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Application of an extended Gronwall-type inequality for a (k, ψ)-Hilfer proportional fractional Ambartsumian system via nonlocal integral conditions

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Abstract

This work obtained a novel nonlinear coupled Ambartsumian system subject to nonlocal integral conditions with the (k, ψ)-Hilfer proportional fractional operator. According to this study, we presented an extended (k, ψ)-Hilfer proportional fractional Gronwall inequality. The results concerning existence and uniqueness were established by employing the fixed point theory of Banach's and Krasnosel'skii's types. Furthermore, a variety of stability in the context of Ulam-Hyers-Mittag-Leffler and Ulam-Hyers-Rassias-Mittag-Leffler were investigated. In addition, two numerical examples were demonstrated to illustrate and apply the main results by using a novel numerical technique based on decomposition formula.

Keywords: (k, ψ)-Hilfer proportional fractional operator, Gronwall inequality, existence and uniqueness, fixed point theorem, Ulam-Hyers-Mittag-Leffler stability.

2020 MSC: 26A33, 26D10, 34A08, 34B10, 33E12.

1. Introduction

Differential equations (DEs) in classical calculus are formulated to describe applications in various domains, including physics, chemistry, mechanics, fluid systems, engineering, etc. The well-known equation, often referred to as the Ambartsumian equation, was introduced by Ambartsumian to investigate the qualitative aspects of surface brightness within the Milky way, which explains the absorption of light by interstellar matter [7]. It is introduced in the context of a differential equation with a linear delay (η) as follows:

$$\begin{cases} u'(\tau) = -u(\tau) + \frac{1}{\eta} u\left(\frac{\tau}{\eta}\right), & \eta \in (1, \infty), \\ u(0) = u_0 \in \mathbb{R}. \end{cases} \tag{1.1}$$

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Many studies have investigated the analytical solution of (1.1) using various methods. For example, in 2017, authors [31] used the Daftardar-Gejji and Jafari technique to achieve the analytical solution under a power series form. Next, in 2019, Ebaid et al. [12] applied the adomian decomposition method to establish the approximate solutions. Adel et al. [1] developed the shifted bernoulli matrix method to get an approximate solution in 2020 while Abdullah et al. [3] investigated the approximate solution by using the homotopy perturbation method. Other works that study this equation include [2, 9, 15, 25, 27] and references therein.

A generalization of classical calculus that deals with non-integer (fractional) order and is concerned with the operations of differentiation and integration is called fractional calculus (FC). It is an effective method for comprehending materials and processes' memory and heredity qualities. Diverse forms of fractional derivative and integral operators involving distinct kernel functions, including Riemann-Liouville (RL), Caputo, Hadamard, Katugampola, Hilfer (H), and so on, which can be found in many literary surveys on FC, see the interested works in [8, 18, 21, 22, 24, 34] and references cited therein. Note that the theory of FC has rapidly developed, mainly as a foundation for several applied subjects. Its applications are extensive in many fields, including fractional differential equations (FDEs) and systems to explain the phenomena of various problems in real-world situations.

Over the past few years, the Ambartsumian equation and its generalized forms have gained increasing attention in modeling real-world problems in physics, engineering, and biology. In 2020, Patade et al. [30] investigated a system of fractional-order Ambartsumian equations motivated by astrophysical modeling and provided convergent analytical solutions using the Picard iterative method. This highlights the equation's applicability in understanding radiative transfer phenomena in astronomy. Subsequently, in 2021, Ebaid and Alqahtani [13] proposed a fractional formulation of the Ambartsumian equation via the Caputo derivative and derived exact solutions in terms of Mittag-Leffler functions. Their work emphasized applications to viscoelastic and biological systems, where memory effects are significant, showcasing the equation's biological relevance. In 2023, Ortigueira [29] introduced a Liouville-type fractional extension and demonstrated its applicability in wave propagation and electromagnetic modeling, supported by efficient computational methods such as the Fast Fourier Transform (FFT), making the model suitable for physical and electrical engineering contexts. Most recently, in 2024, Alyoubi and Hristova [4] analyzed a delay differential form of the Ambartsumian equation with variable coefficients. Their results led to closed-form analytical solutions that are particularly relevant for engineering systems with time-delay dynamics. Collectively, these studies establish that the Ambartsumian-type equations are not only of theoretical interest but also serve as powerful tools for simulating complex systems in applied sciences.

The discussion of coupled systems of FDEs has also attracted much attention. We focus on recent studies on coupled systems of FDEs. For example, in 2022, George et al. [16] examined the assumptions for the existence and uniqueness properties to a nonlinear coupled system comprising two pantograph-type equations within the framework of RL and Caputo formulations. In the same year, Jiang and Bai [20] studied generalized Gronwall inequality under a ψ-fractional operator, while Samadi et al. [33] investigated with the qualitative results for the nonlinear coupled (k,ψ)-Hilfer FDEs under the (k,ψ) RL-fractional boundary conditions. In 2023, Theswan et al. [36] studied the qualitative results for the boundary value problem (BVP) of the nonlinear coupled system under a ψ-Hilfer proportional fractional derivative operator (ψ-HPFDO). Ntouyas et al. [28] established the existence and uniqueness properties for a nonlinear coupled system under the HPFDO. Later, in 2023, Elsayed et al. [15] established the qualitative properties of the Ambartsumian equation with the ψ-HPFDO:

$$\begin{cases} {}^{H}_{a}\mathfrak{D}^{\alpha,\beta,\rho;\psi}u(\tau) = h\left(\tau, u(\tau), u\left(\frac{\tau}{\eta}\right)\right), \quad 0<\rho\leqslant 1, \quad \tau\in(a,T], \\ {}_{a}\mathcal{J}^{1-\gamma,\rho;\psi}u(\tau) = \sum_{i=1}^{m}\mu_i u(\theta_i), \quad \mu_i\in\mathbb{R}, \quad a<\theta_i\leqslant T, \quad i=1,\dots,m, \end{cases} \tag{1.2}$$

where ${}^{H}_{a}\mathfrak{D}^{\alpha,\beta,\rho;\psi}$ denotes the ψ-HPFDO of fractional-order $\alpha\in(0,1)$ and type $\beta\in[0,1]$, ${}_{a}\mathcal{J}^{1-\gamma,\rho;\psi}$ represents the ψ-RL-proportional fractional integral operator (ψ-RL-PFIO) of fractional-order $1-\gamma>0$, $\gamma=\alpha+\beta(1-\alpha)$, $h\in\mathcal{C}([a,b]\times\mathbb{R}^2,\mathbb{R})$, $h(\tau,u(\tau),u(\tau/\eta)) = (1/\eta)u(\tau/\eta)-u(\tau)$, and $1<\eta<\infty$. In

2024, Sudsutad et al. [34] developed the (k,ψ)-HPFDO and analyzed its importance properties. They studied the qualitative properties for the higher-order initial value problem (IVP). Additionally, Sudsutad et al. [35] also studied the qualitative results and approximate solutions of the differential equations under (k,ψ)-Caputo-PFDO for the blood alcohol levels model. A Gronwall-type inequality plays a crucial role in studying the qualitative properties of the solution, which was first given and proved by Bellman [10]. Many researchers often apply Gronwall inequality through the fractional differential and integral equations; see the monographs [5, 6, 20, 39] and references cited therein. In addition, Ulam's type stability analysis is consistently applied in various fractional derivative order research regarding the stability of DEs. Ulam initiated it to study the approximation degree of the numerical and exact solutions of these equation [37]. Then, Hyers [19] defined the stability of Ulam-Hyers (UH) type. Later on, the stability of Ulam-Hyers-Rassias (UHR) type was defined by Rassias [32]. Besides, authors [14] have established the stability of Ulam-Hyers-Mittag-Leffler (UHML) for an integral equation under a fractional-order.

Inspired by [15, 34, 35], this study is driven by realizing the significance of each previously described component, which leads us to create the nonlinear coupled system involving a (k,ψ)-HPFDO of the Ambartsumian model under nonlocal integral conditions, which is in the following coupled system:

$$\begin{cases} {}^{H}_{a,k}\mathfrak{D}^{\alpha_1,\beta,\rho;\psi}u(\tau) = f\left(\tau, u(\tau), v\left(\frac{\tau}{\eta_1}\right)\right), & \frac{\alpha_1}{k}\in(0,1),\ k>0,\ \rho\in(0,1],\ \tau\in(a,b], \\ {}^{H}_{a,k}\mathfrak{D}^{\alpha_2,\beta,\rho;\psi}v(\tau) = g\left(\tau, u\left(\frac{\tau}{\eta_2}\right), v(\tau)\right), & \frac{\alpha_2}{k}\in(0,1),\ k>0,\ \rho\in(0,1],\ \tau\in(a,b], \\ \lim\limits_{\tau\to a} {}_{a,k}\mathfrak{J}^{k-\gamma_1,\rho;\psi}u(\tau) = \sum_{i=1}^{m}\mu_i\ {}_{a,k}\mathfrak{J}^{\sigma_i,\rho;\psi}v(\xi_i), & a<\xi_i\leqslant b,\quad i=1,2,\ldots,m, \\ \lim\limits_{\tau\to a} {}_{a,k}\mathfrak{J}^{k-\gamma_2,\rho;\psi}v(\tau) = \sum_{j=1}^{n}\delta_j\ {}_{a,k}\mathfrak{J}^{\lambda_j,\rho;\psi}u(\theta_j), & a<\theta_j\leqslant b,\quad j=1,2,\ldots,n, \end{cases} \tag{1.3}$$

where ${}^{H}_{a,k}\mathfrak{D}^{\alpha_l,\beta,\rho;\psi}$ denotes the (k,ψ)-HPFDO of α_l and $0\leqslant\beta\leqslant1$, ${}_{a,k}\mathfrak{J}^{q,\rho;\psi}$ denotes the (k,ψ)-RL-PFIO of order $q>0$, where $q\in\{k-\gamma_l,\sigma_i,\lambda_j\}$, $\gamma_l := \alpha_l + (k-\alpha_l)\beta$, $\eta_l>1$, μ_i, $\delta_j\in\mathbb{R}$, for $l=1,2$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, f, $g\in\mathcal{C}([a,b]\times\mathbb{R}^2,\mathbb{R})$, where $f(\tau,u(\tau),v(\tau/\eta_1)) = (1/\eta_1)v(\tau/\eta_1)-u(\tau)$, and $g(\tau,u(\tau/\eta_2),v(\tau)) = (1/\eta_2)u(\tau/\eta_2)-v(\tau)$. The advantages of the proposed system (1.3) generation are that it is a novel system of Ambartsumian equations considered under the (k,ψ)-HPFDO with nonlocal integral conditions. So, this BVPs enriches the literature works on BVPs under (k,ψ)-HPFDO, which is the operator that generalizes various well-known fractional derivative operators. Besides, it also expands the scope of the standard Ambartsumian equations study and generates more general types of systems, which include initial and/or boundary conditions.

