e}^n_{th}\|_1^2 \leqslant M\Delta t(\Delta t^4 + h^{2r+2} + \|\hat{e}^n_{th}\|^2). \tag{4.47}
$$

By summing inequality (4.47) for $k = 1$ to n, we obtain

$$
\|e^n_{th}\|^2 + \Delta t\sum_{k=1}^{n}\|\bar{e}^k_{th}\|_1^2 \leqslant M(\Delta t^4 + h^{2r+2}) + \Delta t M\sum_{k=1}^{n}\|\hat{e}^k_{th}\|^2. \tag{4.48}
$$

To give the error estimate for e^n_{ch} for $n \geqslant 2$, we subtract (4.2) from (4.24) and take $z = z_h = \Delta t\bar{e}^n_{ch}$ to get

$$
\begin{aligned}
&\Delta t([\Phi(x) + E(\hat{T}^n_h, \hat{C}^n_h)]D_t e^n_{ch}, \bar{e}^n_{ch}) + \Delta t(d(x)\Phi(x)(\partial_x\bar{e}^n_{ch}, \partial_x\bar{e}^n_{ch})\\
&\quad + \gamma_1\Delta t\bar{e}^n_{ch}(0)\bar{e}^n_{ch}(0) + \gamma_2\Delta t\bar{e}^n_{ch}(L)\bar{e}^n_{ch}(L)\\
&\ = -\Delta t(E(\hat{T}^n, \hat{C}^n) - E(\hat{T}^n_h, \hat{C}^n_h))D_t C^n, \bar{e}^n_{ch}) - \Delta t([\Phi(x) + E(\hat{T}^n_h, \hat{C}^n_h)]D_t E^n_{ch}, \bar{e}^n_{ch})\\
&\quad + \Delta t\Phi[D_t C^n - \partial_t C^{n-1/2}], \bar{e}^n_{ch}) + \Delta t(E(\hat{T}^n, \hat{C}^n)D_t C^n - E(T^{n-1/2}, C^{n-1/2})\partial_t C^{n-1/2}, \bar{e}^n_{ch})\\
&\quad - \Delta t(d(x)\Phi(x)(\partial_x C^{n-1/2} - \bar{C}^n), \partial_x\bar{e}^n_{ch}) - \gamma_1\Delta t(C^{n-1/2}(0) - \bar{C}^n(0))\bar{e}^n_{ch}(0)\\
&\quad - \gamma_2\Delta t(C^{n-1/2}(L) - \bar{C}^n(L))z_h(L) + \Delta t(H(T^{n-1/2}, C^{n-1/2})\partial_t T^{n-1/2}\\
&\quad - H(\hat{T}^n, \hat{C}^n)D_t T^n, \bar{e}^n_{ch}) + \Delta t(H(\hat{T}^n, \hat{C}^n)D_t T^n - H(\hat{T}^n_h, \hat{C}^n_h)D_t T^n_h, \bar{e}^n_{ch}) = \sum_{i=1}^{8} A_i.
\end{aligned} \tag{4.49}
$$

The left hand side of (4.49) is bounded as follows.

$$
\begin{aligned}
&\Delta t([\Phi(x) + E(\hat{T}^n_h, \hat{C}^n_h)]D_t e^n_{ch}, \bar{e}^n_{ch}) + \Delta t(d(x)\Phi(x)(\partial_x\bar{e}^n_{ch}, \partial_x\bar{e}^n_{ch})\\
&\quad + \gamma_1\Delta t\bar{e}^n_{ch}(0)\bar{e}^n_{ch}(0) + \gamma_2\Delta t\bar{e}^n_{ch}(L)\bar{e}^n_{ch}(L) \geqslant \phi_*\Delta t D_t\|e^n_{ch}\|^2 + \mu\Delta t\|\bar{e}^n_{ch}\|_1^2.
\end{aligned} \tag{4.50}
$$

