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Qualitative dynamics of a higher-order rational difference equation system

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Abstract

This study shows the conditions for local and global asymptotic stability of the equilibrium points in the nonlinear system of difference equations

$$Z_{\eta+1} = \beta_1 Y_{\eta-1} + \frac{\delta_1 Y_{\eta-1} Z_{\eta-4}}{r + Y_{\eta-2} + Z_{-4}}, \quad Y_{\eta+1} = \beta_2 Z_{\eta-1} + \frac{\delta_2 Z_{\eta-1} Y_{\eta-4}}{r + Z_{\eta-2} \pm Y_{\eta-4}}.$$

The boundedness of the positive solutions of the systems is examined. Additionally, the solutions of the systems are investigated. Numerical examples are presented to show the outcomes.

Keywords: Difference equations, solution of difference equation, difference equations system.

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1. Introduction

Dynamic equations are essential in modeling various phenomena across in engineering and natural sciences; see, e.g., [6, 28, 29]. In recent years, interest in studying systems of difference equations has increased. This interest stems from the desire to find analytical solutions, gain a better understanding of dynamic behaviors, and apply these concepts in various fields. For example, Riccati difference equation

$$Z_{\eta+1} = \frac{\delta_1 + \delta_2 Z_{\eta-4}}{\delta_3 + \delta_4 Z_{\eta-4}}.$$

This equation is important in population dynamic, engineering, control theory, and mathematical finance. Most authors who study difference equations provide general forms of solutions. They also investigate the stability properties of equilibria due to the potential difficulties in determining explicit solutions. Nonlinear difference equations have been examined, for instance, in [15], where Elsayed studied the following difference equation:

$$Z_{\eta+1} = \beta_1 Z_{\eta-1} + \frac{\delta_1 Z_{\eta-1} Z_{\eta-4}}{r + Z_{\eta-2} + Z_{\eta-4}}.$$

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Abdul Khaliq and Hassan [25] investigated the properties of a rational equation in one dimension:

$$Z_{\eta+1} = \beta Z_\eta + \delta Z_{\eta-k} + \frac{\alpha + \beta Z_{\eta-k}}{\beta + \delta Z_{\eta-k}}.$$

Furthermore, many authors have investigated systems of difference equations. For example, Althagafi and Ghezal in [16] examined the following system of difference equations:

$$Z_{\eta+1} = \frac{1}{1 + Y_\eta \left(-1 - Z_{\eta-1} Y_{\eta-2}\right)}, \quad Y_{\eta+1} = \frac{Y_{\eta-1} Z_{\eta-3}}{1 + Z_\eta \left(-1 - Y_{\eta-1} Z_{\eta-2}\right)}.$$

Zhang et al. [4] established the existence of closed-form solutions for the system:

$$Z_{\eta+1} = \frac{Z_{\eta-2}}{\beta + Y_\eta Y_{\eta-1-1} Y_{\eta-2}}, \quad Y_{\eta+1} = \frac{Y_{\eta-2}}{\delta + Z_\eta Z_{\eta-1-1} Z_{\eta-2}}.$$

Touafek and Elsayed [33] presented the periodic nature of the solution and derived a formula for the solution of the following system:

$$Z_{1+\eta} = \frac{Z_{\eta-3}}{\pm Z_{\eta-3} Y_{\eta-1} \pm 1}, \quad Y_{1+\eta} = \frac{Y_{\eta-3}}{\pm 1 \pm Y_{\eta-3} Z_{\eta-1}}.$$

The periodic nature and the form of the solutions of the rational difference equation system

$$Z_{\eta+1} = \frac{Y_\eta}{Z_{\eta-1}\left(-1 \pm Y_\eta\right)}, \quad Y_{\eta+1} = \frac{Z_\eta}{Y_{\eta-1}\left(-1 \pm Z_\eta\right)},$$

was obtained by Touafek et al. [34]. For additional examples related to the analysis of the behavior of difference equations and the derivation of solutions, we refer the reader to [1–38]. This paper aims to extend the analysis from a one-dimensional to a two-dimensional system, as

$$Z_{\eta+1} = \beta_1 Y_{\eta-1} + \frac{\delta_1 Y_{\eta-1} Z_{\eta-4}}{r + Y_{\eta-2} + Z_{-4}}, \tag{1.1}$$

$$Y_{\eta+1} = \beta_2 Z_{\eta-1} + \frac{\delta_2 Z_{\eta-1} Y_{\eta-4}}{r + Z_{\eta-2} \pm Y_{\eta-4}}, \tag{1.2}$$

where Z_{-4}, $Z_{-3}, Z_{-2}, Z_{-1}, Z_0, Y_{-4}, Y_{-3}, Y_{-2}, Y_{-1}$, and Y_0 are nonzero real initial conditions. All parameters are real and positive. We examine the stability characteristics and solution forms of the system.

Our work focuses on enhancing the understanding and applications of complex mathematical models across various scientific and technological fields by analyzing the dynamics of discrete nonlinear systems. This study explores systems of difference equations. In Section 1, we provide a detailed review of the relevant literature, highlighting major developments in the field. Section 2 focuses on analyzing the system's behavior and presents empirical findings through numerical simulations and graphical representations that support our theoretical analysis. Finally, Section 3 discusses solutions for specific cases of the system, along with relevant numerical examples.

2. Analysis of the behavior of system solutions

The behavior of the solution to the system (1.1)-(1.2) is discussed in this section, including their local and global stability, as well as boundedness properties, under the assumptions.

2.1. *Boundedness of solution*

We aim to prove that the positive solutions of system (1.1)-(1.2) are bounded in this subsection.

Lemma 2.1. *A condition for the boundedness of the positive solutions of system (1.1)-(1.2) is as follows:*

$$[\beta_2(\beta_1+\delta_1)+\delta_1\delta_2(\beta_1+\delta_1)] \leqslant 1 \quad \textit{and} \quad [\beta_1(\beta_2+\delta_2)+\delta_1\delta_2(\beta_2+\delta_2)] \leqslant 1.$$

Proof. From equation (1.1) we obtain

$$Z_{\eta+1} = \beta_1 Y_{\eta-1} + \frac{\delta_1 Y_{\eta-1} Z_{\eta-4}}{r + Y_{\eta-2} + Z_{\eta-4}},$$

we get from equation (1.2),

$$\begin{aligned}
Z_{\eta+1} &= \beta_1\left(\beta_2 Z_{\eta-1} + \frac{\delta_2 Z_{\eta-1} Y_{\eta-4}}{r + Z_{\eta-2} + Y_{\eta-4}}\right) + \frac{\delta_1 Y_{\eta-1} Z_{\eta-4}}{r + Y_{\eta-2} + Z_{\eta-4}} \\
&\leqslant \beta_1\left(\beta_2 Z_{\eta-1} + \frac{\delta_2 Z_{\eta-1} Y_{\eta-2}}{Y_{\eta-2}}\right) + \frac{\delta_1 Y_{\eta-1} Z_{\eta-4}}{Z_{\eta-4}} \\
&= \beta_1(\beta_2 Z_{\eta-1} + \delta_2 Z_{\eta-1}) + \delta_1 Y_{\eta-1} \\
&= \beta_1(\beta_2 Z_{\eta-1} + \delta_2 Z_{\eta-1}) + \frac{\delta_1\delta_2 Z_{\eta-1} Y_{\eta-4}}{r + Z_{\eta-2} + Y_{\eta-4}} \\
&\leqslant \beta_1(\beta_2 Z_{\eta-1} + \delta_2 Z_{\eta-1}) + \delta_1\delta_2(\beta_2 Z_{\eta-1} + \delta_2 Z_{\eta-1}) \\
&= [\beta_1(\beta_2+\delta_2) + \delta_1\delta_2(\beta_2+\delta_2)] Z_{\eta-1} \\
&\leqslant Z_{\eta-1} \text{ if } [\beta_1(\beta_2+\delta_2) + \delta_1\delta_2(\beta_2+\delta_2)] \leqslant 1.
\end{aligned}$$

Similarly, if $[\beta_2(\beta_1+\delta_1)+\delta_1\delta_2(\beta_1+\delta_1)] \leqslant 1$, we get $Y_{\eta+1} \leqslant Y_{\eta-1}$. Hence $\{Z_\eta, Y_\eta\}_{\eta=0}^{\infty}$ is decreasing. Therefore M is upper bound, where $M = \max\{Z_{-4}, Z_{-3}, Z_{-2}Z_{-1}, Z_0, Z_{-4}, Z_{-3}, Y_{-1}, Y_0\}$. □

2.2. *Analysis of stability*

Consider tenth-dimensional difference equation system of the form:

$$Z_{\eta+1} = F(Z_{\eta-4}, Y_{\eta-4}, Z_{\eta-3}, Y_{\eta-3}, Z_{\eta-2}, Y_{\eta-2}, Z_{\eta-1}, Y_{\eta-1}, Z_\eta, Y_\eta), \tag{2.1}$$
$$Y_{\eta+1} = G(Z_{\eta-4}, Y_{\eta-4}, Z_{\eta-3}, Y_{\eta-3}, Z_{\eta-2}, Y_{\eta-2}, Z_{\eta-1}, Y_{\eta-1}, Z_\eta, Y_\eta), \tag{2.2}$$

where $F : i^5 \times I^5 \to I$ and $G : i^5 \times I^5 \to I$ are continuously differentiable functions and i, I are real numbers intervals. In addition, the system (2.1)-(2.2) is solution unique and determined by initial conditions $(Z_i, Y_i) \in i \times I$ for $i \in \{-4, 0\}$. We consider the corresponding vector map $H = (F, Z_\eta, Z_{\eta-1}, Z_{\eta-3}, G, Y_\eta, Y_{\eta-1}, Y_{\eta-2}, Y_{\eta-3})$, along with system (2.1)-(2.2). The equilibrium point of the map H is the equilibrium point of (2.1)-(2.2). Let's assume $(\bar{Z}, \bar{Y})$ is the equilibrium point which satisfies

$$\bar{Z} = F(\bar{Z}, \bar{Y}, \bar{Z}, \bar{Y}, \bar{Z}, \bar{Y}, \bar{Z}, \bar{Y}, \bar{Z}, \bar{Y}), \quad \bar{Y} = G(\bar{Z}, \bar{Y}, \bar{Z}, \bar{Y}, \bar{Z}, \bar{Y}, \bar{Z}, \bar{Y}, \bar{Z}, \bar{Y}).$$

Consider $(\bar{Z}, \bar{Y})$ is the equilibrium of the system (1.1)-(1.2). That it is

$$\bar{Z} = \beta_1\bar{Y} + \frac{\delta_1\bar{Y}\bar{Z}}{r + \bar{Z} + \bar{Y}}, \quad Y = \beta_2\bar{Z} + \frac{\delta_2\bar{Z}\bar{Y}}{r + \bar{Z} + \bar{Y}}.$$

By solving the system above, we obtain that $(0, 0)$ is the equilibrium point.