The main objective is to investigate the qualitative properties of the solutions for the coupled system (1.3) through the application of the well-known fixed point theorems. In addition, we analyze a variety of the stability for the coupled system (1.3). We provide examples to prefer the qualitative results and guarantee the integrity of the conclusions. To achieve the said goal, all compositions will be presented in sections as follows. Section 2 offers a few essential tools associated with the our main results. Section 3 gives an extended Gronwall-type inequality within the framework of the (k,ψ)-PFIO. In Section 4, we examine the existence and uniqueness properties for the coupled system (1.3) via the well-known fixed point theorems. Moreover, we establish the stability analysis of the solution for the coupled system (1.3) under the UHML stable framework, as well as the stability of the Ulam-Hyers-Rassias-Mittag-Leffler (UHRML) type and its generalizations. Some two numerical examples to present the exactness of the qualitative properties are provided in Section 5. A summary report of our main results is offered in the last part.

2. Preliminary materials

This section contains the definitions, lemmas, and tools required for our study, which can be more detailed in [34, 35]. Now, we give the following notation for simple computation in this paper:

$$ {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s) = (\psi(\tau)-\psi(s))^{\frac{\alpha}{k}-1}\exp\left(\frac{\rho-1}{k\rho}(\psi(\tau)-\psi(s))\right).$$

Definition 2.1 ([34]). Let $\alpha > 0$, $k > 0$, $0 < \rho \leqslant 1$, and $u \in L^1([a, b], \mathbb{R})$. Hence, the (k, ψ)-RL-PFIO of α of u is provided as

$$ {}_{a,k}\mathcal{J}^{\alpha,\rho;\psi}u(\tau) = \frac{1}{\rho^{\frac{\alpha}{k}} k\Gamma_k(\alpha)} \int_a^\tau {}^{\rho}_{k}\Psi_\psi^{\frac{\alpha}{k}-1}(\tau, s)u(s)\psi'(s)ds, $$

where $\Gamma_k(u) = \int_0^\infty s^{u-1} \exp\left(-\frac{s^k}{k}\right) ds$, u is an element in complex number $\mathbb{C}$, that is $Re(u) > 0$, and $\Gamma(u) = \Gamma_k(u)$ as $k \to 1$, $\Gamma_k(z + k) = z\Gamma_k(z)$, $\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$, $\Gamma_k(k) = 1$.

Definition 2.2 ([34]). Let $\alpha > 0$, $k > 0$, $0 < \rho \leqslant 1$, $u \in \mathcal{C}([a, b], \mathbb{R})$, $\psi(\tau) \in \mathcal{C}^n([a, b], \mathbb{R})$ with $\psi'(\tau) \neq 0$, and $n = 1, 2, \ldots$, so that $n = \lfloor \alpha/k \rfloor + 1$. Hence, the (k, ψ)-RL-PFDO of α of u is provided as

$$ {}^{\mathbb{RL}}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}u(\tau) = \frac{{}_k\mathfrak{D}^{n,\rho;\psi}}{\rho^{\frac{nk-\alpha}{k}} k\Gamma_k(nk - \alpha)} \int_a^\tau {}^{\rho}_{k}\Psi_\psi^{\frac{nk-\alpha}{k}-1}(\tau, s)\psi'(\tau)u(s)ds = {}_k\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{J}^{nk-\alpha,\rho;\psi}u(\tau)\right), $$

where ${}_k\mathfrak{D}^{1,\rho;\psi}u(\tau) = {}_k\mathfrak{D}^{\rho;\psi}u(\tau) = (1 - \rho)u(\tau) + k\rho\frac{u'(\tau)}{\psi'(\tau)}$ and ${}_k\mathfrak{D}^{n,\rho;\psi} = \underbrace{{}_k\mathfrak{D}^{\rho;\psi}\,{}_k\mathfrak{D}^{\rho;\psi} \cdots {}_k\mathfrak{D}^{\rho;\psi}}_{n \text{ times}}$.

Definition 2.3 ([34]). Let $\alpha > 0$, $k > 0$, $0 < \rho \leqslant 1$, $u \in \mathcal{C}^n([a, b], \mathbb{R})$, $\psi(\tau) \in \mathcal{C}^n([a, b], \mathbb{R})$ with $\psi'(\tau) \neq 0$, and $n \in 1, 2, \ldots$, so that $n = \lfloor \alpha/k \rfloor + 1$. Hence, the (k, ψ)-Caputo-PFDO of α of u is provided as

$$ {}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}uf(\tau) = \frac{1}{\rho^{\frac{nk-\alpha}{k}} k\Gamma_k(nk - \alpha)} \int_a^\tau {}^{\rho}_{k}\Psi_\psi^{\frac{nk-\alpha}{k}-1}(\tau, s)\psi'(s)\left({}_k\mathfrak{D}^{n,\rho;\psi}u(s)\right)ds = {}_{a,k}\mathcal{J}^{nk-\alpha,\rho;\psi}\left({}_k\mathfrak{D}^{n,\rho;\psi}u(\tau)\right). $$

Definition 2.4 ([34]). Let $\alpha > 0$, $k > 0$, $0 < \rho \leqslant 1$, $0 \leqslant \beta \leqslant 1$, $u \in \mathcal{C}^n([a, b], \mathbb{R})$, $\psi(\tau) \in \mathcal{C}^n([a, b], \mathbb{R})$ with $\psi'(\tau) \neq 0$, and $n \in \mathbb{N}$ so that $n = \lfloor \alpha/k \rfloor + 1$. Hence, the (k, ψ)-HPFDO of α and β of u is provided as

$$ {}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}u(\tau) = {}_{a,k}\mathcal{J}^{\beta(nk-\alpha),\rho;\psi}\left({}_k\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{J}^{(1-\beta)(nk-\alpha),\rho;\psi}u(\tau)\right)\right) = \begin{cases} {}^{\mathbb{RL}}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}u(\tau), & \text{if } \beta = 0, \\ {}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}u(\tau), & \text{if } \beta = 1. \end{cases} $$

Next, we give some important properties that are applied in this work.

Lemma 2.5 ([34]). *Assume $\alpha, \delta \in \mathbb{R}^+ \cup \{0\}$, $k, \gamma \in \mathbb{R}^+$, $\rho \in (0, 1]$, $\omega \in \mathbb{R}$, and $\omega/k > -1$. Hence*

(i) ${}_{a,k}\mathcal{J}^{\alpha,\rho;\psi}\left[{}^{\rho}_{k}\Psi_\psi^{\frac{\omega}{k}-1}(\tau, a)\right] = \frac{\Gamma_k(\omega)}{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega+\alpha)}\, {}^{\rho}_{k}\Psi_\psi^{\frac{\omega+\alpha}{k}-1}(\tau, a)$;

(ii) ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}\left[{}^{\rho}_{k}\Psi_\psi^{\frac{\omega}{k}-1}(\tau, a)\right] = \frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}\, {}^{\rho}_{k}\Psi_\psi^{\frac{\omega-\alpha}{k}-1}(\tau, a)$, *particularly,* $m = 0, \ldots, n - 1$ *with* $n = \lfloor \omega/k \rfloor + 1$, *we obtain* ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}\left[{}^{\rho}_{k}\Psi_\psi^{m}(\tau, a)\right] = 0$;

(iii) ${}_{a,k}\mathcal{J}^{\alpha,\rho;\psi}\left({}_{a,k}\mathcal{J}^{\delta,\rho;\psi}u(\tau)\right) = {}_{a,k}\mathcal{J}^{\delta,\rho;\psi}\left({}_{a,k}\mathcal{J}^{\alpha,\rho;\psi}u(\tau)\right) = {}_{a,k}\mathcal{J}^{\delta+\alpha,\rho;\psi}u(\tau)$;

(iv) ${}^{H}_{a,k}\mathfrak{D}^{\omega,\beta,\rho;\psi}\left({}_{a,k}\mathcal{J}^{\gamma,\rho;\psi}u(\tau)\right) = {}_{a,k}\mathcal{J}^{\gamma-\omega,\rho;\psi}u(\tau)$, *where* $n = \lfloor \omega/k \rfloor + 1$, *and* $\gamma > nk$;

(v) ${}_{a,k}\mathcal{J}^{\alpha,\rho;\psi}\left({}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}u(\tau)\right) = u(\tau) - \sum_{i=1}^{n} \frac{{}^{\rho}_{k}\Psi_\psi^{\frac{\gamma}{k}-i}(\tau,a)}{\rho^{\frac{\gamma-ki}{k}}\Gamma_k(\gamma+k-ki)}\left[{}_k\mathfrak{D}^{n-i,\rho;\psi}\left({}_{a,k}\mathcal{J}^{nk-\gamma,\rho;\psi}u(a^+)\right)\right]$, *where* $\gamma = \alpha + \beta(nk - \alpha)$.

It is worth noting that this work is driven by the fact that the (k, ψ)-HPFDO is broader and specializes to various other types of FDOs by fixing the values of ψ, ρ, k, and β. The details of special cases of the (k, ψ)-HPFDO are shown in Table 1. For more information, see [24, 34].