For the right hand side, we have

$$
\begin{aligned}
A_1 &= \Delta t(E(\hat{T}^n, \hat{C}^n) - E(\hat{T}^n_h, \hat{C}^n_h))D_t C^n, \bar{e}^n_{ch})\\
&\leqslant \frac{\mu\Delta t}{12}\|\bar{e}^n_{ch}\|_1^2 + M\Delta t(\|\hat{T}^n - \hat{T}^n_h\|^2 + \|\hat{C}^n - \hat{C}^n_h\|^2)\|D_t C^n\|^2\\
&\leqslant \frac{\mu\Delta t}{12}\|\bar{e}^n_{ch}\|_1^2 + M\Delta t(h^{2r+2} + \|\hat{e}^n_{th}\|^2 + \|\hat{e}^n_{ch}\|^2),\\
A_2 &= \Delta t([\Phi + E(\hat{T}^n_h, \hat{C}^n_h)]D_t E^n_{ch}, \bar{e}^n_{ch}) \leqslant \frac{\mu\Delta t}{12}\|\bar{e}^n_{ch}\|_1^2 + M\Delta t h^{2r+2},\\
A_3 &= \Delta t(\Phi[D_t C^n - \partial_t C^{n-1/2}], \bar{e}^n_{ch}) \leqslant \frac{\mu\Delta t}{12}\|\bar{e}^n_{ch}\|_1^2 + M\Delta t^5,\\
A_4 &= \Delta t(E(\hat{T}^n, \hat{C}^n)D_t C^n - E(T^{n-1/2}, C^{n-1/2})\partial_t C\,|_{t=t^{n-1/2}}, \bar{e}^n_{ch})\\
&= \Delta t((E(\hat{T}^n, \hat{C}^n) - E(T^{n-1/2}, C^{n-1/2})D_t C^n + E(T^{n-1/2}, C^{n-1/2})[D_t C^1 - \partial_t C^{n-1/2}], \bar{e}^n_{ch})\\
&\leqslant M\Delta t(\|\hat{T}^n - T^{n-1/2}\| + \|\hat{C}^n - C^{n-1/2}\| + \|D_t C^n - \partial_t C^{n-1/2}\|)\|\bar{e}^n_{ch}\| \leqslant \frac{\mu\Delta t}{12}\|\bar{e}^n_{ch}\|_1^2 + M\Delta t^5,
\end{aligned}
$$

$$
\begin{aligned}
A_5 &= \Delta t(d(x)\Phi(x)(\partial_x C^{n-1/2} - \bar{C}^n), \partial_x \bar{e}^n_{ch}) \leqslant \frac{\mu\Delta t}{12}\|\bar{e}^n_{ch}\|_1^2 + M\Delta t^5,\\
A_6 &= \gamma_1\Delta t(C^{n-1/2}(0) - \bar{C}^n(0))\bar{e}^n_{ch}(0) + \gamma_2\Delta t(C^{n-1/2}(L) - \bar{C}^n(L))\bar{e}^n_{ch}(L) \leqslant \frac{\mu\Delta t}{12}\|\bar{e}^n_{ch}\|_1^2 + M\Delta t^5,\\
A_7 &= \Delta t(H(T^{n-1/2}, C^{n-1/2})\partial_t T^{n-1/2} - H(\hat{T}^n, \hat{C}^n)D_t T^n, \bar{e}^n_{ch})\\
&= \Delta t(H(T^{n-1/2}, C^{n-1/2})[\partial_t T^{n-1/2} - D_t T^n] + (H(T^{n-1/2}, C^{n-1/2} - H(\hat{T}^n, \hat{C}^n))D_t T^n, \bar{e}^n_{ch})\\
&\leqslant M\Delta t(\|\partial_t T^{n-1/2} - D_t T^n\| + \|\hat{T}^n - T^{n-1/2}\| + \|\hat{C}^n - C^{n-1/2}\|)\|\bar{e}^n_{ch}\| \leqslant \frac{\mu\Delta t}{12}\|\bar{e}^n_{ch}\|_1^2 + M\Delta t^5,\\
A_8 &= \Delta t(H(\hat{T}^n, \hat{C}^n)D_t T^n - H(\hat{T}^n_h, \hat{C}^n_h)D_t T^n_h, \bar{e}^n_{ch})\\
&= \Delta t([H(\hat{T}^n, \hat{C}^n) - H(\hat{T}^n_h, \hat{C}^n_h)]D_t T^n, \bar{e}^n_{ch}) + \Delta t(H(\hat{T}^n_h, \hat{C}^n_h)(D_t E^n_t + D_t e^n_t), \bar{e}^n_{ch})\\
&\leqslant M\Delta t(\|\hat{T}^n - \hat{T}^n_h\|^2 + \|\hat{C}^n - \hat{C}^n_h\|^2 + \|E^n_{th}\|^2 + \|e^n_{th}\|^2) + \frac{\mu\Delta t}{12}\|\bar{e}^n_{ch}\|_1^2\\
&\leqslant M\Delta t(h^{2r+2} + \|e^n_{th}\|^2) + \frac{\mu\Delta t}{12}\|\bar{e}^n_{ch}\|_1^2.
\end{aligned}
$$