Theorem 2.2. *The condition for local asymptotic stability of the equilibrium point $(0, 0)$ for the system (1.1)-(1.2) is $\beta_1\beta_2 < 1$ while the equilibrium point is unstable if $\beta_1\beta_2 > 1$.*

Proof. We obtain the linearized system by creating the transformation

$$(Z_\eta, Y_\eta, Z_{\eta-1}, Y_{\eta-1}, Z_{\eta-2}, Y_{\eta-2}, Z_{\eta-3}, Y_{\eta-3}, Z_{\eta-4}, Y_{\eta-4}) \to (F, G, F_1, G_1, F_2, G_2, F_3, G_3, F_4, G_4),$$

where $F = Z_{\eta+1}, G = Y_{\eta+1}, F_1 = Z_\eta, G_1 = Y_\eta, F_2 = Z_{\eta-1}, G_2 = Y_{\eta-1}, F_3 = Z_{\eta-2}, G_3 = Y_{\eta-2}, F_4 = Z_{\eta-3}, G_4 = Y_{\eta-3}$. The linearized form of system (1.1)-(1.2) is as $W_{\eta+1} = J(\bar{Z}, \bar{Y})W_\eta$, where

$$W_\eta = \begin{pmatrix} Z_\eta \\ Y_\eta \\ Z_{\eta-1} \\ Y_{\eta-1} \\ Z_{\eta-2} \\ Y_{\eta-2} \\ Z_{\eta-3} \\ Y_{\eta-3} \\ Z_{\eta-4} \\ Y_{\eta-4} \end{pmatrix}$$

and $J(\bar{Z}, \bar{Y})$ is Jacobian matrix in $(\bar{Z}, \bar{Y})$. The Jacobian matrix for system (1.1)-(1.2) is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \Psi_1 & 0 & \Psi_2 & \Psi_3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{F}_1 & \mathcal{F}_2 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{F}_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where

$$\Psi_1 = \frac{\delta_1 Y_{-1}}{(r + Y_{-2} + Z_{-4})} - \frac{\delta_1 Y_{-1} Z_{-4}}{(r + Y_{-2} + Z_{-4})^2}, \quad \Psi_2 = \beta_1 + \frac{\delta_1 Z_{-4}}{(r + Y_{-2} + Z_{-4})}, \quad \Psi_3 = -\frac{\delta_1 Y_{-1} Z_{-4}}{(r + Y_{-2} + Z_{-4})^2},$$

$$\mathcal{F}_1 = \beta_2 + \frac{\delta_1 Y_{-4}}{(r + Z_{-2} + Y_{-4})}, \quad \mathcal{F}_2 = -\frac{\delta_1 Z_{-1} Y_{-4}}{(r + Z_{-2} + Y_{-4})^2}, \quad \mathcal{F}_3 = \frac{\delta_2 Y_{-1}}{(r + Y_{-2} + Z_{-4})} - \frac{\delta_2 Y_{-1} Z_{-4}}{(r + Y_{-2} + Z_{-4})^2}.$$

The characteristic equation about equilibrium point $(0, 0)$ is given by $\lambda^{10} - \beta_1\beta_2\lambda^2 = 0$. Then, all eigenvalues lie inside open unit disk if $\beta_1\beta_2 < 1$. Hence the point $(0, 0)$ is locally asymptotically stable according to Rouche's theorem. Define the following function: $\Omega(\lambda) = \lambda^{10} - \beta_1\beta_2\lambda^2 = 0$. We have $\lim_{\lambda\to\infty} \Omega(\lambda) = \infty$, if $\Omega(1) = -\beta_1\beta_2 + 1 < 0$, indicating that there is one at least eigenvalue greater than one. This implies that the equilibrium point is unstable. □

2.3. *Analysis of the global attractor*

The character of global attractor of the system (1.1)-(1.2) will be discussed in this subsection.

Theorem 2.3. *If* $\mathcal{K} = \frac{(r^2\beta_1 + \delta_1 M^2)(r^2\beta_2 + \delta_2 M^2)}{r^4} < 1$, *then the equilibrium point of system* (1.1)-(1.2) *is a global attractor.*

Proof. Letting $F(Y, Y_2, Z_1) = \beta_1 Y + \frac{\delta_1 Y Z_1}{r + Y_2 + Z_1}$, $G(Z, Z_2, Y_1) = \beta_2 Z + \frac{\delta_2 Z Y_1}{r + Z_2 + Y_1}$, hence

$$\frac{\partial F}{\partial Y} = \beta_1 + \frac{\delta_1 Z_1}{r + Y_2 + Z_1}, \qquad \frac{\partial F}{\partial Z_1} = \frac{\delta_1 Y(r + Y_2)}{(r + Y_2 + Z_1)^2}, \qquad \frac{\partial F}{\partial Y_2} = \frac{-\delta_1 Y Z_1}{(r + Y_2 + Z_1)^2},$$

$$\frac{\partial G}{\partial Z}=\beta_2+\frac{\delta_2 Y_1}{r+Z_2+Y_1},\qquad \frac{\partial G}{\partial Y_1}=\frac{\delta_2 Z(r+Z_2)}{(r+Z_2+Y_1)^2},\qquad \frac{\partial G}{\partial Z_2}=\frac{-\delta_2 Z Y_1}{(r+Y_1+Z_2)^2}.$$

Hence, F is decreasing in Y_2, and increasing in Y, Z_1. G is decreasing in Z_2, and increasing in Y_1, Z. Suppose that (u_1, U_1, u_2, U_2) is a solution of

$$U_1=\beta_1 U_2+\frac{\delta_1 U_1 U_2}{(r+U_1+u_2)},\qquad U_2=\beta_2 U_1+\frac{\delta_2 U_1 U_2}{(r+u_1+U_2)}, \tag{2.3}$$

$$u_1=\beta_1 u_2+\frac{\delta_1 u_1 u_2}{(r+u_1+U_2)},\qquad u_2=\beta_2 u_1+\frac{\delta_2 u_1 u_2}{(r+U_1+u_2)}. \tag{2.4}$$

Multiplying Eqs. (2.3) by $\frac{1}{U_1 U_2}$ and Eqs. (2.4) by $\frac{1}{u_1 u_2}$, we obtain

$$\frac{1}{U_2}=\frac{\beta_1}{U_1}+\frac{\delta_1}{(r+U_1+u_2)},\qquad \frac{1}{U_1}=\frac{\beta_2}{U_2}+\frac{\delta_2}{(r+u_1+U_2)},$$
$$\frac{1}{u_2}=\frac{\beta_1}{u_1}+\frac{\delta_1}{(r+u_1+U_2)},\qquad \frac{1}{u_1}=\frac{\beta_2}{u_2}+\frac{\delta_2}{(r+U_1+u_2)}.$$

Subtracting the equations,

$$\frac{U_2-u_2}{U_2u_2}=\frac{\beta_1(U_1-u_1)}{U_1u_1}+\frac{\delta_1(r+U_1+u_2-r-u_1-U_2)}{(r+U_1+u_2)(r+u_1+U_2)},$$
$$\frac{U_1-u_1}{U_1u_1}=\frac{\beta_2(U_2-u_2)}{U_2u_2}+\frac{\delta_2(r+u_1+U_2-r-U_1-u_2)}{(r+U_1+u_2)(r+U_1+u_2)},$$
$$\frac{U_2-u_2}{U_2u_2}=\frac{\beta_1(U_1-u_1)}{U_1u_1}+\frac{\delta_1(r+U_1+u_2-r-u_1-U_2)}{r^2},$$
$$\frac{U_1-u_1}{U_1u_1}=\frac{\beta_2(U_2-u_2)}{U_2u_2}+\frac{\delta_2(r+u_1+U_2-r-U_1-u_2)}{r^2},$$
$$(U_2-u_2)\left(\frac{r^2+\delta_1U_2u_2}{r^2U_2u_2}\right)\leqslant(U_1-u_1)\left(\frac{r^2\beta_1+\delta_1U_1u_1}{r^2U_1u_1}\right),$$
$$(U_1-u_1)\left(\frac{r^2+\delta_2U_1u_1}{r^2U_1u_1}\right)\leqslant(U_2-u_2)\left(\frac{r^2\beta_2+\delta_2U_2u_2}{r^2U_2u_2}\right),$$

$$(U_2-u_2)\leqslant(U_1-u_1)\left(\frac{r^2\beta_1+\delta_1U_1u_1}{r^2U_1u_1}\right)\left(\frac{r^2U_2u_2}{r^2+\delta_1U_2u_2}\right), \tag{2.5}$$

$$(U_1-u_1)\leqslant(U_2-u_2)\left(\frac{r^2\beta_2+\delta_2U_2u_2}{r^2U_2u_2}\right)\left(\frac{r^2U_1u_1}{r^2+\delta_2U_1u_1}\right). \tag{2.6}$$

From (2.5) and (2.6), we get

$$(U_2-u_2)\leqslant(U_2-u_2)\left(\frac{r^2\beta_1+\delta_1U_1u_1}{r^2+\delta_2U_1u_1}\right)\left(\frac{r^2\beta_2+\delta_2U_2u_2}{r^2+\delta_1U_2u_2}\right)\leqslant(U_2-u_2)\left(\frac{r^2\beta_1+\delta_1M^2}{r^2}\right)\left(\frac{r^2\beta_2+\delta_2M^2}{r^2}\right),$$

therefore, $(U_2-u_2)(1-\mathcal{K})\leqslant 0$, where $\mathcal{K}=\frac{(r^2\beta_1+\delta_1M^2)(r^2\beta_2+\delta_2M^2)}{r^4}<1$, it follows $U_2=u_2$. Similarly, we get $U_1=u_1$. □

2.4. *Numerical examples*

We confirm the results of this section by providing examples.

Example 2.4. Consider $\beta_1=0.2, \beta_2=0.2, \delta_1=0.3, \delta_2=0.3, r=0.2$. Hence, the system becomes as follows:

$$Z_{\eta+1}=0.2Y_\eta+\frac{0.3Y_\eta Z_{\eta-1}}{0.2+Y_\eta+Z_{\eta-1}},\quad Y_{\eta+1}=0.2Z_\eta+\frac{0.3Z_\eta Y_{\eta-1}}{0.2+Z_\eta\pm Y_{\eta-1}},$$

with initial conditions $Z_{-4}=0.6, Z_{-3}=0.6, Z_{-2}=0.5, Z_{-1}=0.9, Z_0=0.5, Y_{-4}=0.1, Y_{-3}=0.1, Y_{-2}=0.7, Y_{-1}=0.3, Y_0=0.3$. Since $\beta_1\beta_2<1$. Then, $(0,0)$ is locally stable (see Figure 1).

Example 2.5. Assume that $\beta_1 = 0.2, \beta_2 = 0.2, \delta_1 = 0.3, \delta_2 = 0.3, r = 0.2$. Then the system can be written as:

$$Z_{\eta+1} = 0.5Y_\eta + \frac{1.4Y_\eta Z_{\eta-1}}{10 + Y_\eta + Z_{\eta-1}}, \quad Y_{\eta+1} = 0.1Z_\eta + \frac{5Z_\eta Y_{\eta-1}}{10 + Z_\eta \pm Y_{\eta-1}},$$

with initial conditions

IC1: $Z_{-4} = 10, Z_{-3} = 6, Z_{-2} = 5, Z_{-1} = 9, Z_0 = 5, Y_{-4} = 1, Y_{-3} = 1, Y_{-2} = 7, Y_{-1} = 3, Y_0 = 3$;
IC2: $Z_{-4} = 9, Z_{-3} = 5, Z_{-2} = 4, Z_{-1} = 8, Z_0 = 4, Y_{-4} = 4, Y_{-3} = 4, Y_{-2} = 6, Y_{-1} = 2, Y_0 = 2$;
IC3: $Z_{-4} = 11, Z_{-3} = 7, Z_{-2} = 6, Z_{-1} = 10, Z_0 = 6, Y_{-4} = 1, Y_{-3} = 6, Y_{-2} = 8, Y_{-1} = 4, Y_0 = 4$.

Since $\mathcal{K}$ =0.9979 < 1. Then $(0, 0)$ is globally stable (see Figure 2).