Table 1: The details of special cases of the (k, ψ)-Hilfer-PFDO.

${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}$		$0<\rho<1$		$\rho=1$	
$\psi(\tau)$	β	$k>0$	$k=1$	$k>0$	$k=1$
$\psi(\tau)$	0	(k,ψ)-RL-PFDO	ψ-RL-PFDO	(k,ψ)-RL-FDO	ψ-RL-FDO
	1	(k,ψ)-Caputo-PFDO	ψ-Caputo-PFDO	(k,ψ)-Caputo-FDO	ψ-Caputo-FDO
	β	(k,ψ)-Hilfer-PFDO	ψ-Hilfer-PFDO	(k,ψ)-Hilfer-FDO	ψ-Hilfer-FDO
τ	0	k-RL-PFDO	RL-PFDO	k-RL-FDO	RL-FDO
	1	k-Caputo-PFDO	Caputo-PFDO	k-Caputo-FDO	Caputo-FDO
	β	k-Hilfer-PFDO	Hilfer-PFDO	k-Hilfer-FDO	Hilfer-FDO
τ^{μ}, $\mu>0$	0	k-RL-Katugampola-PFDO	RL-Katugampola-PFDO	k-RL-Katugampola-FDO	RL-Katugampola-FDO
	1	k-Caputo-Katugampola-PFDO	Caputo-Katugampola-PFDO	k-Caputo-Katugampola-FDO	Caputo-Katugampola-FDO
	β	k-Hilfer-Katugampola-PFDO	Hilfer-Katugampola-PFDO	k-Hilfer-Katugampola-FDO	Hilfer-Katugampola-FDO
$\log\tau$	0	k-RL-Hadamard-PFDO	RL-Hadamard-PFDO	k-RL-Hadamard-FDO	RL-Hadamard-FDO
	1	k-Caputo-Hadamard-PFDO	Caputo-Hadamard-PFDO	k-Caputo-Hadamard-FDO	Caputo-Hadamard-FDO
	β	k-Hilfer-Hadamard-PFDO	Hilfer-Hadamard-PFDO	k-Hilfer-Hadamard-FDO	Hilfer-Hadamard-FDO
e^{τ}	0	k-RL-Exponential-PFDO	RL-Exponential-PFDO	k-RL-Exponential-FDO	RL-Exponential-FDO
	1	k-Caputo-Exponential-PFDO	Caputo-Exponential-PFDO	k-Caputo-Exponential-FDO	Caputo-Exponential-FDO
	β	k-Hilfer-Exponential-PFDO	Hilfer-Exponential-PFDO	k-Hilfer-Exponential-FDO	Hilfer-Exponential-FDO

Now, we offer a support result that will help us turn the nonlinear coupled system (1.3) to a fixed-point problem.

Lemma 2.6. *Let $k>0$, $0<\rho\leqslant 1$, $0\leqslant\beta\leqslant 1$, $u, v\in\mathcal{AC}^2([a,b],\mathbb{R})$, $h_l\in\mathcal{AC}([a,b],\mathbb{R})$, $\alpha_l\in(0,1)$, $\gamma_l=\alpha_l+\beta(k-\alpha_l)$, and $\eta_l>1$, $l=1,2$. If $\omega_1\omega_2\neq 1$, hence, a unique solution of*

$$\begin{cases} {}^{H}_{a,k}\mathfrak{D}^{\alpha_1,\beta,\rho;\psi}u(\tau)=h_1(\tau), \qquad {}^{H}_{a,k}\mathfrak{D}^{\alpha_2,\beta,\rho;\psi}v(\tau)=h_2(\tau), \quad \tau\in(a,b], \\ \lim\limits_{\tau\to a}{}_{a,k}\mathfrak{J}^{k-\gamma_1,\rho;\psi}u(\tau)=\sum\limits_{i=1}^{m}\mu_i\,{}_{a,k}\mathfrak{J}^{\sigma_i,\rho;\psi}v(\xi_i), \quad \xi_i\in(a,b],\ i=1,2,\dots,m, \\ \lim\limits_{\tau\to a}{}_{a,k}\mathfrak{J}^{k-\gamma_2,\rho;\psi}v(\tau)=\sum\limits_{j=1}^{n}\delta_j\,{}_{a,k}\mathfrak{J}^{\lambda_j,\rho;\psi}u(\theta_j), \quad \theta_j\in(a,b],\ j=1,2,\dots,n, \end{cases} \tag{2.1}$$

can be presented in the context of the below equations

$$\begin{aligned} u(\tau)&={}_{a,k}\mathfrak{J}^{\alpha_1,\rho;\psi}h_1(\tau)+\frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)} \\ &\quad\times\left[\sum_{i=1}^{m}\mu_i\,{}_{a,k}\mathfrak{J}^{\alpha_2+\sigma_i,\rho;\psi}h_2(\xi_i)+\omega_1\sum_{j=1}^{n}\delta_j\,{}_{a,k}\mathfrak{J}^{\alpha_1+\lambda_j,\rho;\psi}h_1(\theta_j)\right], \\ v(\tau)&={}_{a,k}\mathfrak{J}^{\alpha_2,\rho;\psi}h_2(\tau)+\frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\rho^{\frac{\gamma_2}{k}-1}\Gamma_k(\gamma_2)} \\ &\quad\times\left[\sum_{j=1}^{n}\delta_j\,{}_{a,k}\mathfrak{J}^{\alpha_1+\lambda_j,\rho;\psi}h_1(\theta_j)+\omega_2\sum_{i=1}^{m}\mu_i\,{}_{a,k}\mathfrak{J}^{\alpha_2+\sigma_i,\rho;\psi}h_2(\xi_i)\right], \end{aligned} \tag{2.2}$$

where

$$\omega_1=\sum_{i=1}^{m}\frac{\mu_i\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2+\sigma_i}{k}-1}(\xi_i,a)}{\rho^{\frac{\gamma_2+\sigma_i}{k}-1}\Gamma_k(\gamma_2+\sigma_i)} \quad \text{and} \quad \omega_2=\sum_{j=1}^{n}\frac{\delta_j\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1+\lambda_j}{k}-1}(\theta_j,a)}{\rho^{\frac{\gamma_1+\lambda_j}{k}-1}\Gamma_k(\gamma_1+\lambda_j)}.$$

Proof. Let (u, v) represents a solution to the coupled system (2.1). Applying Lemma 2.5 (v), we get the following results

$$u(\tau)=c_1\frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)}{\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}+{}_{a,k}\mathfrak{J}^{\alpha_1,\rho;\psi}h_1(\tau) \quad \text{and} \quad v(\tau)=c_2\frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)}{\rho^{\frac{\gamma_2}{k}-1}\Gamma_k(\gamma_2)}+{}_{a,k}\mathfrak{J}^{\alpha_2,\rho;\psi}h_2(\tau), \tag{2.3}$$

where $c_l \in \mathbb{R}$, for $l = 1, 2$. Now, taking the (k,ψ)-RL-PFIO of order α_l for $l = 1, 2$, into (2.3), we have

$$ {}_{a,k}\mathcal{J}^{k-\gamma_1,\rho;\psi}u(\tau) = c_1 + {}_{a,k}\mathcal{J}^{\alpha_1+k-\gamma_1,\rho;\psi}h_1(\tau) \quad \text{and} \quad {}_{a,k}\mathcal{J}^{k-\gamma_2,\rho;\psi}v(\tau) = c_2 + {}_{a,k}\mathcal{J}^{\alpha_2+k-\gamma_2,\rho;\psi}h_2(\tau). $$

Similarly, taking the (k,ψ)-RL-PFIO of orders σ_i and λ_j into (2.3), one has

$$ \begin{aligned} {}_{a,k}\mathcal{J}^{\lambda_j,\rho;\psi}u(\tau) &= c_1 \frac{{}_{k}^{\rho}\Psi_{\psi}^{\frac{\gamma_1+\lambda_j}{k}-1}(\tau,a)}{\rho^{\frac{\gamma_1+\lambda_j}{k}-1}\Gamma_k(\gamma_1+\lambda_j)} + {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}h_1(\tau), \\ {}_{a,k}\mathcal{J}^{\sigma_i,\rho;\psi}v(\tau) &= c_2 \frac{{}_{k}^{\rho}\Psi_{\psi}^{\frac{\gamma_2+\sigma_i}{k}-1}(\tau,a)}{\rho^{\frac{\gamma_2+\sigma_i}{k}-1}\Gamma_k(\gamma_2+\sigma_i)} + {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}h_2(\tau). \end{aligned} \tag{2.4} $$

By using the nonlocal integral conditions in the coupled system (2.1) into (2.4), we get

$$ c_1 - c_2 \sum_{i=1}^{m} \frac{\mu_i \, {}_{k}^{\rho}\Psi_{\psi}^{\frac{\gamma_2+\sigma_i}{k}-1}(\xi_i,a)}{\rho^{\frac{\gamma_2+\sigma_i}{k}-1}\Gamma_k(\gamma_2+\sigma_i)} = \sum_{i=1}^{m} \mu_i \, {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}h_2(\xi_i), \tag{2.5} $$

$$ -c_1 \sum_{j=1}^{n} \frac{\delta_j \, {}_{k}^{\rho}\Psi_{\psi}^{\frac{\gamma_1+\lambda_j}{k}-1}(\theta_j,a)}{\rho^{\frac{\gamma_1+\lambda_j}{k}-1}\Gamma_k(\gamma_1+\lambda_j)} + c_2 = \sum_{j=1}^{n} \delta_j \, {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}h_1(\theta_j). \tag{2.6} $$

Solving the system (2.5)-(2.6), we obtain two values

$$ \begin{aligned} c_1 &= \frac{1}{1-\omega_1\omega_2}\left[\sum_{i=1}^{m} \mu_i \, {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}h_2(\xi_i) + \omega_1 \sum_{j=1}^{n} \delta_j \, {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}h_1(\theta_j)\right], \\ c_2 &= \frac{1}{1-\omega_2\omega_2}\left[\sum_{j=1}^{n} \delta_j \, {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}h_1(\theta_j) + \omega_2 \sum_{i=1}^{m} \mu_i \, {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}h_2(\xi_i)\right]. \end{aligned} $$

Substituting two constants c_1 and c_2 into (2.3), we achieve the equations (2.2). The reverse follows by direct calculation. □

When $\beta = 1$, the (k,ψ)-HPFDO reduces to the (k,ψ)-Caputo-PFDO of order α. The following lemma presents an approximation of this operator for $\alpha \in (0, 1]$, which is subsequently applied in Section 5 to solve a nonlinear coupled system of Ambartsumian-type equations.