Taking A_1–A_8 together with (4.50) and (4.49), we end up with

$$\|e^n_{ch}\|^2 - \|e^{n-1}_{ch}\|^2 + \Delta t\|\bar{e}^n_{ch}\|_1^2 \leqslant M\Delta t(\Delta t^4 + h^{2r+2} + \|\hat{e}^n_{th}\|^2 + \|\hat{e}^n_{ch}\|^2). \tag{4.51}$$

By summing inequality (4.51) for $k = 1$ to n, we obtain:

$$\|e^n_{ch}\|^2 + \Delta t\sum_{k=1}^{n}\|\bar{e}^k_{ch}\|_1^2 \leqslant M(\Delta t^4 + h^{2r+2}) + M\Delta t\sum_{k=1}^{n}(\|\hat{e}^k_{th}\|_1^2 + \|\hat{e}^k_{ch}\|^2). \tag{4.52}$$

Applying Gronwall's discrete lemma on inequality (4.48) and (4.52), we conclude (4.38). □

5. Numerical example

We consider the following artificial example to confirm our theoretical findings. The problem is simulated on uniform meshes with a linear finite element approximation. The system is defined as follows:

$$
\begin{aligned}
G(x)\partial_t T - \partial_x(a(x)\partial_x T) &= \Theta(T) + f, \quad x \in (0,1), \quad t \in (0,1],\\
[\Phi(x) + E(T,C)]\partial_t C - \partial_x\left(d(x)\Phi(x)\partial_x C\right) &= H(T,C)\partial_t T + g,\\
a(x)\partial_x T(0,t) = \frac{1}{4}T(0,t), \quad a(x)\partial_x T(L,t) &= -\frac{1}{2}T(L,t),\\
d(x)\Phi(x)\partial_x C(0,t) = C(0,t), \quad d(x)\Phi(x)\partial_x C(L,t) &= -C(L,t),\\
T(x,0) = T^0(x), \quad C(x,0) &= C^0(x), \quad x \in (0,1),
\end{aligned}
$$

where $a(x) = 1 - x^2$, $G(x) = 1 + x$, $\Phi(x) = 1 - x$, $d(x) = 1 + x$, and $E(x,y) = H(x,x) = xy$. The functions f and g are chosen correspondingly to the exact solution

$$T(x,t) = 2 + e^{x^2 - 0.2t}, \quad C(x,t) = 1 + e^{2x - 0.2t}.$$

The L^2-error at final time $T = 1$ is computed and shown in Figure 1 on a log-log scale. Note for the time rate of convergence, the mesh size h is chosen as $h = 1/128$ so that it should be negligible compare to the time discretization error. Also for the spatial rate of convergence, the time step is fixed as $\Delta t = 1/100$. One can see that for L^2-norm, the slope is almost 2 for each component T and C which are in good agreement with our theoretical analysis that is, in Theorem 4.4, we obtain the optimal L^2 error estimate $O(h^2 + \Delta t^2)$ unconditionally. To show the unconditional stability result, we choose $h = 1/128$ and the large time steps $\Delta t = kh$ with $k \in \{1, 4, 16, 32, 64\}$. We present the numerical results in Tables 1 and 2, which suggest that the scheme is stable for large time steps.