Example 2.6. Suppose that $\beta_1 = 5, \beta_2 = 1, \delta_1 = 3, \delta_2 = 2, r = 1$, then the system is expressed as:

$$Z_{\eta+1} = 2Y_\eta + \frac{1.4Y_\eta Z_{\eta-1}}{10 + Y_\eta + Z_{\eta-1}}, \quad Y_{\eta+1} = Z_\eta + \frac{5Z_\eta Y_{\eta-1}}{10 + Z_\eta \pm Y_{\eta-1}},$$

and IC1-IC3 are initial conditions. Since $\beta_1\beta_2 > 1$, then $(0, 0)$ is unstable (see Figure 3).

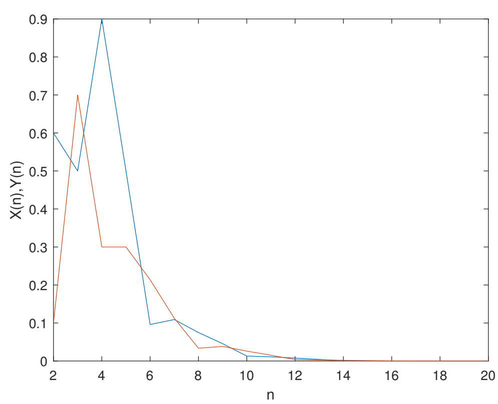

Figure 1: This figure illustrates that the point $(0, 0)$ is unstable for system (1.1)-(1.2).

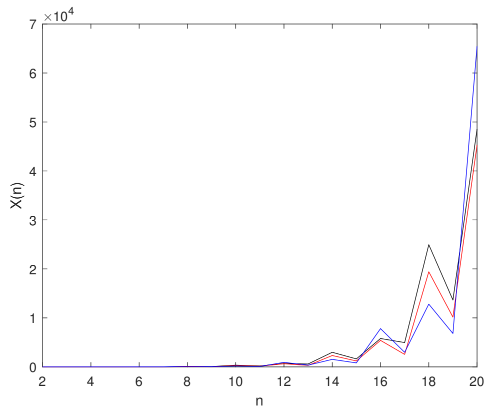

Figure 2: This figure illustrates Unstable of the point $(0, 0)$ of system (1.1)-(1.2) with IC1-IC3.

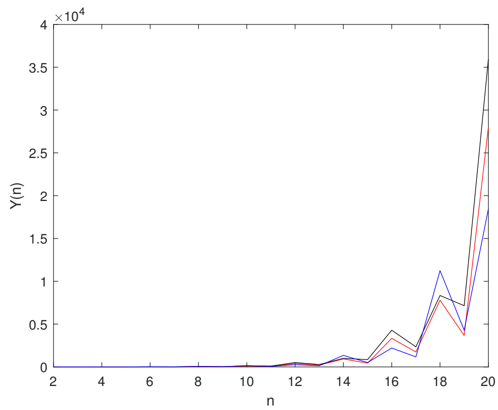

Figure 3: This figure illustrates Unstable of the point $(0, 0)$ of system (1.1)-(1.2) with IC1-IC3.

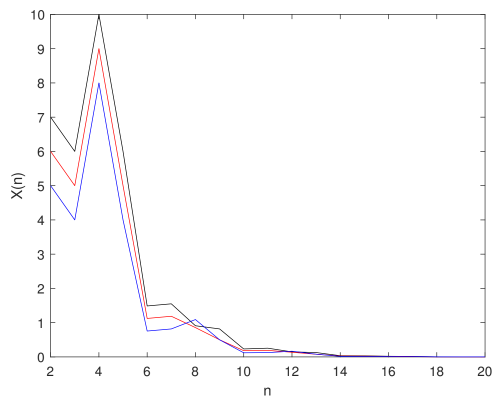

Figure 4: This figure illustrates that $(0, 0)$ is globally stabe for the system (1.1)-(1.2) with IC1-IC3.

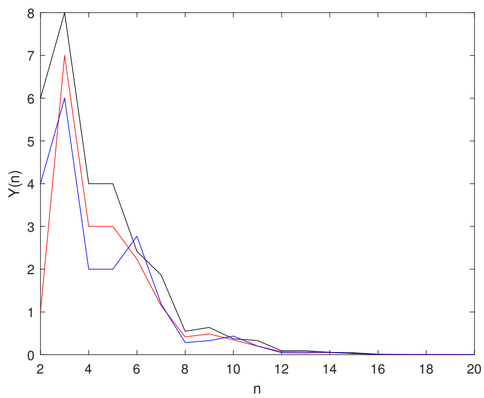

Figure 5: This figure illustrates that $(0,0)$ is globally stabe for the system (1.1)-(1.2) with IC1-IC3.

3. Solutions formulation for system (1.1)-(1.2)

In this part, our goal is to solve two cases of system (1.1)-(1.2) where all initial values are nonzero numbers and $\eta \in {}_0$.

3.1. *Solution formulations for system* $Z_{\eta+1} = Y_{\eta-1} + \frac{Z_{\eta-4}Y_{\eta-1}}{Y_{\eta-2}-Z_{\eta-4}}$, $Y_{\eta+1} = Z_{\eta-1} + \frac{Z_{\eta-1}Y_{\eta-4}}{Z_{\eta-2}-Y_{\eta-4}}$

The solution of the system

$$Z_{\eta+1} = Y_{\eta-1} + \frac{Z_{\eta-4}Y_{\eta-1}}{Y_{\eta-2}-Z_{\eta-4}}, \quad Y_{\eta+1} = Z_{\eta-1} + \frac{Z_{\eta-1}Y_{\eta-4}}{Z_{\eta-2}-Y_{\eta-4}}, \tag{3.1}$$

is investigated in this subsection where all initial conditions are nonzero real arbitrary numbers and $\eta \in {}_0$.

Theorem 3.1. *For* $\eta = 1, 2, \ldots$, *the solution of difference equations system* (3.1), $\{Z_\eta Y_\eta\}_{\eta=1}^{\infty}$, *is expressed as the following equations*

$$\begin{aligned}
Z_{12\eta-11} &= \frac{-Z_{-2}^{\eta-1}Z_{-1}^{\eta-1}Z_{0}^{\eta-1}Y_{-2}^{\eta}Y_{-1}^{\eta}Y_{0}^{\eta-1}}{Z_{-3}^{\eta-1}\left(Y_{-4}-Z_{-2}\right)^{\eta-1}\left(Y_{-2}-Z_{0}\right)^{\eta-1}Y_{-3}^{\eta-1}\left(Z_{-4}-Y_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta-1}},\\
Z_{12\eta-10} &= \frac{-Z_{-1}^{\eta-1}Z_{0}^{\eta-1}Y_{-1}^{\eta}Y_{0}^{\eta}}{Z_{-4}^{\eta-1}\left(Y_{-3}-Z_{-1}\right)^{\eta-1}Z_{-4}^{\eta-1}\left(Z_{-3}-Y_{-1}\right)^{\eta}},\\
Z_{12\eta-9} &= \frac{Z_{-2}^{\eta}Z_{-1}^{\eta}Z_{0}^{\eta-1}Y_{-2}^{\eta-1}Y_{-1}^{\eta-1}Y_{0}^{\eta}}{Z_{-3}^{\eta-1}\left(Z_{-4}-Y_{-2}\right)^{\eta-1}\left(Y_{-2}-Z_{0}\right)^{\eta-1}Y_{-3}^{\eta-1}\left(Y_{-4}-Z_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta}},\\
Z_{12\eta-8} &= \frac{-Z_{-2}Z_{-1}^{\eta}Z_{0}^{\eta}(-1)^{\eta}Y_{-1}^{\eta-1}Y_{0}^{\eta-1}}{Z_{-4}^{\eta-1}\left(Z_{-3}-Y_{-1}\right)^{\eta-1}Y_{-4}^{\eta}\left(Y_{-3}-Z_{-1}\right)^{\eta}},\\
Z_{12\eta-7} &= \frac{Z_{-2}^{\eta-1}Z_{-1}^{\eta}Z_{0}^{\eta}Y_{-2}^{\eta}Y_{0}^{\eta-1}Y_{-1}^{\eta}}{Z_{-3}^{\eta-1}\left(Y_{-4}-Z_{-2}\right)^{\eta-1}\left(Y_{-2}-Z_{0}\right)^{\eta}Y_{-3}^{\eta}\left(Z_{-4}-Y_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta-1}},\\
Z_{12\eta-6} &= \frac{-Z_{-1}^{\eta-1}Z_{0}^{\eta}Y_{-1}^{\eta}Y_{0}^{\eta}}{Z_{-4}^{\eta}\left(Z_{-3}-Y_{-1}\right)^{\eta}Y_{-4}^{\eta-1}\left(Y_{-3}-Z_{-1}\right)^{\eta-1}},\\
Z_{12\eta-5} &= \frac{-Z_{-2}^{\eta}Z_{-1}^{\eta}Z_{0}^{\eta-1}Y_{-2}^{\eta}Y_{-1}^{\eta}Y_{0}^{\eta}}{Z_{-3}^{\eta}\left(Z_{-4}-Y_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta}Y_{-3}^{\eta-1}\left(Y_{-4}-Z_{-2}\right)^{\eta}\left(Y_{-2}-Z_{0}\right)^{\eta-1}},\\
Z_{12\eta-4} &= \frac{Z_{-1}^{\eta}Z_{0}^{\eta}Y_{-1}^{\eta}Y_{0}^{\eta}}{Z_{-4}^{\eta-1}\left(Y_{-3}-Z_{-1}\right)^{\eta}Y_{-4}^{\eta}\left(Z_{-3}-Y_{-1}\right)^{\eta}},
\end{aligned}$$