Lemma 2.7 ([35])**.** *Assume that* $N = 1, 2, \ldots,$ *and* $u \in \mathcal{AC}^2([a,b],\mathbb{R})$*. Let*

$$ \begin{aligned} A_N^{\alpha} &= \frac{1}{\rho^{1-\frac{\alpha}{k}}\Gamma_k(2k-\alpha)} \sum_{i=0}^{N} \frac{\Gamma\left(i+\frac{\alpha}{k}-1\right)}{i!\,\Gamma\left(\frac{\alpha}{k}-1\right)}, \\ B_{N,i}^{\alpha} &= \frac{\Gamma\left(i+\frac{\alpha}{k}-1\right)}{\rho^{1-\frac{\alpha}{k}}(i-1)!\,\Gamma_k(2k-\alpha)\Gamma\left(\frac{\alpha}{k}-1\right)}, \quad i = 1, 2, \ldots, N, \end{aligned} $$

and let $\mathcal{V}_i : [a,b] \to \mathbb{R}$ *be a function, which is given by*

$$ \mathcal{V}_i(\tau) = \int_a^{\tau} (\psi(s)-\psi(a))^{i-1}\psi'(s) e^{\frac{1-\rho}{k\rho}\psi(s)} {}_{k}\mathfrak{D}^{\rho;\psi}u(s)ds, \quad i = 1, 2, \ldots, N. $$

Hence,

$$ \begin{aligned} {}_{a,k}^{C}\mathfrak{D}^{\alpha,\rho;\psi}u(\tau) \approx{}& A_N^{\alpha}(\psi(\tau)-\psi(a))^{1-\frac{\alpha}{k}} {}_{k}\mathfrak{D}^{\rho;\psi}u(\tau) \\ &- e^{\frac{\rho-1}{k\rho}\psi(\tau)} \sum_{i=1}^{N} B_{N,i}^{\alpha}(\psi(\tau)-\psi(a))^{1-\frac{\alpha}{k}-i}\mathcal{V}_i(\tau) + E_{tr}(\tau), \end{aligned} \tag{2.7} $$

where ${}_{k}\mathfrak{D}^{\rho;\psi}u(\tau) = (1-\rho)u(\tau) + k\rho\frac{u'(\tau)}{\psi'(\tau)}$ *and* $\lim\limits_{N\to\infty} E_{tr}(\tau) = 0$, $\tau \in [a,b]$.

Let $\mathcal{C}([a,b],\mathbb{R})$ be the Banach space of all continuous functions u, v from $[a,b]$ to $\mathbb{R}$ supplemented with $\|u\| = \sup_{\tau\in[a,b]}\{|u(\tau)|\}$ and $\|v\| = \sup_{\tau\in[a,b]}\{|v(\tau)|\}$. Next, the weighted space $\mathcal{C}_{\psi}^{1-\frac{\gamma}{k}}([a,b],\mathbb{R})$ of continuous functions u on $[a,b]$ is given by as follows:

$$\mathbb{W} := \mathcal{C}_{\psi}^{1-\frac{\gamma}{k}}([a,b],\mathbb{R}) = \left\{u(\tau) : (a,b] : {}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma}{k}}(\tau,a)u(\tau) \in \mathcal{C}([a,b],\mathbb{R})\right\},$$

supplemented with $\|u\|_{\mathbb{W}} = \sup_{\tau\in[a,b]}\{|{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)u(\tau)|\}$ and $\|v\|_{\mathbb{W}} = \sup_{\tau\in[a,b]}\{|{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau,a)v(\tau)|\}$. Hence, the product space $(\mathbb{W}\times\mathbb{W}, \|(u,v)\|_{\mathbb{W}})$ is the Banach space supplemented with $\|(u,v)\|_{\mathbb{W}} = \|u\|_{\mathbb{W}} + \|v\|_{\mathbb{W}}$.

3. An extended (k,ψ)-proportional fractional Gronwall inequality

Theorem 3.1 (An extended (k,ψ)-proportional fractional Gronwall inequality)**.** *Let* k, $\alpha_1 > 0$, $\alpha_2 > 0$, $\rho \in (0,1]$, *and the function* $\psi \in \mathcal{C}^1([a,b],\mathbb{R})$ *be an increasing such that the property* $\psi'(\tau) \neq 0$, $\tau \in [a,b]$. *Suppose that the following properties hold.*

(H_1) *Let three functions* $u(\tau)$, $v(\tau)$, κ_1, *and* κ_2 *be locally integrable non-negative on* $[a,b]$.

(H_2) *Let the functions* $\phi_i(\tau)$ *be non-negative, non-decreasing, and continuous, which defined on* $\tau\in[a,b]$, *so that* $\phi_i(\tau) \leqslant \phi_i^*$ *with* $\phi_i^* \in \mathbb{R}$, $i = 1,2$.

If

$$\begin{cases} u(\tau) \leqslant \kappa_1(\tau) + \frac{\Gamma_k(\alpha_1)}{k}\phi_1(\tau)\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}v(\tau), \\ v(\tau) \leqslant \kappa_2(\tau) + \frac{\Gamma_k(\alpha_2)}{k}\phi_2(\tau)\,{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}u(\tau), \end{cases} \tag{3.1}$$

then

$$\begin{aligned} u(\tau) \leqslant{}& \kappa_1(\tau) + \frac{\Gamma_k(\alpha_1)}{k}\phi_1(\tau)\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\kappa_2(\tau) + \int_a^\tau \sum_{n=1}^{\infty} \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^n}{\rho^{\frac{n(\alpha_1+\alpha_2)}{k}}k^{2n+1}\Gamma_k(n(\alpha_1+\alpha_2))}\,{}_{k}^{\rho}\Psi_{\psi}^{\frac{n(\alpha_1+\alpha_2)}{k}-1}(\tau,s) \\ &\times\psi'(s)\left(\kappa_1(s) + \frac{\Gamma_k(\alpha_1)}{k}\phi_1(s)\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\kappa_2(s)\right)ds, \end{aligned} \tag{3.2}$$

$$\begin{aligned} v(\tau) \leqslant{}& \kappa_2(\tau) + \frac{\Gamma_k(\alpha_2)}{k}\phi_2(\tau)\,{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}\kappa_1(\tau) + \int_a^\tau \sum_{n=1}^{\infty} \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^n}{\rho^{\frac{n(\alpha_1+\alpha_2)}{k}}k^{2n+1}\Gamma_k(n(\alpha_1+\alpha_2))}\,{}_{k}^{\rho}\Psi_{\psi}^{\frac{n(\alpha_1+\alpha_2)}{k}-1}(\tau,s) \\ &\times\psi'(s)\left(\kappa_2(s) + \frac{\Gamma_k(\alpha_2)}{k}\phi_2(s)\,{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}\kappa_1(s)\right)ds. \end{aligned} \tag{3.3}$$

Proof. Define two operators

$$(\mathcal{A}_1 v)(\tau) = \frac{\phi_1(\tau)}{\rho^{\frac{\alpha_1}{k}}k^2}\int_a^\tau {}_{k}^{\rho}\Psi_{\psi}^{\frac{\alpha_1}{k}-1}(\tau,s)\psi'(s)v(s)ds \quad\text{and}\quad (\mathcal{A}_2 u)(\tau) = \frac{\phi_2(\tau)}{\rho^{\frac{\alpha_2}{k}}k^2}\int_a^\tau {}_{k}^{\rho}\Psi_{\psi}^{\frac{\alpha_2}{k}-1}(\tau,s)\psi'(s)u(s)ds.$$

From (3.1), implies that

$$\begin{cases} u(\tau) \leqslant \kappa_1(\tau) + (\mathcal{A}_1 v)(\tau), \\ v(\tau) \leqslant \kappa_2(\tau) + (\mathcal{A}_2 u)(\tau). \end{cases} \tag{3.4}$$

Using (3.4) with a monotonic property of the operators $\mathcal{A}_1$ and $\mathcal{A}_2$, one has

$$\begin{aligned} u(\tau) &\leqslant \kappa_1(\tau) + (\mathcal{A}_1 v)(\tau) \leqslant \kappa_1(\tau) + (\mathcal{A}_1\kappa_2)(\tau) + (\mathcal{A}_1\mathcal{A}_2 u)(\tau) \\ &\leqslant \kappa_1(\tau) + (\mathcal{A}_1\kappa_2)(\tau) + (\mathcal{A}_1\mathcal{A}_2\kappa_1)(\tau) + (\mathcal{A}_1\mathcal{A}_2\mathcal{A}_1\kappa_2)(\tau) + (\mathcal{A}_1\mathcal{A}_2)^2 u(\tau). \end{aligned}$$