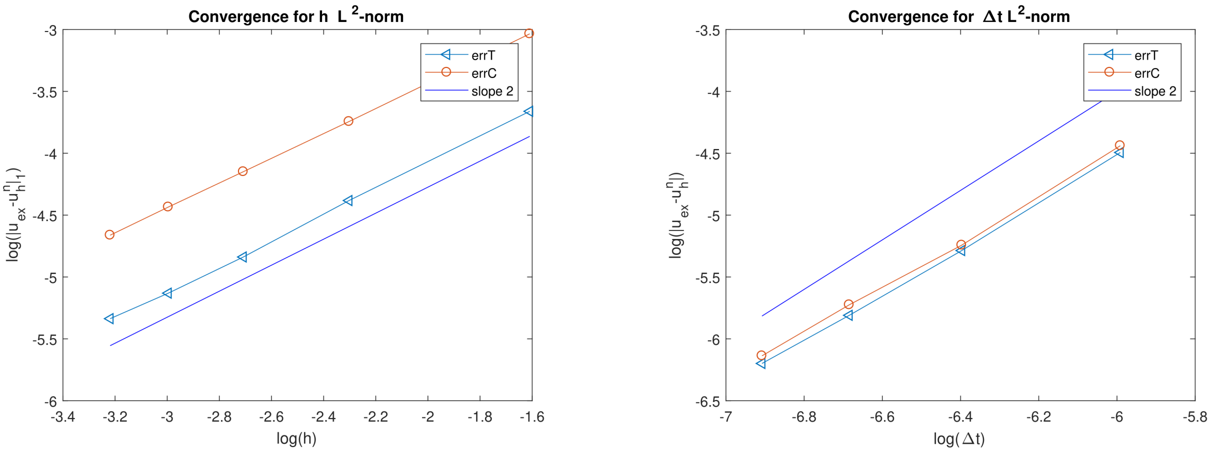

Figure 1: Convergence rate with respect to the mesh size h and time step Δt in L^2 norm, respectively.

Table 1: Convergence result for $\|T^n - T_h^n\|$ for $h = 1/128$ and $\Delta t = kh$.

t	$k = 1$	k=4	k=16	k=32	k=64
0.5	7.512e-4	7.514e-4	7.533e-4	2.732e-3	2.723e-3
1.0	2.414e-3	2.424e-3	2.622e-3	2.639e-3	2.642e-3

Table 2: Convergence result for $\|C^n - C_h^n\|$ for $h = 1/128$ and $\Delta t = kh$.

t	$k = 1$	k=4	k=16	k=32	k=64
0.5	9.862e-4	9.863e-4	9.883e-4	1.929e-3	2.594e-3
1.0	2.587e-3	2.598e-3	2.600e-3	2.704e-3	4.297e-3

6. Conclusion

The comprehensive analysis conducted in this study validates the effectiveness of the linearized Crank-Nicolson finite element method (FEM) for solving coupled heat and moisture transfer equations in fibrous insulation materials. The established optimal error estimates for both L^2 and energy norms in the semi-discrete formulation, as well as the stability and accuracy of the fully discrete scheme, underscore the robustness of the method. The consideration of factors such as fiber orientation, density, and environmental conditions has demonstrated the method's adaptability to real-world scenarios. These findings are instrumental for future advancements in the design and optimization of fibrous insulation materials, ensuring enhanced thermal performance and durability. Future research could extend this work by incorporating experimental validation and exploring more complex insulation structures. The outcomes of this study contribute significantly to the body of knowledge in the field of thermal insulation and numerical methods.

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