$$Z_{12\eta-3}=\frac{Z_{-2}^{\eta}Z_{-1}^{\eta}Z_{0}^{\eta}Y_{-2}^{\eta}Y_{-1}^{\eta}Y_{0}^{\eta}}{Z_{-3}^{\eta-1}\left(Y_{-4}-Y_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta}Y_{-3}^{\eta}\left(Z_{-4}-Z_{-2}\right)^{\eta}\left(Y_{-2}-Z_{0}\right)^{\eta}},$$
$$Z_{12\eta-2}=\frac{Z_{-2}Z_{-1}^{\eta}Z_{0}^{\eta}Y_{-1}^{\eta}Y_{0}^{\eta}}{Z_{-4}^{\eta}\left(Z_{-3}-Y_{-1}\right)^{\eta}Y_{-4}^{\eta}\left(Y_{-3}-Z_{-1}\right)^{\eta}},$$
$$Z_{12\eta-1}=\frac{Z_{-2}^{\eta}Z_{-1}^{\eta+1}Z_{0}^{\eta}Y_{-2}^{\eta}Y_{-1}^{\eta}Y_{0}^{\eta}}{Z_{-3}^{\eta}\left(Z_{-4}-Y_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta}Y_{-3}^{\eta}\left(Y_{-4}-Z_{-2}\right)^{\eta}\left(Y_{-2}-Z_{0}\right)^{\eta}},$$
$$Z_{12\eta}=\frac{Z_{-1}^{\eta}Z_{0}^{\eta+1}Y_{-1}^{\eta}Y_{0}^{\eta}}{Z_{-4}^{\eta}\left(Z_{-3}-Y_{-1}\right)^{\eta}Y_{-4}^{\eta}\left(Y_{-3}-Z_{-1}\right)^{\eta}},$$
$$Y_{12\eta-11}=\frac{-Y_{-2}^{\eta-1}Y_{-1}^{\eta-1}Y_{0}^{\eta-1}Z_{-2}^{\eta}Z_{-1}^{\eta}Z_{0}^{\eta-1}}{Y_{-3}^{\eta-1}\left(Z_{-4}-Y_{-2}\right)^{\eta-1}\left(Z_{-2}-Y_{0}\right)^{\eta-1}Z_{-3}^{\eta-1}\left(Y_{-4}-Z_{-2}\right)^{\eta}\left(Y_{-2}-Z_{0}\right)^{\eta-1}},$$
$$Y_{12\eta-10}=\frac{-Y_{-1}^{\eta-1}Y_{0}^{\eta-1}Z_{-1}^{\eta}Z_{0}^{\eta}}{Y_{-4}^{\eta-1}\left(Z_{-3}-Y_{-1}\right)^{\eta-1}Y_{-4}^{\eta-1}\left(Y_{-3}-Z_{-1}\right)^{\eta}},$$
$$Y_{12\eta-9}=\frac{Y_{-2}^{\eta}Y_{-1}^{\eta}Y_{0}^{\eta-1}Z_{-2}^{\eta-1}Z_{-1}^{\eta-1}Z_{0}^{\eta}}{Y_{-3}^{\eta-1}\left(Y_{-4}-Z_{-2}\right)^{\eta-1}\left(Z_{-2}-Y_{0}\right)^{\eta-1}Z_{-3}^{\eta-1}\left(Z_{-4}-Y_{-2}\right)^{\eta}\left(Y_{-2}-Z_{0}\right)^{\eta}},$$
$$Y_{12\eta-8}=\frac{-Y_{-2}Y_{-1}^{\eta}Y_{0}^{\eta}(-1)^{\eta}Z_{-1}^{\eta-1}Z_{0}^{\eta-1}}{Y_{-4}^{\eta-1}\left(Y_{-3}-Z_{-1}\right)^{\eta-1}Z_{-4}^{\eta}\left(Z_{-3}-Y_{-1}\right)^{\eta}},$$
$$Y_{12\eta-7}=\frac{Y_{-2}^{\eta-1}Y_{-1}^{\eta}Y_{0}^{\eta}Z_{-2}^{\eta}Z_{0}^{\eta-1}Z_{-1}^{\eta}}{Y_{-3}^{\eta-1}\left(Z_{-4}-Y_{-2}\right)^{\eta-1}\left(Z_{-2}-Y_{0}\right)^{\eta}Z_{-3}^{\eta}\left(Y_{-4}-Z_{-2}\right)^{\eta}\left(Y_{-2}-Z_{0}\right)^{\eta-1}},$$
$$Y_{12\eta-6}=\frac{-Y_{-1}^{\eta-1}Y_{0}^{\eta}Z_{-1}^{\eta}Z_{0}^{\eta}}{Y_{-4}^{\eta}\left(Y_{-3}-Z_{-1}\right)^{\eta}Z_{-4}^{\eta-1}\left(Z_{-3}-Y_{-1}\right)^{\eta-1}},$$
$$Y_{12\eta-5}=\frac{-Y_{-2}^{\eta}Y_{-1}^{\eta}Y_{0}^{\eta-1}Z_{-2}^{\eta}Z_{-1}^{\eta}Z_{0}^{\eta}}{Y_{-3}^{\eta}\left(Y_{-4}-Z_{-2}\right)^{\eta}\left(Y_{-2}-Z_{0}\right)^{\eta}Z_{-3}^{\eta-1}\left(Z_{-4}-Y_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta-1}},$$
$$Y_{12\eta-4}=\frac{Y_{-1}^{\eta}Y_{0}^{\eta}Z_{-1}^{\eta}Z_{0}^{\eta}}{Y_{-4}^{\eta-1}\left(Z_{-3}-Y_{-1}\right)^{\eta}Z_{-4}^{\eta}\left(Y_{-3}-Z_{-1}\right)^{\eta}},$$
$$Y_{12\eta-3}=\frac{Y_{-2}^{\eta}Y_{-1}^{\eta}Y_{0}^{\eta}Z_{-2}^{\eta}Z_{-1}^{\eta}Z_{0}^{\eta}}{Y_{-3}^{\eta-1}\left(Z_{-4}-Z_{-2}\right)^{\eta}\left(Y_{-2}-Z_{0}\right)^{\eta}Z_{-3}^{\eta}\left(Y_{-4}-Y_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta}},$$
$$Y_{12\eta-2}=\frac{Y_{-2}Y_{-1}^{\eta}Y_{0}^{\eta}Z_{-1}^{\eta}Z_{0}^{\eta}}{Y_{-4}^{\eta}\left(Y_{-3}-Z_{-1}\right)^{\eta}Z_{-4}^{\eta}\left(Z_{-3}-Y_{-1}\right)^{\eta}},$$
$$Y_{12\eta-1}=\frac{Y_{-2}^{\eta}Y_{-1}^{\eta+1}Y_{0}^{\eta}Z_{-2}^{\eta}Z_{-1}^{\eta}Z_{0}^{\eta}}{Y_{-3}^{\eta}\left(Y_{-4}-Z_{-2}\right)^{\eta}\left(Y_{-2}-Z_{0}\right)^{\eta}Z_{-3}^{\eta}\left(Z_{-4}-Y_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta}},$$
$$Y_{12\eta}=\frac{Y_{-1}^{\eta}Y_{0}^{\eta+1}Z_{-1}^{\eta}Z_{0}^{\eta}}{Y_{-4}^{\eta}\left(Y_{-3}-Z_{-1}\right)^{\eta}Z_{-4}^{\eta}\left(Z_{-3}-Y_{-1}\right)^{\eta}},$$

where Z_{-4}, Z_{-3}, Z_{-2}, Z_{-1}, Z_0, Y_{-4}, Y_{-3}, Y_{-2}, Y_{-1}, *and* Z_0 *are initial values.*

Proof. Our result holds for $\eta = 1$. Suppose that $\eta > 0$ and for $\eta - 1$, our assumption holds which is

$$Z_{12\eta-17}=\frac{-Z_{-2}^{\eta-1}Z_{-1}^{\eta-1}Z_{0}^{\eta-2}Y_{-2}^{\eta-1}Y_{-1}^{\eta-1}Y_{0}^{\eta-1}}{Z_{-3}^{\eta-1}\left(Z_{-4}-Y_{-2}\right)^{\eta-1}\left(Z_{-2}-Y_{0}\right)^{\eta-1}Y_{-3}^{\eta-2}\left(Y_{-4}-Z_{-2}\right)^{\eta-2}\left(Y_{-2}-Z_{0}\right)^{\eta-2}},$$
$$Z_{12\eta-16}=\frac{Z_{-1}^{\eta-1}Z_{0}^{\eta-1}Y_{-1}^{\eta-1}Y_{0}^{\eta-1}}{Z_{-4}^{\eta-2}\left(Y_{-3}-Z_{-1}\right)^{\eta-1}Y_{-4}^{\eta-1}\left(Z_{-3}-Y_{-1}\right)^{\eta-1}},$$

$$
\begin{aligned}
Z_{12\eta-15} &= \frac{Z_{-2}^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta} Y_{-2}^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1}}{Z_{-3}^{\eta-2} \left(Y_{-4}-Y_{-2}\right)^{\eta-1} \left(Z_{-2}-Y_0\right)^{\eta-1} Y_{-3}^{\eta-1} \left(Z_{-4}-Z_{-2}\right)^{\eta-1} \left(Y_{-2}-Z_0\right)^{\eta-1}},\\
Z_{12\eta-14} &= \frac{(-1)^{\eta-1} Z_{-2} Z_{-1}^{\eta-1} Z_0^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1}}{Z_{-4}^{\eta-1} \left(Z_{-3}-Y_{-1}\right)^{\eta-} Y_{-4}^{\eta-1} \left(Y_{-3}-Z_{-1}\right)^{\eta-1}},\\
Z_{12\eta-13} &= \frac{Z_{-2}^{\eta-1} Z_{-1}^{1} Z_0^{\eta-1} Y_{-2}^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1}}{Z_{-3}^{\eta-1} \left(Z_{-4}-Y_{-2}\right)^{\eta-1} \left(Z_{-2}-Y_0\right)^{\eta-1} Y_{-3}^{\eta-1} \left(Y_{-4}-Z_{-2}\right)^{\eta-1} \left(Y_{-2}-Z_0\right)^{\eta-1}},\\
Z_{12\eta-12} &= \frac{(-1)^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta} Y_{-1}^{\eta-1} Y_0^{\eta-1}}{Z_{-4}^{\eta-1} \left(Z_{-3}-Y_{-1}\right)^{\eta-1} Y_{-4}^{\eta-1} \left(Y_{-3}-Z_{-1}\right)^{\eta-1}},\\
Y_{12\eta-17} &= \frac{-Y_{-2}^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-2} Z_{-2}^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1}}{Y_{-3}^{\eta-1} \left(Y_{-4}-Z_{-2}\right)^{\eta-1} \left(Y_{-2}-Z_0\right)^{\eta-1} Z_{-3}^{\eta-2} \left(Z_{-4}-Y_{-2}\right)^{\eta-2} \left(Z_{-2}-Y_0\right)^{\eta-2}},\\
Y_{12\eta-16} &= \frac{(-1)^{\eta}-1 Y_{-1}^{\eta-1} Y_0^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1}}{Y_{-4}^{\eta-2} \left(Z_{-3}-Y_{-1}\right)^{\eta-1} Z_{-4}^{\eta-1} \left(Y_{-3}-Z_{-1}\right)^{\eta-2}},\\
Y_{12\eta-15} &= \frac{Y_{-2}^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta} Z_{-2}^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1}}{Y_{-3}^{\eta-2} \left(Z_{-4}-Z_{-2}\right)^{\eta-1} \left(Y_{-2}-Z_0\right)^{\eta-1} Z_{-3}^{\eta-1} \left(Y_{-4}-Y_{-2}\right)^{\eta-1} \left(Z_{-2}-Y_0\right)^{\eta-1}},\\
Y_{12\eta-14} &= \frac{Y_{-2} Y_{-1}^{\eta-1} Y_0^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1}}{Y_{-4}^{\eta-1} \left(Y_{-3}-Z_{-1}\right)^{\eta-} Z_{-4}^{\eta-1} \left(Z_{-3}-Y_{-1}\right)^{\eta-1}},\\
Y_{12\eta-13} &= \frac{Y_{-2}^{\eta-1} Y_{-1}^{1} Y_0^{\eta-1} Z_{-2}^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1}}{Y_{-3}^{\eta-1} \left(Y_{-4}-Z_{-2}\right)^{\eta-1} \left(Y_{-2}-Z_0\right)^{\eta-1} Z_{-3}^{\eta-1} \left(Z_{-4}-Y_{-2}\right)^{\eta-1} \left(Z_{-2}-Y_0\right)^{\eta-1}},\\
Y_{12\eta-12} &= \frac{(-1)^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta} Z_{-1}^{\eta-1} Z_0^{\eta-1}}{Y_{-4}^{\eta-1} \left(Y_{-3}-Z_{-1}\right)^{\eta-1} Z_{-4}^{\eta-1} \left(Z_{-3}-Y_{-1}\right)^{\eta-1}}.
\end{aligned}
$$