By iterative process, for $m = 1,2,\ldots$, it yields that

$$u(\tau) \leqslant \sum_{n=0}^{m-1}(\mathcal{A}_1\mathcal{A}_2)^n\kappa_1(\tau) + \sum_{n=0}^{m-1}(\mathcal{A}_1\mathcal{A}_2)^n(\mathcal{A}_1\kappa_2)(\tau) + (\mathcal{A}_1\mathcal{A}_2)^m u(\tau), \quad \tau\in[a,b],$$

where $(\mathcal{A}_1\mathcal{A}_2)^0\kappa_1(\tau) = \kappa_1(\tau)$. In the same ways, for $\tau \in [a, b]$, then

$$\nu(\tau) \leqslant \sum_{n=0}^{m-1} (\mathcal{A}_2\mathcal{A}_1)^n\kappa_2(\tau) + \sum_{n=0}^{m-1} (\mathcal{A}_2\mathcal{A}_1)^n(\mathcal{A}_2\kappa_1)(\tau) + (\mathcal{A}_2\mathcal{A}_1)^m\nu(\tau),$$

where $(\mathcal{A}_2\mathcal{A}_1)^0\kappa_2(\tau) = \kappa_2(\tau)$. Subsequently, it will be demonstrated that

$$(\mathcal{A}_1\mathcal{A}_2)^m u(\tau) \leqslant \int_a^\tau \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^m}{\rho^{\frac{m(\alpha_1+\alpha_2)}{k}} k^{2m+1}\Gamma_k(m(\alpha_1+\alpha_2))} {}^{\rho}_{k}\Psi_{\psi}^{\frac{m(\alpha_1+\alpha_2)}{k}-1}(\tau, s)\psi'(s)u(s)ds, \tag{3.5}$$

$$(\mathcal{A}_2\mathcal{A}_1)^m \nu(\tau) \leqslant \int_a^\tau \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^m}{\rho^{\frac{m(\alpha_1+\alpha_2)}{k}} k^{2m+1}\Gamma_k(m(\alpha_1+\alpha_2))} {}^{\rho}_{k}\Psi_{\psi}^{\frac{m(\alpha_1+\alpha_2)}{k}-1}(\tau, s)\psi'(s)\nu(s)ds, \tag{3.6}$$

and $(\mathcal{A}_1\mathcal{A}_2)^m u(\tau) \to 0$, and $(\mathcal{A}_2\mathcal{A}_1)^m\nu(\tau) \to 0$ as $m \to \infty$, $\tau \in [a, b]$. It is straightforward to verify that the inequality (3.5) holds true if $m = 1$ with $\phi_2(\tau)$ is a non-decreasing, $\phi_2(s) \leqslant \phi_2(\tau)$, for any $s \leqslant \tau$, that is,

$$\begin{aligned}
(\mathcal{A}_1\mathcal{A}_2 u)(\tau) &= \mathcal{A}_1(\mathcal{A}_2 u)(\tau) \\
&= \frac{\phi_1(\tau)}{\rho^{\frac{\alpha_1}{k}} k^2} \int_a^\tau {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1}{k}-1}(\tau, s)\psi'(s) \left[\frac{\phi_2(s)}{\rho^{\frac{\alpha_2}{k}} k^2} \int_a^s {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_2}{k}-1}(s, r)\psi'(r)u(r)dr \right] ds \\
&\leqslant \frac{\phi_1(\tau)\phi_2(\tau)}{\rho^{\frac{\alpha_1}{k}}\rho^{\frac{\alpha_2}{k}} k^4} \int_a^\tau \left[\int_r^\tau {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1}{k}-1}(\tau, s) {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_2}{k}-1}(s, r)\psi'(s)ds \right] \psi'(r)u(r)dr \\
&\leqslant \frac{\phi_1(\tau)\phi_2(\tau)}{\rho^{\frac{\alpha_1}{k}}\rho^{\frac{\alpha_2}{k}} k^4} \int_a^\tau \left[\int_r^\tau \big(\psi(\tau) - \psi(s)\big)^{\frac{\alpha_1}{k}-1} \big(\psi(s) - \psi(r)\big)^{\frac{\alpha_2}{k}-1} \psi'(s)ds \right] \\
&\quad \times e^{\frac{\rho-1}{k\rho}(\psi(\tau)-\psi(r))}\psi'(r)u(r)dr.
\end{aligned} \tag{3.7}$$

By setting $\psi(s) - \psi(r) = z(\psi(\tau) - \psi(r))$, the inequality (3.7) reduces to the following result

$$(\mathcal{A}_1\mathcal{A}_2 u)(\tau) = \int_a^\tau \frac{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)}{\rho^{\frac{\alpha_1}{k}}\rho^{\frac{\alpha_2}{k}} k^3\Gamma_k(\alpha_1+\alpha_2)} {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}-1}(\tau, s)\psi'(s)u(s)ds.$$

Assume that the inequality (3.5) is hold if $m = n$, $n = 1, 2, \ldots$, we have

$$(\mathcal{A}_1\mathcal{A}_2)^n u(\tau) \leqslant \int_a^\tau \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^n}{\rho^{\frac{n(\alpha_1+\alpha_2)}{k}} k^{2n+1}\Gamma_k(n(\alpha_1+\alpha_2))} {}^{\rho}_{k}\Psi_{\psi}^{\frac{n(\alpha_1+\alpha_2)}{k}-1}(\tau, s)\psi'(s)u(s)ds.$$

Applying mathematical induction technique, for $m = n + 1$, we get

$$\begin{aligned}
(\mathcal{A}_1\mathcal{A}_2)^{n+1} u(\tau) &= \mathcal{A}_1\mathcal{A}_2(\mathcal{A}_1\mathcal{A}_2)^n u(\tau) \\
&\leqslant \mathcal{A}_1\mathcal{A}_2 \left[\int_a^\tau \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^n}{\rho^{\frac{n(\alpha_1+\alpha_2)}{k}} k^{2n+1}\Gamma_k(n(\alpha_1+\alpha_2))} {}^{\rho}_{k}\Psi_{\psi}^{\frac{n(\alpha_1+\alpha_2)}{k}-1}(\tau, s)\psi'(s)u(s)ds \right] \\
&\leqslant \int_a^\tau \left[\int_a^s \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(s)\phi_2(s)]^n}{\rho^{\frac{n(\alpha_1+\alpha_2)}{k}} k^{2n+1}\Gamma_k(n(\alpha_1+\alpha_2))} {}^{\rho}_{k}\Psi_{\psi}^{\frac{n(\alpha_1+\alpha_2)}{k}-1}(s, r)\psi'(r)u(r)dr \right] \\
&\quad \times \frac{\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)}{\rho^{\frac{\alpha_1}{k}}\rho^{\frac{\alpha_2}{k}} k^3\Gamma_k(\alpha_1+\alpha_2)} {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}-1}(\tau, s)\psi'(s)ds \\
&\leqslant \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^{n+1}}{\rho^{\frac{(n+1)(\alpha_1+\alpha_2)}{k}} k^{2n+4}\Gamma_k(\alpha_1+\alpha_2)\Gamma_k(n(\alpha_1+\alpha_2))} \\
&\quad \times \int_a^\tau \left[\int_r^\tau {}^{\rho}_{k}\Psi_{\psi}^{\frac{(\alpha_1+\alpha_2)}{k}-1}(\tau, s) {}^{\rho}_{k}\Psi_{\psi}^{\frac{n(\alpha_1+\alpha_2)}{k}-1}(s, r)\psi'(s)ds \right] \psi'(r)u(r)dr
\end{aligned}$$

$$
\begin{aligned}
&= \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^{n+1}}{\rho^{\frac{(n+1)(\alpha_1+\alpha_2)}{k}} k^{2n+4}\Gamma_k(\alpha_1+\alpha_2)\Gamma_k(n(\alpha_1+\alpha_2))} \int_a^\tau {}_k^\rho\Psi_\psi^{\frac{(n+1)(\alpha_1+\alpha_2)}{k}-1}(\tau,r) \\
&\quad \times \left[\int_0^1 [1-z]^{\frac{\alpha_1+\alpha_2}{k}-1} z^{\frac{n(\alpha_1+\alpha_2)}{k}-1} dz\right] \psi'(r)u(r)dr \\
&= \int_a^\tau \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^{n+1}}{\rho^{\frac{(n+1)(\alpha_1+\alpha_2)}{k}} k^{2n+3}\Gamma_k((n+1)(\alpha_1+\alpha_2))} {}_k^\rho\Psi_\psi^{\frac{(n+1)(\alpha_1+\alpha_2)}{k}-1}(\tau,r)\psi'(r)u(r)dr.
\end{aligned}
$$

In the similar procedure of (3.6) with $\phi_1(\tau)$ being non-decreasing, $\phi_1(s) \leqslant \phi_1(\tau)$, $s \leqslant \tau$, we have

$$
(\mathcal{A}_2\mathcal{A}_1)^{n+1}v(\tau) \leqslant \int_a^\tau \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^{n+1}}{\rho^{\frac{(n+1)(\alpha_1+\alpha_2)}{k}} k^{2n+3}\Gamma_k((n+1)(\alpha_1+\alpha_2))} {}_k^\rho\Psi_\psi^{\frac{(n+1)(\alpha_1+\alpha_2)}{k}-1}(\tau,r)\psi'(r)v(r)dr.
$$

Now, we prove that $(\mathcal{A}_1\mathcal{A}_2)^m u(\tau) \to 0$ and $(\mathcal{A}_2\mathcal{A}_1)^m v(\tau) \to 0$ as $m \to \infty$ for any $\tau \in [a,b]$. Since ϕ_1, $\phi_2 \in \mathcal{C}([a,b],\mathbb{R})$, thus, by the theorem of Weierstrass's [11, 26], there exist two constants $\phi_1^* > 0$ and $\phi_2^* > 0$ so that $\phi_1(\tau) \leqslant \phi_1^*$ and $\phi_2(\tau) \leqslant \phi_2^*$, for any $\tau \in [a,b]$, then

$$
(\mathcal{A}_1\mathcal{A}_2)^m u(\tau) \leqslant \int_a^\tau \frac{\left[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1^*\phi_2^*\right]^m}{\rho^{\frac{m(\alpha_1+\alpha_2)}{k}} k^{2m+1}\Gamma_k(m(\alpha_1+\alpha_2))} {}_k^\rho\Psi_\psi^{\frac{m(\alpha_1+\alpha_2)}{k}-1}(\tau,s)\psi'(s)u(s)ds.
$$