Now we find from Eq. (3.1) that

$$
\begin{aligned}
Z_{12\eta-11} &= Y_{12\eta-13} + \frac{Y_{12\eta-13} Z_{12\eta-16}}{Y_{12\eta-14} - Z_{12\eta-16}}\\
&= Y_{12\eta-13} + \frac{\frac{Y_{-2}^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1} Z_{-2}^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1}}{Y_{-3}^{\eta-1}(Y_{-4}-Z_{-2})^{\eta-1}(Y_{-2}-Z_0)^{\eta-1} Z_{-3}^{\eta-1}(Z_{-4}-Y_{-2})^{\eta-1}(Z_{-2}-Y_0)^{\eta-1}}}{\frac{Y_{-2} Y_{-1}^{\eta-1} Y_0^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1}}{Y_{-4}^{\eta-1}(Y_{-3}-Z_{-1})^{\eta-} Z_{-4}^{\eta-1}(Z_{-3}-Y_{-1})^{\eta-1}} - \frac{Z_{-1}^{\eta-1} Z_0^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1}}{Z_{-4}^{\eta-2}(Y_{-3}-Z_{-1})^{\eta-1} Y_{-4}^{\eta-1}(Z_{-3}-Y_{-1})^{\eta-2}}}\\
&\quad \times \frac{Z_{-1}^{\eta-1} Z_0^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1}}{Z_{-4}^{\eta-2}\left(Y_{-3}-Z_{-1}\right)^{\eta-1} Y_{-4}^{\eta-1}\left(Z_{-3}-Y_{-1}\right)^{\eta-2}}\\
&= Y_{12\eta-13} + \frac{\frac{Y_{-2}^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1} Z_{-2}^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1}}{Y_{-3}^{\eta-1}(Y_{-4}-Z_{-2})^{\eta-1}(Y_{-2}-Z_0)^{\eta-1} Z_{-3}^{\eta-1}(Z_{-4}-Y_{-2})^{\eta-1}(Z_{-2}-Y_0)^{\eta-1}}}{\frac{Y_{-2} Y_{-1}^{\eta-1} Y_0^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1} - Z_{-1}^{\eta-1} Z_0^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1} Z_{-4}}{Y_{-4}^{\eta-1}(Y_{-3}-Z_{-1})^{\eta-} Z_{-4}^{\eta-1}(Z_{-3}-Y_{-1})^{\eta-1}}}\\
&\quad \times \frac{Z_{-1}^{\eta-1} Z_0^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1}}{Z_{-4}^{\eta-2}\left(Y_{-3}-Z_{-1}\right)^{\eta-1} Y_{-4}^{\eta-1}\left(Z_{-3}-Y_{-1}\right)^{\eta-2}}\\
&= Y_{12\eta-13} + \frac{\frac{Y_{-2}^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1} Z_{-2}^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1} Z_{-4}}{Y_{-3}^{\eta-1}(Y_{-4}-Z_{-2})^{\eta-1}(Y_{-2}-Z_0)^{\eta-1} Z_{-3}^{\eta-1}(Z_{-4}-Y_{-2})^{\eta-1}(Z_{-2}-Y_0)^{\eta-1}} \times \frac{(-1)^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1}}{1}}{\frac{Y_{-2} Y_{-1}^{\eta-1} Y_0^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1} - Z_{-1}^{\eta-1} Z_0^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1} Z_{-4}}{1}}\\
&= \frac{Y_{-2}^{\eta-1} Y_{-1}^{\eta-1} Y_0^{\eta-1} Z_{-2}^{\eta-1} Z_{-1}^{\eta-1} Z_0^{\eta-1}}{Y_{-3}^{\eta-1}\left(Y_{-4}-Z_{-2}\right)^{\eta-1}\left(Y_{-2}-Z_0\right)^{\eta-1} Z_{-3}^{\eta-1}\left(Z_{-4}-Y_{-2}\right)^{\eta-1}\left(Z_{-2}-Y_0\right)^{\eta-1}}
\end{aligned}
$$

$$-\frac{Y_{-2}^{\eta-1}Z_{-2}^{\eta-1}Z_{0}^{\eta-1}Z_{-1}^{\eta-1}Y_{-1}^{\eta-1}Y_{0}^{\eta-1}Z_{-4}}{Y_{-3}^{\eta-1}\left(Y_{-4}-Z_{-2}\right)^{\eta-1}\left(Y_{-2}-Z_{0}\right)^{\eta-1}Z_{-3}^{\eta-1}\left(Z_{-4}-Y_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta-1}}$$
$$=\frac{Y_{-2}^{\eta-1}Y_{-1}^{\eta-1}Y_{0}^{\eta-1}Z_{-2}^{\eta-1}Z_{-1}^{\eta-1}Z_{0}^{\eta-1}\left(Z_{-4}-Y_{-2}-Z_{-4}\right)}{Y_{-3}^{\eta-1}\left(Y_{-4}-Z_{-2}\right)^{\eta-1}\left(Y_{-2}-Z_{0}\right)^{\eta-1}Z_{-3}^{\eta-1}\left(Z_{-4}-Y_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta-1}}$$
$$=\frac{Y_{-2}^{\eta}Y_{-1}^{\eta-1}Y_{0}^{\eta-1}Z_{-2}^{\eta-1}Z_{-1}^{\eta-1}Z_{0}^{\eta-1}}{Y_{-3}^{\eta-1}\left(Y_{-4}-Z_{-2}\right)^{\eta-1}\left(Y_{-2}-Z_{0}\right)^{\eta-1}Z_{-3}^{\eta-1}\left(Z_{-4}-Y_{-2}\right)^{\eta}\left(Z_{-2}-Y_{0}\right)^{\eta-1}}.$$

All remaining relations can be proved in the same manner. □

Example 3.2. Let the difference system (3.1) with the nonzero initial values $Z_{-4}=1, Z_{-3}=3, Z_{-2}=1, Z_{-1}=3, Z_{0}=1, Y_{-4}=4, Y_{-3}=2, Y_{-2}=4, Y_{-1}=2, Y_{0}=4$ (see Figure 6).

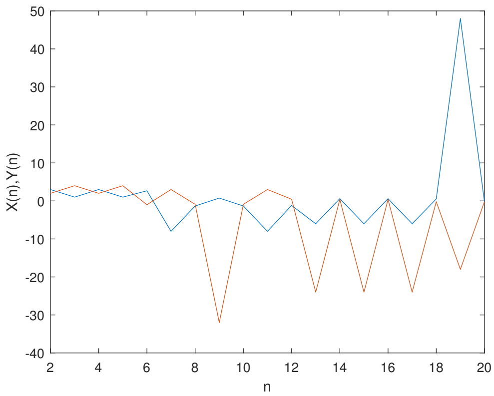

Figure 6: Solution of system (3.1).

3.2. *Solution formulations for system* $Z_{\eta+1}=Y_{\eta-1}+\frac{Z_{\eta-4}Y_{\eta-1}}{Z_{\eta-4}+Y_{\eta-2}}$, $Y_{\eta+1}=Z_{\eta-1}+\frac{Z_{\eta-1}Y_{\eta-4}}{Z_{\eta-2}+Y_{\eta-4}}$

The solution of the following system

$$Z_{\eta+1}=Y_{\eta-1}+\frac{Z_{\eta-4}Y_{\eta-1}}{Z_{\eta-4}+Y_{\eta-2}},\ Y_{\eta+1}=Z_{\eta-1}+\frac{Z_{\eta-1}Y_{\eta-4}}{Z_{\eta-2}+Y_{\eta-4}}, \tag{3.2}$$

is discussed in the subsection where $\eta\in{}_0$ and the initial conditions are real nonzero arbitrary numbers.

Theorem 3.3. *For* $\eta=1,2,\ldots$, *the solution of difference equations system* (3.2), $\{Z_\eta Y_\eta\}_{\eta=1}^{\infty}$, *is expressed as follows*

$$Z_{12\eta-11}=\frac{Y_{-1}\prod_{j=1}^{\eta-1}\left((X_{2j-1}+2W_{2j-1})Y_{-3}+(X_{2j-1}+W_{2j-1})Z_{-1}\right)\left(X_{2j}Z_{-2}+W_{2j}Y_{0}\right)\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Z_{-3}+(X_{2j}+W_{2j})Y_{-1}\right)}{\prod_{j=1}^{\eta-1}\left((X_{2j-1}+W_{2j-1})Y_{-3}+Z_{2j-1}Z_{-1}\right)\left(W_{2j-1}Y_{-2}+\frac{X_{2j-1}}{2}Z_{0}\right)\prod_{j=1}^{\eta-1}\left(W_{2j}Y_{-4}+\frac{X_{2j}}{2}Z_{-2}\right)}$$
$$\times\frac{\prod_{j=1}^{\eta}\left(X_{2j-1}Z_{-4}+W_{2j-1}Y_{-2}\right)\prod_{j=1}^{\eta-1}\left(X_{2j}Y_{-4}+W_{2j}Z_{-2}\right)\left(X_{2j-1}Y_{-2}+W_{2j-1}Z_{0}\right)}{\prod_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})Z_{-3}+Z_{2j}Y_{-1}\right)\left(W_{2j}Z_{-2}+\frac{X_{2j}}{2}Y_{0}\right)\prod_{j=1}^{\eta}\left(W_{2j-1}Z_{-4}+\frac{X_{2j-1}}{2}Y_{-2}\right)},$$

$$Z_{12\eta-10}=\frac{Y_{0}\prod_{j=1}^{\eta-1}\left((X_{2j-1}+2W_{2j-1})Z_{-4}+(X_{2j-1}+W_{2j-1})Y_{-2}\right)\left((X_{j}+2W_{j})_{j}Y_{-2}+(X_{2j-1}+W_{2j-1})Z_{0}\right)\prod_{j=1}^{\eta}\left(X_{2j-1}Z_{-3}+W_{2j-1}Y_{-1}\right)}{\prod_{j=1}^{\eta}\left(W_{2j-1}Z_{-3}+\frac{X_{2j-1}}{2}Y_{-1}\right)\prod_{j=1}^{\eta-1}\left(\left(X_{2j-1}+W_{2j-1}\right)Z_{-4}+Z_{j}Y_{-2}\right)\left((X_{2j-1}+W_{2j-1})Y_{-2}+Z_{2j-1}Z_{0}\right)\prod_{j=1}^{\eta-1}\left(W_{2j}Y_{-3}+\frac{X_{2j}}{2}Z_{-1}\right)}$$
$$\times\frac{\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Y_{-4}+(X_{2j}+W_{2j})Z_{-2}\right)\left((X_{2j}+2W_{2j})_{j}Z_{-2}+(X_{2j}+W_{2j})Y_{0}\right)}{\prod_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})Y_{-4}+Z_{2j}Z_{-2}\right)\left((X_{2j-1}+W_{2j-1})Z_{-2}+Z_{2j-1}Y_{0}\right)}\prod_{j=1}^{\eta-1}\left(X_{2j}Y_{-3}+W_{2j}Z_{-1}\right),$$

$$
\begin{aligned}
Z_{12\eta-9} = {} & \frac{Z_{-1}\prod\limits_{j=1}^{\eta-1}\left((X_{2j-1}+2W_{2j-1})Z_{-3}+(X_{2j-1}+W_{2j-1})Y_{-1}\right)\prod\limits_{j=1}^{\eta-1}\left(X_{2j}Z_{-4}+W_{2j}Y_{-2}\right)\left(X_{2j}Y_{-2}+W_{2j}Z_{0}\right)}{\prod\limits_{j=1}^{\eta-1}\left((X_{2j-1}+W_{2j-1})Z_{-3}+Z_{j}Y_{-1}\right)\prod\limits_{j=1}^{\eta-1}\left(W_{2j}Y_{-2}+\frac{X_{2j}}{2}Z_{0}\right)\left(W_{2j}Z_{-4}+\frac{X_{2j}}{2}Y_{-2}\right)} \\
& \times \frac{\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Y_{-3}+(X_{2j}+W_{2j})Z_{-1}\right)\prod\limits_{j=1}^{\eta}\left(X_{2j-1}Y_{-4}+W_{2j-1}Z_{-2}\right)\left(X_{2j-1}Z_{-2}+W_{2j-1}Y_{0}\right)}{\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})Y_{-3}+Z_{2j}Z_{-1}\right)\prod\limits_{j=1}^{\eta}\left(W_{2j-1}Z_{-2}+\frac{X_{2j-1}}{2}Y_{0}\right)\left(W_{2j-1}Y_{-4}+\frac{X_{2j-1}}{2}Z_{-2}\right)},
\end{aligned}
$$

$$
\begin{aligned}
Z_{12\eta-8} = {} & \frac{Z_{0}\prod\limits_{j=1}^{\eta-1}\left((X_{2j-1}+2W_{2j-1})Z_{-2}+(X_{2j-1}+W_{2j-1})Y_{0}\right)\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Z_{-4}+(X_{2j}+W_{2j})Y_{-2}\right)\left(X_{2j}Z_{-3}+W_{2j}Y_{-1}\right)}{\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})\,Z_{-4}+Z_{2j}Y_{-2}\right)\left(W_{2j}Z_{-3}+\frac{X_{2j}}{2}Y_{-1}\right)\prod\limits_{j=1}^{\eta-1}\left((X_{2j-1}+W_{2j-1})\,Z_{-2}+Z_{2j-1}Y_{0}\right)} \\
& \times \frac{\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Y_{-2}+(X_{2j}+W_{2j})Z_{0}\right)}{\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+W_{2j-1})\,Y_{-4}+Z_{2j-1}Z_{-2}\right)\left(W_{2j-1}Y_{-3}+\frac{X_{2j-1}}{2}Z_{-1}\right)} \\
& \times \frac{\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Y_{-4}+(X_{2j-1}+W_{2j-1})Z_{-2}\right)\prod\limits_{j=1}^{\eta}\left(X_{2j-1}Y_{-3}+W_{2j-1}Z_{-1}\right)}{\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})\,Y_{-2}+Z_{2j}Z_{0}\right)},
\end{aligned}
$$

$$
\begin{aligned}
Z_{12\eta-7} = {} & \frac{Y_{-1}\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Z_{-3}+(X_{2j}+W_{2j})Y_{-1}\right)\prod\limits_{j=1}^{\eta-1}\left(X_{2j}Z_{-2}+W_{2j}Y_{0}\right)\left(X_{2j}Y_{-4}+W_{2j}Z_{-2}\right)}{\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})Z_{-3}+Z_{2j}Y_{-1}\right)\left(W_{2j}Z_{-2}+\frac{X_{2j}}{2}Y_{0}\right)\left(W_{2j}Y_{-4}+\frac{X_{2j}}{2}Z_{-2}\right)} \\
& \times \frac{\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Y_{-3}+(X_{2j-1}+W_{2j-1})Z_{-1}\right)\left(X_{2j-1}Z_{-4}+W_{2j-1}Y_{-2}\right)}{\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+W_{2j-1})Y_{-3}+Z_{2j-1}Z_{-1}\right)\left(W_{2j-1}Y_{-2}+\frac{X_{2j-1}}{2}Z_{0}\right)\left(W_{2j-1}Z_{-4}+\frac{X_{2j-1}}{2}Y_{-2}\right)}\prod_{j=1}^{\eta}\left(X_{2j-1}Y_{-2}+W_{2j-1}Z_{0}\right),
\end{aligned}
$$

$$
\begin{aligned}
Z_{12\eta-6} = {} & \frac{Y_{0}\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Y_{-4}+(X_{2j}+W_{2j})Z_{-2}\right)\left((X_{2j}+2W_{2j})_{j}Z_{-2}+(X_{2j}+W_{2j})Y_{0}\right)\left(X_{2j}Y_{-3}+W_{2j}Z_{-1}\right)}{\prod\limits_{j=1}^{\eta-1}\left(W_{2j}Y_{-3}+\frac{X_{2j}}{2}Z_{-1}\right)\left((X_{2j}+W_{2j})\,Z_{-2}+Z_{2j}Y_{0}\right)\left((X_{2j}+W_{2j})\,Y_{-4}+Z_{2j}Z_{-2}\right)} \\
& \times \frac{\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})_{j}Y_{-2}+(X_{2j-1}+W_{2j-1})Z_{0}\right)\left((X_{2j-1}+2W_{2j-1})Z_{-4}+(X_{2j-1}+W_{2j-1})Y_{-2}\right)}{\prod\limits_{j=1}^{\eta}\left(W_{2j-1}Z_{-3}+\frac{X_{2j-1}}{2}Y_{-1}\right)\left((X_{2j-1}+W_{2j-1})\,Z_{-4}+Z_{2j-1}Y_{-2}\right)} \\
& \times \frac{\left(X_{2j-1}Z_{-3}+W_{2j-1}Y_{-1}\right)}{\left((X_{2j-1}+W_{2j-1})\,Y_{-2}+Z_{2j-1}Z_{0}\right)},
\end{aligned}
$$

$$
\begin{aligned}
Z_{12\eta-5} = {} & \frac{Z_{-1}\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Y_{-3}+(X_{2j}+W_{2j})Z_{-1}\right)\left(X_{2j}Y_{-2}+W_{2j}Z_{0}\right)\prod\limits_{j=1}^{\eta}\left(X_{2j-1}Y_{-4}+W_{2j-1}Z_{-2}\right)}{\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})Y_{-3}+Z_{2j}Z_{-1}\right)\left(W_{2j}Z_{-2}+\frac{X_{2j}}{2}Y_{0}\right)\prod\limits_{j=1}^{\eta}\left(W_{j}Y_{-4}+\frac{X_{j}}{2}Z_{-2}\right)} \\
& \times \frac{\prod\limits_{j=1}^{\eta}\left(X_{2j-1}Z_{-2}+W_{2j-1}Y_{0}\right)\left((X_{2j-1}+2W_{2j-1})Z_{-3}+(X_{2j-1}+W_{2j-1})Y_{-1}\right)\left(X_{2j}Z_{-4}+W_{2j}Y_{-2}\right)}{\prod\limits_{j=1}^{\eta}\left(W_{2j-1}Y_{-2}+\frac{X_{2j-1}}{2}Z_{0}\right)\left((X_{2j-1}+W_{2j-1})Z_{-3}+Z_{2j-1}Y_{-1}\right)},
\end{aligned}
$$

$$Z_{12\eta-4}=\frac{Z_0\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Z_{-4}+(X_{2j}+W_{2j})Y_{-2}\right)\left((X_{2j}+2W_{2j})Y_{-2}+(X_{2j}+W_{2j})Z_0\right)\prod_{j=1}^{\eta}\left(X_{2j-1}Y_{-3}+W_{2j-1}Z_{-1}\right)}{\prod_{j=1}^{\eta}\left(W_{2j-1}Y_{-3}+\frac{X_{2j-1}}{2}Z_{-1}\right)\prod_{j=1}^{\eta-1}\left(\left(X_{2j}+W_{2j}\right)Z_{-4}+Z_{2j}Y_{-2}\right)\left(\left(X_{2j}+W_{2j}\right)Y_{-2}+Z_{2j}Z_0\right)}$$
$$\times\frac{\prod_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})_jZ_{-2}+(X_{2j-1}+W_{2j-1})Y_0\right)\left((X_{2j-1}+2W_{2j-1})Y_{-4}+(X_{2j-1}+W_{2j-1})Z_{-2}\right)}{\prod_{j=1}^{\eta}\left(\left(X_{2j-1}+W_{2j-1}\right)Z_{-2}+Z_{2j-1}Y_0\right)\left(\left(X_{2j-1}+W_{2j-1}\right)Z_{-2}+Z_{2j-1}Y_{-4}\right)}$$
$$\times\frac{\prod_{j=1}^{\eta}\left(X_{2j}Z_{-3}+W_{2j}Z_{-1}\right)}{\prod_{j=1}^{\eta}\left(W_{2j}Z_{-3}+\frac{X_{2j}}{2}Y_{-1}\right)},$$

$$Z_{12\eta-3}=\frac{Y_{-1}\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Z_{-3}+(X_{2j}+W_{2j})Y_{-1}\right)\prod_{j=1}^{\eta}\left(X_{2j-1}Z_{-4}+W_{2j-1}Y_{-2}\right)\left(X_{2j-1}Y_{-2}+W_{2j-1}Z_0\right)}{\prod_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})Z_{-3}+Z_{2j}Y_{-1}\right)\prod_{j=1}^{\eta}\left(W_{2j-1}Y_{-2}+\frac{X_{2j-1}}{2}Z_0\right)\left(W_{2j-1}Z_{-4}+\frac{X_{2j-1}}{2}Y_{-2}\right)}$$
$$\times\frac{\prod_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Y_{-3}+(X_{2j-1}+W_{2j-1})Z_{-1}\right)}{\prod_{j=1}^{\eta}\left((X_{2j-1}+W_{2j-1})Y_{-3}+Z_{2j-1}Z_{-1}\right)}\frac{\prod_{j=1}^{\eta}\left(X_{2j}Y_{-4}+Y_{2j}Z_{-2}\right)}{\prod_{j=1}^{\eta}\left(W_{2j}Z_{-2}+\frac{X_{2j}}{2}Y_0\right)\left(X_{2j}Y_{-4}+W_{2j}Z_{-2}\right)},$$

$$Z_{12\eta-2}=\frac{Y_0\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Z_{-2}+(X_{2j}+W_{2j})Y_0\right)\prod_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Z_{-4}+(X_{2j-1}+W_{2j-1})Y_{-2}\right)\left(X_{2j-1}Z_{-3}+W_{2j-1}Y_{-1}\right)}{\prod_{j=1}^{\eta}\left(\left(X_{2j-1}+W_{2j-1}\right)Z_{-4}+Z_{2j-1}Y_{-2}\right)\left(W_{2j-1}Z_{-3}+\frac{X_{2j-1}}{2}Y_{-1}\right)\prod_{j=1}^{\eta-1}\left(\left(X_{2j}+W_{2j}\right)Z_{-2}+Z_{2j}Y_0\right)}$$
$$\times\frac{\prod_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Y_{-2}+(X_{2j-1}+W_{2j-1})Z_0\right)}{\prod_{j=1}^{\eta}\left(\left(X_{2j-1}+W_{2j-1}\right)Y_{-2}+Z_{2j-1}Z_0\right)}\frac{\prod_{j=1}^{\eta}\left((X_{2j}+2W_{2j})Y_{-4}+(X_{2j}+W_{2j})Z_{-2}\right)\left(X_{2j}Y_{-3}+W_{2j}Z_{-1}\right)}{\prod_{j=1}^{\eta}\left(W_{2j}Y_{-3}+\frac{X_{2j}}{2}Z_{-1}\right)\left(\left(X_{2j}+W_{2j}\right)Y_{-4}+Z_{2j}Z_{-2}\right)},$$