Consider the infinite series

$$
\sum_{m=1}^{\infty} \frac{\left[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1^*\phi_2^*\right]^m}{\rho^{\frac{m(\alpha_1+\alpha_2)}{k}} k^{2m+1}\Gamma_k(m(\alpha_1+\alpha_2))}. \tag{3.8}
$$

Applying the ratio test technique to (3.8) with the asymptotic approximation [38], we have

$$
\lim_{m\to\infty} \frac{\Gamma_k(m(\alpha_1+\alpha_2))}{k^2\Gamma_k((m+1)(\alpha_1+\alpha_2))} = 0.
$$

Then, (3.8) converges, indicating that

$$
\begin{aligned}
u(\tau) &\leqslant \kappa_1(\tau) + \frac{\phi_1(\tau)}{\rho^{\frac{\alpha_1}{k}}k^2}\int_a^\tau {}_k^\rho\Psi_\psi^{\frac{\alpha_1}{k}-1}(\tau,s)\psi'(s)\kappa_2(s)ds + \int_a^\tau \sum_{n=1}^{\infty} \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^n}{\rho^{\frac{n(\alpha_1+\alpha_2)}{k}} k^{2n+1}\Gamma_k(n(\alpha_1+\alpha_2))} \\
&\quad \times {}_k^\rho\Psi_\psi^{\frac{n(\alpha_1+\alpha_2)}{k}-1}(\tau,s)\psi'(s)\left[\kappa_1(s) + \frac{\phi_1(s)}{\rho^{\frac{\alpha_1}{k}}k^2}\int_a^s {}_k^\rho\Psi_\psi^{\frac{\alpha_1}{k}-1}(s,r)\psi'(r)\kappa_2(r)dr\right]ds.
\end{aligned}
$$

The inequality (3.2) is obtained. Similar process can be used for inequality (3.3). □

Corollary 3.2. *Let* $k, \alpha_i \in \mathbb{R}^+$, $i = 1,2$, $\rho \in (0,1]$, *and* $\psi \in \mathcal{C}^1([a,b],\mathbb{R})$ *be the increasing so that* $\psi'(\tau) \neq 0$, *for any* $\tau \in [a,b]$. *Let the functions* $u(\tau)$, $v(\tau)$, *and* $\kappa_i(\tau)$ *be locally integrable non-negative on* $[a,b]$, *and* $\phi_i(\tau) \equiv b \geqslant 0$ *for any* $i = 1,2$. *If*

$$
\begin{cases}
u(\tau) \leqslant \kappa_1(\tau) + \frac{b\Gamma_k(\alpha_1)}{k}\, {}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}v(\tau), \\
v(\tau) \leqslant \kappa_2(\tau) + \frac{b\Gamma_k(\alpha_2)}{k}\, {}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}u(\tau),
\end{cases}
$$

then,

$$
\begin{aligned}
u(\tau) &\leqslant \kappa_1(\tau) + \frac{b\Gamma_k(\alpha_1)}{k}\, {}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\kappa_2(\tau) + \int_a^\tau \sum_{n=1}^{\infty} \frac{\left[b^2\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\right]^n}{\rho^{\frac{n(\alpha_1+\alpha_2)}{k}} k^{2n+1}\Gamma_k(n(\alpha_1+\alpha_2))} {}_k^\rho\Psi_\psi^{\frac{n(\alpha_1+\alpha_2)}{k}-1}(\tau,s) \\
&\quad \times \left(\kappa_1(s) + \frac{b\Gamma_k(\alpha_1)}{k}\, {}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\kappa_2(s)\right)\psi'(s)ds, \quad \tau \in [a,b],
\end{aligned}
$$

$$\begin{aligned}v(\tau) \leqslant \kappa_2(\tau) + \frac{b\Gamma_k(\alpha_2)}{k} {}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}\kappa_1(\tau) + \int_a^\tau \sum_{n=1}^{\infty} \frac{\left[b^2\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\right]^n}{\rho^{\frac{n(\alpha_1+\alpha_2)}{k}} k^{2n+1}\Gamma_k(n(\alpha_1+\alpha_2))} {}^{\rho}_{k}\Psi_{\psi}^{\frac{n(\alpha_1+\alpha_2)}{k}-1}(\tau,s) \\ \times \left(\kappa_2(s) + \frac{b\Gamma_k(\alpha_2)}{k} {}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}\kappa_1(s)\right)\psi'(s)ds, \quad \tau \in [a,b].\end{aligned}$$

Corollary 3.3. *Based on all conditions in Theorem 3.1, let the functions $\kappa_i(\tau)$ be non-decreasing for $i = 1,2$ and $\tau \in [a,b]$. Hence, we obtain the following inequalities:*

$$\begin{aligned}u(\tau) \leqslant &\left[\kappa_1(\tau) + \frac{(\rho^{\frac{\alpha_1}{k}} k)^{-1}}{\alpha_1}\phi_1(\tau)\kappa_2(\tau){}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1}{k}}(b,a)\right] \\ &\times \mathbb{E}_{k,\alpha_1+\alpha_2,k}\left((\rho^{\frac{\alpha_1+\alpha_2}{k}}k^2)^{-1}\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau){}^{\rho}_{k}\Psi_{\psi}^{\frac{(\alpha_1+\alpha_2)}{k}}(\tau,a)\right),\end{aligned} \tag{3.9}$$

$$\begin{aligned}v(\tau) \leqslant &\left[\kappa_2(\tau) + \frac{(\rho^{\frac{\alpha_2}{k}} k)^{-1}}{\alpha_2}\phi_2(\tau)\kappa_1(\tau){}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_2}{k}}(b,a)\right] \\ &\times \mathbb{E}_{k,\alpha_1+\alpha_2,k}\left((\rho^{\frac{\alpha_1+\alpha_2}{k}}k^2)^{-1}\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau){}^{\rho}_{k}\Psi_{\psi}^{\frac{(\alpha_1+\alpha_2)}{k}}(\tau,a)\right),\end{aligned} \tag{3.10}$$

where $\mathbb{E}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(n\alpha+\beta)}$, $z \in \mathbb{R}$, α, $\beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, *and* $k > 0$.

Proof. By applying (3.2)-(3.3) and the property of $\phi_i(\tau)$ are non-decreasing functions for $\tau \in [a,b]$ with $\kappa_i(s) \leqslant \kappa_i(\tau)$ for $i = 1,2$, then

$$\begin{aligned}u(\tau) &\leqslant \kappa_1(\tau) + \frac{\Gamma_k(\alpha_1)}{k}\phi_1(\tau){}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\kappa_2(\tau) + \int_a^\tau \sum_{n=1}^{\infty} \frac{[\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^n}{\rho^{\frac{n(\alpha_1+\alpha_2)}{k}} k^{2n+1}\Gamma_k(n(\alpha_1+\alpha_2))} {}^{\rho}_{k}\Psi_{\psi}^{\frac{n(\alpha_1+\alpha_2)}{k}-1}(\tau,s) \\ &\quad\times \left(\kappa_1(s) + \frac{\Gamma_k(\alpha_1)}{k}\phi_1(\tau){}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\kappa_2(s)\right)\psi'(s)ds \\ &\leqslant \kappa_1(\tau) + \frac{\Gamma_k(\alpha_1)}{k}\phi_1(\tau)\kappa_2(\tau){}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}[1] + \left(\kappa_1(\tau) + \frac{\Gamma_k(\alpha_1)}{k}\phi_1(\tau)\kappa_2(\tau){}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}[1]\right) \\ &\quad\times \sum_{n=1}^{\infty}\left[k^{-2}\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^n {}_{a,k}\mathcal{J}^{n(\alpha_1+\alpha_2),\rho;\psi}[1] \\ &\leqslant \left[\kappa_1(\tau) + \frac{(\rho^{\frac{\alpha_1}{k}}k)^{-1}}{\alpha_1}\phi_1(\tau)\kappa_2(\tau){}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1}{k}}(\tau,a)\right] \\ &\quad\times\left[1 + \sum_{n=1}^{\infty}\frac{\left[(\rho^{\frac{\alpha_1+\alpha_2}{k}}k^2)^{-1}\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^n}{\Gamma_k(n(\alpha_1+\alpha_2)+k)}{}^{\rho}_{k}\Psi_{\psi}^{\frac{n(\alpha_1+\alpha_2)}{k}}(\tau,a)\right] \\ &= \left[\kappa_1(\tau) + \frac{(\rho^{\frac{\alpha_1}{k}}k)^{-1}}{\alpha_1}\phi_1(\tau)\kappa_2(\tau){}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1}{k}}(\tau,a)\right] \\ &\quad\times\sum_{n=0}^{\infty}\frac{\left[(\rho^{\frac{\alpha_1+\alpha_2}{k}}k^2)^{-1}\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^n {}^{\rho}_{k}\Psi_{\psi}^{\frac{n(\alpha_1+\alpha_2)}{k}}(\tau,a)}{\Gamma_k(n(\alpha_1+\alpha_2)+k)} \\ &\leqslant \left[\kappa_1(\tau) + \frac{(\rho^{\frac{\alpha_1}{k}}k)^{-1}}{\alpha_1}\phi_1(\tau)\kappa_2(\tau){}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1}{k}}(b,a)\right] \\ &\quad\times\mathbb{E}_{k,\alpha_1+\alpha_2,k}\left((\rho^{\frac{\alpha_1+\alpha_2}{k}}k^2)^{-1}\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)\phi_1(\tau)\phi_2(\tau){}^{\rho}_{k}\Psi_{\psi}^{\frac{(\alpha_1+\alpha_2)}{k}}(\tau,a)\right).\end{aligned}$$

The inequality (3.9) is obtained. The case of the inequality (3.10) is a similar process. □