$$Z_{12\eta-1}=\frac{Z_{-1}\prod_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Z_{-3}+(X_{2j-1}+W_{2j-1})Y_{-1}\right)\prod_{j=1}^{\eta}\left(X_{2j-1}Y_{-4}+W_{2j-1}Z_{-2}\right)\left(X_{2j-1}Z_{-2}+W_{2j-1}Y_0\right)}{\prod_{j=1}^{\eta}\left((X_{2j-1}+W_{2j-1})Z_{-3}+Z_{2j-1}Y_{-1}\right)\left(W_{2j-1}Z_{-2}+\frac{X_{2j-1}}{2}Y_0\right)\left(W_{2j-1}Y_{-4}+\frac{X_{2j-1}}{2}Z_{-2}\right)}$$
$$\times\frac{\prod_{j=1}^{\eta}\left((X_{2j}+2W_{2j})Y_{-3}+(X_{2j}+W_{2j})Z_{-1}\right)\left(X_{2j}Z_{-4}+W_{2j}Y_{-2}\right)\left(X_{2j}Y_{-2}+W_{2j}Z_0\right)}{\prod_{j=1}^{\eta}\left((X_{2j}+W_{2j})Y_{-3}+Z_{2j}Z_{-1}\right)\left(W_{2j}Y_{-2}+\frac{X_{2j}}{2}Z_0\right)\left(W_{2j}Z_{-4}+\frac{X_{2j}}{2}Y_{-2}\right)},$$

$$Z_{12\eta}=\frac{Z_0\prod_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Z_{-2}+(X_{2j-1}+W_{2j-1})Y_0\right)\left((X_{2j-1}+2W_{2j-1})Y_{-4}+(X_{2j-1}+W_{2j-1})Z_{-2}\right)\left(X_{2j}Z_{-3}+W_{2j}Y_{-1}\right)}{\prod_{j=1}^{\eta}\left(W_{2j-1}Y_{-3}+\frac{X_{2j-1}}{2}Z_{-1}\right)\left(\left(X_{2j-1}+2W_{2j-1}\right)Y_{-4}+\left(X_{2j-1}+W_{2j-1}\right)Z_{-2}\right)\left(\left(X_{2j-1}+W_{2j-1}\right)Z_{-2}+Z_{2j-1}Y_0\right)}$$
$$\times\frac{\prod_{j=1}^{\eta}\left((X_{2j}+2W_{2j})Y_{-2}+(X_{2j}+W_{2j})Z_0\right)\left((X_{2j}+2W_{2j})Z_{-4}+(X_{2j}+W_{2j})Y_{-2}\right)\left(X_{2j-1}Y_{-3}+W_{2j-1}Z_{-1}\right)}{\prod_{j=1}^{\eta}\left(W_{2j}Z_{-3}+\frac{X_{2j}}{2}Y_{-1}\right)\left(\left(X_{2j}+2W_{2j}\right)Z_{-4}+\left(X_{2j}+W_{2j}\right)Y_{-2}\right)\left(\left(X_{2j}+W_{2j}\right)Y_{-2}+Z_{2j}Z_0\right)},$$

$$Y_{12\eta-11}=\frac{Z_{-1}\prod_{j=1}^{\eta-1}\left((X_{2j-1}+2W_{2j-1})Z_{-3}+(X_{2j-1}+W_{2j-1})Y_{-1}\right)\left(X_{2j}Y_{-2}+W_{2j}Z_0\right)\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Y_{-3}+(X_{2j}+W_{2j})Z_{-1}\right)}{\prod_{j=1}^{\eta-1}\left((X_{2j-1}+W_{2j-1})Z_{-3}+Z_{2j-1}Y_{-1}\right)\left(W_{2j-1}Z_{-2}+\frac{X_{2j-1}}{2}Y_0\right)\prod_{j=1}^{\eta-1}\left(W_{2j}Z_{-4}+\frac{X_{2j}}{2}Y_{-2}\right)}$$

$$\times \frac{\prod_{j=1}^{\eta}\left(X_{2j-1}Y_{-4}+W_{2j-1}Z_{-2}\right)\prod_{j=1}^{\eta-1}\left(X_{2j}Z_{-4}+W_{2j}Y_{-2}\right)\left(X_{2j-1}Z_{-2}+W_{2j-1}Y_{0}\right)}{\prod_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})Y_{-3}+Z_{2j}Z_{-1}\right)\left(W_{2j}Y_{-2}+\frac{X_{2j}}{2}Z_{0}\right)\prod_{j=1}^{\eta}\left(W_{2j-1}Y_{-4}+\frac{X_{2j-1}}{2}Z_{-2}\right)},$$

$$Y_{12\eta-10}=\frac{Z_{0}\prod_{j=1}^{\eta-1}\left((X_{2j-1}+2W_{2j-1})Y_{-4}+(X_{2j-1}+W_{2j-1})Z_{-2}\right)\left((X_{j}+2W_{j})_{j}Z_{-2}+(X_{2j-1}+W_{2j-1})Y_{0}\right)\prod_{j=1}^{\eta}\left(X_{2j-1}Y_{-3}+W_{2j-1}Z_{-1}\right)}{\prod_{j=1}^{\eta}\left(W_{2j-1}Y_{-3}+\frac{X_{2j-1}}{2}Z_{-1}\right)\prod_{j=1}^{\eta-1}\left(\left(X_{2j-1}+W_{2j-1}\right)Y_{-4}+Z_{j}Z_{-2}\right)\left((X_{2j-1}+W_{2j-1})Z_{-2}+Z_{2j-1}Y_{0}\right)\prod_{j=1}^{\eta-1}\left(W_{2j}Z_{-3}+\frac{X_{2j}}{2}Y_{-1}\right)}$$

$$\times \frac{\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Z_{-4}+(X_{2j}+W_{2j})Y_{-2}\right)\left((X_{2j}+2W_{2j})_{j}Y_{-2}+(X_{2j}+W_{2j})Z_{0}\right)}{\prod_{j=1}^{\eta-1}\left(\left(X_{2j}+W_{2j}\right)Z_{-4}+Z_{2j}Y_{-2}\right)\left(\left(X_{2j-1}+W_{2j-1}\right)Y_{-2}+Z_{2j-1}Z_{0}\right)}\prod_{j=1}^{\eta-1}\left(X_{2j}Z_{-3}+W_{2j}Y_{-1}\right),$$

$$Y_{12\eta-9}=\frac{Y_{-1}\prod_{j=1}^{\eta-1}\left((X_{2j-1}+2W_{2j-1})Y_{-3}+(X_{2j-1}+W_{2j-1})Z_{-1}\right)\prod_{j=1}^{\eta-1}\left(X_{2j}Y_{-4}+W_{2j}Z_{-2}\right)\left(X_{2j}Z_{-2}+W_{2j}Y_{0}\right)}{\prod_{j=1}^{\eta-1}\left((X_{2j-1}+W_{2j-1})Y_{-3}+Z_{j}Z_{-1}\right)\prod_{j=1}^{\eta-1}\left(W_{2j}Z_{-2}+\frac{X_{2j}}{2}Y_{0}\right)\left(W_{2j}Y_{-4}+\frac{X_{2j}}{2}Z_{-2}\right)}$$

$$\times \frac{\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Z_{-3}+(X_{2j}+W_{2j})Y_{-1}\right)\prod_{j=1}^{\eta}\left(X_{2j-1}Z_{-4}+W_{2j-1}Y_{-2}\right)\left(X_{2j-1}Y_{-2}+W_{2j-1}Z_{0}\right)}{\prod_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})Z_{-3}+Z_{2j}Y_{-1}\right)\prod_{j=1}^{\eta}\left(W_{2j-1}Y_{-2}+\frac{X_{2j-1}}{2}Z_{0}\right)\left(W_{2j-1}Z_{-4}+\frac{X_{2j-1}}{2}Y_{-2}\right)},$$

$$Y_{12\eta-8}=\frac{Y_{0}\prod_{j=1}^{\eta-1}\left((X_{2j-1}+2W_{2j-1})Y_{-2}+(X_{2j-1}+W_{2j-1})Z_{0}\right)\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Y_{-4}+(X_{2j}+W_{2j})Z_{-2}\right)\left(X_{2j}Y_{-3}+W_{2j}Z_{-1}\right)}{\prod_{j=1}^{\eta-1}\left(\left(X_{2j}+W_{2j}\right)Y_{-4}+Z_{2j}Z_{-2}\right)\left(W_{2j}Y_{-3}+\frac{X_{2j}}{2}Z_{-1}\right)\prod_{j=1}^{\eta-1}\left(\left(X_{2j-1}+W_{2j-1}\right)Y_{-2}+Z_{2j-1}Z_{0}\right)}$$

$$\times \frac{\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Z_{-2}+(X_{2j}+W_{2j})Y_{0}\right)}{\prod_{j=1}^{\eta}\left(\left(X_{2j-1}+W_{2j-1}\right)Z_{-4}+Z_{2j-1}Y_{-2}\right)\left(W_{2j-1}Z_{-3}+\frac{X_{2j-1}}{2}Y_{-1}\right)}$$

$$\times \frac{\prod_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Z_{-4}+(X_{2j-1}+W_{2j-1})Y_{-2}\right)\prod_{j=1}^{\eta}\left(X_{2j-1}Z_{-3}+W_{2j-1}Y_{-1}\right)}{\prod_{j=1}^{\eta-1}\left(\left(X_{2j}+W_{2j}\right)Z_{-2}+Z_{2j}Y_{0}\right)},$$

$$Y_{12\eta-7}=\frac{Z_{-1}\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Y_{-3}+(X_{2j}+W_{2j})Z_{-1}\right)\prod_{j=1}^{\eta-1}\left(X_{2j}Y_{-2}+W_{2j}Z_{0}\right)\left(X_{2j}Z_{-4}+W_{2j}Y_{-2}\right)}{\prod_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})Y_{-3}+Z_{2j}Z_{-1}\right)\left(W_{2j}Y_{-2}+\frac{X_{2j}}{2}Z_{0}\right)\left(W_{2j}Z_{-4}+\frac{X_{2j}}{2}Y_{-2}\right)}$$

$$\times \frac{\prod_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Z_{-3}+(X_{2j-1}+W_{2j-1})Y_{-1}\right)\left(X_{2j-1}Y_{-4}+W_{2j-1}Z_{-2}\right)}{\prod_{j=1}^{\eta}\left((X_{2j-1}+W_{2j-1})Z_{-3}+Z_{2j-1}Y_{-1}\right)\left(W_{2j-1}Z_{-2}+\frac{X_{2j-1}}{2}Y_{0}\right)\left(W_{2j-1}Y_{-4}+\frac{X_{2j-1}}{2}Z_{-2}\right)}\prod_{j=1}^{\eta}\left(X_{2j-1}Z_{-2}+W_{2j-1}Y_{0}\right),$$

$$Y_{12\eta-6}=\frac{Z_{0}\prod_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Z_{-4}+(X_{2j}+W_{2j})Y_{-2}\right)\left((X_{2j}+2W_{2j})_{j}Y_{-2}+(X_{2j}+W_{2j})Z_{0}\right)\left(X_{2j}Z_{-3}+W_{2j}Y_{-1}\right)}{\prod_{j=1}^{\eta-1}\left(W_{2j}Z_{-3}+\frac{X_{2j}}{2}Y_{-1}\right)\left(\left(X_{2j}+W_{2j}\right)Y_{-2}+Z_{2j}Z_{0}\right)\left(\left(X_{2j}+W_{2j}\right)Z_{-4}+Z_{2j}Y_{-2}\right)}$$

$$\times \frac{\prod_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})_{j}Z_{-2}+(X_{2j-1}+W_{2j-1})Y_{0}\right)\left((X_{2j-1}+2W_{2j-1})Y_{-4}+(X_{2j-1}+W_{2j-1})Z_{-2}\right)}{\prod_{j=1}^{\eta}\left(W_{2j-1}Y_{-3}+\frac{X_{2j-1}}{2}Z_{-1}\right)\left(\left(X_{2j-1}+W_{2j-1}\right)Y_{-4}+Z_{2j-1}Z_{-2}\right)}$$

$$\times \frac{\left(X_{2j-1}Y_{-3}+W_{2j-1}Z_{-1}\right)}{\left(\left(X_{2j-1}+W_{2j-1}\right)Z_{-2}+Z_{2j-1}Y_{0}\right)},$$

$$Y_{12\eta-5}=\frac{Y_{-1}\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Z_{-3}+(X_{2j}+W_{2j})Y_{-1}\right)\left(X_{2j}Z_{-2}+W_{2j}Y_{0}\right)\prod\limits_{j=1}^{\eta}\left(X_{2j-1}Z_{-4}+W_{2j-1}Y_{-2}\right)}{\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})Z_{-3}+Z_{2j}Y_{-1}\right)\left(W_{2j}Y_{-2}+\frac{X_{2j}}{2}Z_{0}\right)\prod\limits_{j=1}^{\eta}\left(W_{j}Z_{-4}+\frac{X_{j}}{2}Y_{-2}\right)}$$
$$\times\frac{\prod\limits_{j=1}^{\eta}\left(X_{2j-1}Y_{-2}+W_{2j-1}Z_{0}\right)\left((X_{2j-1}+2W_{2j-1})Y_{-3}+(X_{2j-1}+W_{2j-1})Z_{-1}\right)\left(X_{2j}Y_{-4}+W_{2j}Z_{-2}\right)}{\prod\limits_{j=1}^{\eta}\left(W_{2j-1}Z_{-2}+\frac{X_{2j-1}}{2}Y_{0}\right)\left((X_{2j-1}+W_{2j-1})Y_{-3}+Z_{2j-1}Z_{-1}\right)},$$

$$Y_{12\eta-4}=\frac{Y_{0}\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Y_{-4}+(X_{2j}+W_{2j})Z_{-2}\right)\left((X_{2j}+2W_{2j})Z_{-2}+(X_{2j}+W_{2j})Y_{0}\right)\prod\limits_{j=1}^{\eta}\left(X_{2j-1}Z_{-3}+W_{2j-1}Y_{-1}\right)}{\prod\limits_{j=1}^{\eta}\left(W_{2j-1}Z_{-3}+\frac{X_{2j-1}}{2}Y_{-1}\right)\prod\limits_{j=1}^{\eta-1}\left(\left(X_{2j}+W_{2j}\right)Y_{-4}+Z_{2j}Z_{-2}\right)\left(\left(X_{2j}+W_{2j}\right)Z_{-2}+Z_{2j}Y_{0}\right)}$$
$$\times\frac{\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})_{j}Y_{-2}+(X_{2j-1}+W_{2j-1})Z_{0}\right)\left((X_{2j-1}+2W_{2j-1})Z_{-4}+(X_{2j-1}+W_{2j-1})Y_{-2}\right)}{\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+W_{2j-1})Y_{-2}+Z_{2j-1}Z_{0}\right)\left((X_{2j-1}+W_{2j-1})Y_{-2}+Z_{2j-1}Z_{-4}\right)}\frac{\prod\limits_{j=1}^{\eta}\left(X_{2j}W_{-3}+W_{2j}Y_{-1}\right)}{\prod\limits_{j=1}^{\eta}\left(W_{2j}Y_{-3}+\frac{X_{2j}}{2}Z_{-1}\right)},$$

$$Y_{12\eta-3}=\frac{Z_{-1}\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Y_{-3}+(X_{2j}+W_{2j})Z_{-1}\right)\prod\limits_{j=1}^{\eta}\left(X_{2j-1}Y_{-4}+W_{2j-1}Z_{-2}\right)\left(X_{2j-1}Z_{-2}+W_{2j-1}Y_{0}\right)}{\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+W_{2j})Y_{-3}+Z_{2j}Z_{-1}\right)\prod\limits_{j=1}^{\eta}\left(W_{2j-1}Z_{-2}+\frac{X_{2j-1}}{2}Y_{0}\right)\left(W_{2j-1}Y_{-4}+\frac{X_{2j-1}}{2}Z_{-2}\right)}$$
$$\times\frac{\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Z_{-3}+(X_{2j-1}+W_{2j-1})Y_{-1}\right)}{\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+W_{2j-1})Z_{-3}+Z_{2j-1}Y_{-1}\right)}\frac{\prod\limits_{j=1}^{\eta}\left(X_{2j}Z_{-4}+W_{2j}Y_{-2}\right)}{\prod\limits_{j=1}^{\eta}\left(W_{2j}Y_{-2}+\frac{X_{2j}}{2}Z_{0}\right)\left(X_{2j}Z_{-4}+W_{2j}Y_{-2}\right)},$$

$$Y_{12\eta-2}=\frac{Z_{0}\prod\limits_{j=1}^{\eta-1}\left((X_{2j}+2W_{2j})Y_{-2}+(X_{2j}+W_{2j})Z_{0}\right)\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Y_{-4}+(X_{2j-1}+W_{2j-1})Z_{-2}\right)\left(X_{2j-1}Y_{-3}+W_{2j-1}Z_{-1}\right)}{\prod\limits_{j=1}^{\eta}\left(\left(X_{2j-1}+W_{2j-1}\right)Y_{-4}+Z_{2j-1}Z_{-2}\right)\left(W_{2j-1}Y_{-3}+\frac{X_{2j-1}}{2}Z_{-1}\right)\prod\limits_{j=1}^{\eta-1}\left(\left(X_{2j}+W_{2j}\right)Y_{-2}+Z_{2j}Z_{0}\right)}$$
$$\times\frac{\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Z_{-2}+(X_{2j-1}+W_{2j-1})Y_{0}\right)}{\prod\limits_{j=1}^{\eta}\left(\left(X_{2j-1}+W_{2j-1}\right)Z_{-2}+Z_{2j-1}Y_{0}\right)}\frac{\prod\limits_{j=1}^{\eta}\left((X_{2j}+2W_{2j})Z_{-4}+(X_{2j}+W_{2j})Y_{-2}\right)\left(X_{2j}Z_{-3}+W_{2j}Y_{-1}\right)}{\prod\limits_{j=1}^{\eta}\left(W_{2j}Z_{-3}+\frac{X_{2j}}{2}Y_{-1}\right)\left(\left(X_{2j}+W_{2j}\right)Z_{-4}+Z_{2j}Y_{-2}\right)},$$

$$Y_{12\eta-1}=\frac{Y_{-1}\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Y_{-3}+(X_{2j-1}+W_{2j-1})Z_{-1}\right)\prod\limits_{j=1}^{\eta}\left(X_{2j-1}Z_{-4}+W_{2j-1}Y_{-2}\right)\left(X_{2j-1}Y_{-2}+W_{2j-1}Z_{0}\right)}{\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+W_{2j-1})Y_{-3}+Z_{2j-1}Z_{-1}\right)\left(W_{2j-1}Y_{-2}+\frac{X_{2j-1}}{2}Z_{0}\right)\left(W_{2j-1}Z_{-4}+\frac{X_{2j-1}}{2}Y_{-2}\right)}$$
$$\times\frac{\prod\limits_{j=1}^{\eta}\left((X_{2j}+2W_{2j})Z_{-3}+(X_{2j}+W_{2j})Y_{-1}\right)\left(X_{2j}Y_{-4}+W_{2j}Z_{-2}\right)\left(X_{2j}Z_{-2}+W_{2j}Y_{0}\right)}{\prod\limits_{j=1}^{\eta}\left((X_{2j}+W_{2j})Z_{-3}+Z_{2j}Y_{-1}\right)\left(W_{2j}Z_{-2}+\frac{X_{2j}}{2}Y_{0}\right)\left(W_{2j}Y_{-4}+\frac{X_{2j}}{2}Z_{-2}\right)},$$

$$Y_{12\eta}=\frac{Y_{0}\prod\limits_{j=1}^{\eta}\left((X_{2j-1}+2W_{2j-1})Y_{-2}+(X_{2j-1}+W_{2j-1})Z_{0}\right)\left((X_{2j-1}+2W_{2j-1})Z_{-4}+(X_{2j-1}+W_{2j-1})Y_{-2}\right)\left(X_{2j}Y_{-3}+W_{2j}Z_{-1}\right)}{\prod\limits_{j=1}^{\eta}\left(W_{2j-1}Z_{-3}+\frac{X_{2j-1}}{2}Y_{-1}\right)\left(\left(X_{2j-1}+2W_{2j-1}\right)Z_{-4}+\left(X_{2j-1}+W_{2j-1}\right)Y_{-2}\right)\left(\left(X_{2j-1}+W_{2j-1}\right)Y_{-2}+Z_{2j-1}Z_{0}\right)},$$
$$\times\frac{\prod\limits_{j=1}^{\eta}\left((X_{2j}+2W_{2j})Z_{-2}+(X_{2j}+W_{2j})Y_{0}\right)\left((X_{2j}+2W_{2j})Y_{-4}+(X_{2j}+W_{2j})Z_{-2}\right)\left(X_{2j-1}Z_{-3}+W_{2j-1}Y_{-1}\right)}{\prod\limits_{j=1}^{\eta}\left(W_{2j}Y_{-3}+\frac{X_{2j}}{2}Z_{-1}\right)\left(\left(X_{2j}+2W_{2j}\right)Y_{-4}+\left(X_{2j}+W_{2j}\right)Z_{-2}\right)\left(\left(X_{2j}+W_{2j}\right)Z_{-2}+Z_{2j}Y_{0}\right)},$$

where Z_{-4}, Z_{-3}, Z_{-2}, Z_{-1}, Z_0, Y_{-4}, Y_{-3}, Y_{-2}, Y_{-1}, *and* Z_0 *are nonzero real numbers*, $\{X_\eta\}_{\eta=1}^{\infty}=\{2,10,58,338,\ldots\}$, $\{W_\eta\}_{\eta=1}^{\infty}=\{1,7,41,239,\ldots\},\{Z_\eta\}_{\eta=1}^{\infty}=\{2,12,70,408,\ldots\}$, *such that for* $j \geqslant 2$, $Z_j = \sum_{j=1}^{j} Xj$, $W_j = 2X_{j-1} + 3W_{j-1}$, $\frac{X_j}{2} = X_{j-1}+Z_{j-1}+W_{j-1}$, $W_j = \frac{X_j}{2}+Z_{j-1}$, $X_j = W_j+W_{j-1}+X_{j-1}$, $\frac{X_j}{2}+Z_j+ = +X_j+W_j$, $Z_j = \frac{X_j}{2}+W_j$.

Proof. The proof is similar to the previous theorem. □

Example 3.4. Consider the difference system (3.2) with initial conditions $Z_{-4} = 1$, $Z_{-3} = 3$, $Z_{-2} = 1$, $Z_{-1} = 3$, $Z_0 = 1$, $Y_{-4} = 4$, $Y_{-3} = 2$, $Y_{-2} = 4$, $Y_{-1} = 2$, $Y_0 = 4$ (see Figure 5).

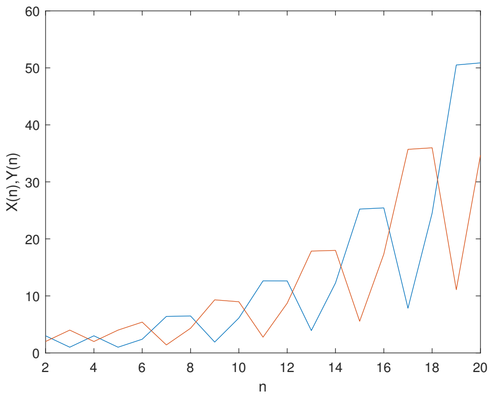

Figure 7: Solution of system (3.2).

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