4. Main results

From Lemma 2.1, we provide an operator $\mathcal{F} : \mathbb{W} \times \mathbb{W} \to \mathbb{W} \times \mathbb{W}$ related to the coupled system (1.3) as follows:

$$\mathcal{F}(u,v)(\tau) = \begin{pmatrix} \mathcal{F}_1(u,v)(\tau) \\ \mathcal{F}_2(u,v)(\tau) \end{pmatrix}, \tag{4.1}$$

where

$$\begin{aligned} \mathcal{F}_1(u,v)(\tau) = & \frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)} \Bigg[\sum_{i=1}^{m} \mu_i \, {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi} g\Big(\xi_i, u\Big(\frac{\xi_i}{\eta_2}\Big), v(\xi_i)\Big) \\ & + \omega_1 \sum_{j=1}^{n} \delta_j \, {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi} f\Big(\theta_j, u(\theta_j), v\Big(\frac{\theta_j}{\eta_1}\Big)\Big) \Bigg] + {}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi} f\Big(\tau, u(\tau), v\Big(\frac{\tau}{\eta_1}\Big)\Big), \end{aligned} \tag{4.2}$$

$$\begin{aligned} \mathcal{F}_2(u,v)(\tau) = & \frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\rho^{\frac{\gamma_2}{k}-1}\Gamma_k(\gamma_2)} \Bigg[\sum_{j=1}^{n} \delta_j \, {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi} f\Big(\theta_j, u(\theta_j), v\Big(\frac{\theta_j}{\eta_1}\Big)\Big) \\ & + \omega_2 \sum_{i=1}^{m} \mu_i \, {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi} g\Big(\xi_i, u\Big(\frac{\xi_i}{\eta_2}\Big), v(\xi_i)\Big) \Bigg] + {}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi} g\Big(\tau, u\Big(\frac{\tau}{\eta_2}\Big), v(\tau)\Big). \end{aligned} \tag{4.3}$$

For the ease of computation, we give some symbols

$$\Delta_1^{\ell} := \frac{|\omega_1|\,\Gamma_k(\ell)}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1-k+\alpha_1}{k}}\Gamma_k(\gamma_1)} \sum_{j=1}^{n} \frac{|\delta_j|\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\ell-k+\alpha_1+\lambda_j}{k}}(\theta_j,a)}{\rho^{\frac{\lambda_j}{k}}\Gamma_k(\ell+\alpha_1+\lambda_j)}, \quad \ell \in \{\gamma_1,\gamma_2,k\},$$

$$\Theta_1^{\ell} := \frac{|\omega_2|\,\Gamma_k(\ell)}{|1-\omega_1\omega_2|\,\rho^{\frac{\gamma_2-k+\alpha_2}{k}}\Gamma_k(\gamma_2)} \sum_{i=1}^{m} \frac{|\mu_i|{}^{\rho}_{k}\Psi_{\psi}^{\frac{\ell-k+\alpha_2+\sigma_i}{k}}(\xi_i,a)}{\rho^{\frac{\sigma_i}{k}}\Gamma_k(\ell+\alpha_2+\sigma_i)}, \quad \ell \in \{\gamma_1,\gamma_2,k\},$$

$$\Omega_1^{\ell} := \Delta_1^{\ell} + \frac{\Gamma_k(\ell){}^{\rho}_{k}\Psi_{\psi}^{\frac{\ell+\alpha_1-\gamma_1}{k}}(b,a)}{\rho^{\frac{\alpha_1}{k}}\Gamma_k(\ell+\alpha_1)}, \quad \ell \in \{\gamma_1,\gamma_2,k\},$$

$$\Omega_2^{\ell} := \frac{\Gamma_k(\ell)}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1-k+\alpha_2}{k}}\Gamma_k(\gamma_1)} \sum_{i=1}^{m} \frac{|\mu_i|\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\ell-k+\alpha_2+\sigma_i}{k}}(\xi_i,a)}{\rho^{\frac{\sigma_i}{k}}\Gamma_k(\ell+\alpha_2+\sigma_i)}, \quad \ell \in \{\gamma_1,\gamma_2,k\},$$

$$\Xi_1^{\ell} := \Theta_1^{\ell} + \frac{\Gamma_k(\ell){}^{\rho}_{k}\Psi_{\psi}^{\frac{\ell+\alpha_2-\gamma_2}{k}}(b,a)}{\rho^{\frac{\alpha_2}{k}}\Gamma_k(\ell+\alpha_2)}, \quad \ell \in \{\gamma_1,\gamma_2,k\},$$

$$\Xi_2^{\ell} := \frac{\Gamma_k(\ell)}{|1-\omega_1\omega_2|\,\rho^{\frac{\gamma_2-k+\alpha_1}{k}}\Gamma_k(\gamma_2)} \sum_{j=1}^{n} \frac{|\delta_j|{}^{\rho}_{k}\Psi_{\psi}^{\frac{\ell-k+\alpha_1+\lambda_j}{k}}(\theta_j,a)}{\rho^{\frac{\lambda_j}{k}}\Gamma_k(\ell+\alpha_1+\lambda_j)}, \quad \ell \in \{\gamma_1,\gamma_2,k\}.$$

4.1. Existence and uniqueness results

This part proves the existence and uniqueness of the solutions for the coupled system (1.3) by utilizing Banach's contraction principle and Krasnoselskii's fixed-point theorem.

Theorem 4.1. *Suppose that* $f, g \in \mathcal{C}([a,b] \times \mathbb{R}^2, \mathbb{R})$ *verify the following conditions:*

($\mathcal{H}_1$) *there are two numbers* $\mathcal{L}_{i,f} > 0$ *and* $\mathcal{K}_{i,g} > 0$, $i = 1,2$, *so that*

$$\begin{aligned} |f(\tau,u_1,v_1) - f(\tau,u_2,v_2)| &\leqslant \mathcal{L}_{1,f}\,|u_1-u_2| + \mathcal{L}_{2,f}\,|v_1-v_2|\,, \ (u_i,v_i) \in \mathbb{R}\times\mathbb{R}, \\ |g(\tau,v_1,u_1) - g(\tau,u_2,v_2)| &\leqslant \mathcal{K}_{1,g}\,|u_1-u_2| + \mathcal{K}_{2,g}\,|v_1-v_2|\,, \ (u_i,v_i) \in \mathbb{R}\times\mathbb{R}. \end{aligned}$$

Therefore the coupled system (1.3) *has a unique solution, given that*

$$\sum_{i=1}^{2}\left[(\Omega_1^{\gamma_i}+\Xi_2^{\gamma_i})\mathcal{L}_{i,f}+(\Omega_2^{\gamma_i}+\Xi_1^{\gamma_i})\mathcal{K}_{i,g}\right]<1. \tag{4.4}$$

Proof. Assume that $\sup_{\tau\in[a,b]}\{|f(\tau,0,0)|\}:=\mathcal{F}^*<+\infty$ and $\sup_{\tau\in[a,b]}\{|g(\tau,0,0)|\}:=\mathcal{G}^*<+\infty$. Suppose that $\mathcal{B}_{r_1}:=\{(u,v)\in\mathbb{W}\times\mathbb{W}:\|(u,v)\|_{\mathbb{W}}\leqslant r_1\}$ such that

$$r_1\geqslant\frac{(\Omega_1^k+\Xi_2^k)\mathcal{F}^*+(\Omega_2^k+\Xi_1^k)\mathcal{G}^*}{1-\sum_{i=1}^2[(\Omega_1^{\gamma_i}+\Xi_2^{\gamma_i})\mathcal{L}_{i,f}+(\Omega_2^{\gamma_i}+\Xi_1^{\gamma_i})\mathcal{K}_{i,g}]}>0.$$

Notice that $\mathcal{B}_{r_1}$ is a bounded, closed, and convex subset of $\mathbb{W}\times\mathbb{W}$.

Step 1. We show that $\mathcal{F}\mathcal{B}_{r_1}\subset\mathcal{B}_{r_1}$. Applying $\mathbb{H}_1$, for $(u,v)\in\mathcal{B}_{r_1}$, it follows

$$\begin{aligned}\left|f\left(\tau,u(\tau),v\left(\frac{\tau}{\eta_1}\right)\right)\right|&\leqslant\left|f\left(\tau,u(\tau),v\left(\frac{\tau}{\eta_1}\right)\right)-f(\tau,0,0)\right|+|f(\tau,0,0)|\\&\leqslant\mathcal{L}_{1,f}|u(\tau)|+\mathcal{L}_{2,f}\left|v\left(\frac{\tau}{\eta_1}\right)\right|+\mathcal{F}^*\\&\leqslant\mathcal{L}_{1,f}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)\|u\|_{\mathbb{W}}+\mathcal{L}_{2,f}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)\|v\|_{\mathbb{W}}+\mathcal{F}^*,\\\left|g\left(\tau,u\left(\frac{\tau}{\eta_2}\right),v(\tau)\right)\right|&\leqslant\left|g\left(\tau,u\left(\frac{\tau}{\eta_2}\right),v(\tau)\right)-g(\tau,0,0)\right|+|g(\tau,0,0)|\\&\leqslant\mathcal{K}_{1,g}\left|u\left(\frac{\tau}{\eta_2}\right)\right|+\mathcal{K}_{2,g}|v(\tau)|+\mathcal{G}^*\\&\leqslant\mathcal{K}_{1,g}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)\|u\|_{\mathbb{W}}+\mathcal{K}_{2,g}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)\|v\|_{\mathbb{W}}+\mathcal{G}^*.\end{aligned}\tag{4.5}$$

By applying Lemma 2.5 with (4.5), we obtain

$$\begin{aligned}&\left|{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\mathcal{F}_1(u,v)(\tau)\right|\\&\leqslant\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left[\sum_{i=1}^{m}|\mu_i|\,{}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}\left|g\left(\xi_i,u\left(\frac{\xi_i}{\eta_2}\right),v(\xi_i)\right)\right|\right.\\&\quad\left.+|\omega_1|\sum_{j=1}^{n}|\delta_j|\,{}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}\left|f\left(\theta_j,u(\theta_j),v\left(\frac{\theta_j}{\eta_1}\right)\right)\right|\right]\\&\quad+{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left|f\left(\tau,u(\tau),v\left(\frac{\tau}{\eta_1}\right)\right)\right|\\&\leqslant\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left[\sum_{i=1}^{m}|\mu_i|\,{}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}\left(\left|g\left(\xi_i,u\left(\frac{\xi_i}{\eta_2}\right),v(\xi_i)\right)-g(\xi_i,0,0)\right|+|g(\xi_i,0,0)|\right)\right.\\&\quad\left.+|\omega_1|\sum_{j=1}^{n}|\delta_j|\,{}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}\left(\left|f\left(\theta_j,u(\theta_j),v\left(\frac{\theta_j}{\eta_1}\right)\right)-f(\theta_j,0,0)\right|+|f(\theta_j,0,0)|\right)\right]\\&\quad+{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left(\left|f\left(b,u(b),v\left(\frac{b}{\eta_1}\right)\right)-f(b,0,0)\right|+|f(b,0,0)|\right)\\&\leqslant\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left[\sum_{i=1}^{m}|\mu_i|\,{}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}\left(\mathcal{K}_{1,g}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\xi_i,a)\|u\|_{\mathbb{W}}\right.\right.\\&\quad\left.+\mathcal{K}_{2,g}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\xi_i,a)\|v\|_{\mathbb{W}}+\mathcal{G}^*\right)+|\omega_1|\sum_{j=1}^{n}|\delta_j|\,{}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}\left(\mathcal{L}_{1,f}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\theta_j,a)\|u\|_{\mathbb{W}}\right.\end{aligned}$$

$$
\begin{aligned}
&\left.\left.+\mathcal{L}_{2,f}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\theta_j,a)\|v\|_{\mathbb{W}}+\mathcal{F}^*\right)\right]+{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\Big(\mathcal{L}_{1,f}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)\|u\|_{\mathbb{W}}\\
&\quad+\mathcal{L}_{2,f}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)\|v\|_{\mathbb{W}}+\mathcal{F}^*\Big)\\
&\leqslant\Bigg[\mathcal{L}_{1,f}\Bigg(\frac{|\omega_1|\,\Gamma_k(\gamma_1)}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1-k+\alpha_1}{k}}\Gamma_k(\gamma_1)}\sum_{j=1}^{n}\frac{|\delta_j|\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1-k+\alpha_1+\lambda_j}{k}}(\theta_j,a)}{\rho^{\frac{\lambda_j}{k}}\Gamma_k(\gamma_1+\alpha_1+\lambda_j)}+\frac{\Gamma_k(\gamma_1)\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1}{k}}(b,a)}{\rho^{\frac{\alpha_1}{k}}\Gamma_k(\gamma_1+\alpha_1)}\Bigg)\\
&\quad+\mathcal{K}_{1,g}\frac{\Gamma_k(\gamma_1)}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1-k+\alpha_2}{k}}\Gamma_k(\gamma_1)}\sum_{i=1}^{m}\frac{|\mu_i|\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1-k+\alpha_2+\sigma_i}{k}}(\xi_i,a)}{\rho^{\frac{\sigma_i}{k}}\Gamma_k(\gamma_1+\alpha_2+\sigma_i)}\Bigg]\|u\|_{\mathbb{W}}\\
&\quad+\Bigg[\mathcal{L}_{2,f}\Bigg(\frac{|\omega_1|\,\Gamma_k(\gamma_2)}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1-k+\alpha_1}{k}}\Gamma_k(\gamma_1)}\sum_{j=1}^{n}\frac{|\delta_j|\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2-k+\alpha_1+\lambda_j}{k}}(\theta_j,a)}{\rho^{\frac{\lambda_j}{k}}\Gamma_k(\gamma_2+\alpha_1+\lambda_j)}+\frac{\Gamma_k(\gamma_2)\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2+\alpha_1-\gamma_1}{k}}(b,a)}{\rho^{\frac{\alpha_1}{k}}\Gamma_k(\gamma_2+\alpha_1)}\Bigg)\\
&\quad+\mathcal{K}_{2,g}\frac{\Gamma_k(\gamma_2)}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1-k+\alpha_2}{k}}\Gamma_k(\gamma_1)}\sum_{i=1}^{m}\frac{|\mu_i|\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2-k+\alpha_2+\sigma_i}{k}}(\xi_i,a)}{\rho^{\frac{\sigma_i}{k}}\Gamma_k(\gamma_2+\alpha_2+\sigma_i)}\Bigg]\|v\|_{\mathbb{W}}\\
&\quad+\mathcal{F}^*\Bigg(\frac{|\omega_1|}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1-k+\alpha_1}{k}}\Gamma_k(\gamma_1)}\sum_{j=1}^{n}\frac{|\delta_j|\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1+\lambda_j}{k}}(\theta_j,a)}{\rho^{\frac{\lambda_j}{k}}\Gamma_k(\alpha_1+\lambda_j+k)}+\frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{k+\alpha_1-\gamma_1}{k}}(b,a)}{\rho^{\frac{\alpha_1}{k}}\Gamma_k(k+\alpha_1)}\Bigg)\\
&\quad+\mathcal{G}^*\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1-k+\alpha_2}{k}}\Gamma_k(\gamma_1)}\sum_{i=1}^{m}\frac{|\mu_i|\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_2+\sigma_i}{k}}(\xi_i,a)}{\rho^{\frac{\sigma_i}{k}}\Gamma_k(k+\alpha_2+\sigma_i)}\\
&\leqslant\left[\mathcal{L}_{1,f}\Omega_1^{\gamma_1}+\mathcal{K}_{1,g}\Omega_2^{\gamma_1}\right]\|u\|_{\mathbb{W}}+\left[\mathcal{L}_{2,f}\Omega_1^{\gamma_2}+\mathcal{K}_{2,g}\Omega_2^{\gamma_2}\right]\|v\|_{\mathbb{W}}+\mathcal{F}^*\Omega_1^{k}+\mathcal{G}^*\Omega_2^{k}.
\end{aligned}
$$

Similarly, we obtain the following result

$$
\left|{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau,a)\mathcal{F}_2(u,v)(\tau)\right|\leqslant\left[\mathcal{L}_{1,f}\Xi_2^{\gamma_1}+\mathcal{K}_{1,g}\Xi_1^{\gamma_1}\right]\|u\|_{\mathbb{W}}+\left[\mathcal{L}_{2,f}\Xi_2^{\gamma_2}+\mathcal{K}_{2,g}\Xi_1^{\gamma_2}\right]\|v\|_{\mathbb{W}}+\mathcal{F}^*\Xi_2^{k}+\mathcal{G}^*\Xi_1^{k}.
$$

Then,

$$
\begin{aligned}
\|\mathcal{F}(u,v)(\tau)\|_{\mathbb{W}}&=\|\mathcal{F}_1(u,v)(\tau)\|_{\mathbb{W}}+\|\mathcal{F}_2(u,v)(\tau)\|_{\mathbb{W}}\\
&\leqslant\left[\mathcal{L}_{1,f}\Omega_1^{\gamma_1}+\mathcal{K}_{1,g}\Omega_2^{\gamma_1}\right]\|u\|_{\mathbb{W}}+\left[\mathcal{L}_{2,f}\Omega_1^{\gamma_2}+\mathcal{K}_{2,g}\Omega_2^{\gamma_2}\right]\|v\|_{\mathbb{W}}+\mathcal{F}^*\Omega_1^{k}+\mathcal{G}^*\Omega_2^{k}\\
&\quad+\left[\mathcal{L}_{1,f}\Xi_2^{\gamma_1}+\mathcal{K}_{1,g}\Xi_1^{\gamma_1}\right]\|u\|_{\mathbb{W}}+\left[\mathcal{L}_{2,f}\Xi_2^{\gamma_2}+\mathcal{K}_{2,g}\Xi_1^{\gamma_2}\right]\|v\|_{\mathbb{W}}+\mathcal{F}^*\Xi_2^{k}+\mathcal{G}^*\Xi_1^{k}\\
&\leqslant r_1\sum_{i=1}^{2}\left[(\Omega_1^{\gamma_i}+\Xi_2^{\gamma_i})\mathcal{L}_{i,f}+(\Omega_2^{\gamma_i}+\Xi_1^{\gamma_i})\mathcal{K}_{i,g}\right]+(\Omega_1^{k}+\Xi_2^{k})\mathcal{F}^*+(\Omega_2^{k}+\Xi_1^{k})\mathcal{G}^*\leqslant r_1,
\end{aligned}
$$

which implies that $\mathcal{F}\mathcal{B}_{r_1}\subset\mathcal{B}_{r_1}$.

Step 2. We show that $\mathcal{F}$ is a contraction. For every $(u_1,v_1),(u_2,v_2)\in\mathbb{W}\times\mathbb{W}$ and $\tau\in[a,b]$, it follows that

$$
|f(\tau,u_1,v_1)-f(\tau,u_2,v_2)|\leqslant\mathcal{L}_{1,f}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)\,\|u_1-u_2\|_{\mathbb{W}}+\mathcal{L}_{2,f}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)\,\|v_1-v_2\|_{\mathbb{W}},\tag{4.6}
$$

$$
|g(\tau,u_1,v_1)-g(\tau,u_2,v_2)|\leqslant\mathcal{K}_{1,g}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)\,\|u_1-u_2\|_{\mathbb{W}}+\mathcal{K}_{2,g}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)\,\|v_1-v_2\|_{\mathbb{W}}.\tag{4.7}
$$

From Lemma 2.5 with (4.6)-(4.7), it yields that

$$
\left|{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\left(\mathcal{F}_1(u_1,v_1)(\tau)-\mathcal{F}_1(u_2,v_2)(\tau)\right)\right|
$$

$$
\begin{aligned}
&\leqslant \frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(k\gamma_1)} \left[\sum_{i=1}^{m} |\mu_i|_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi} \left|g\left(\xi_i, u_1\left(\frac{\xi_i}{\eta_2}\right), v_1(\xi_i)\right) - g\left(\xi_i, u_2\left(\frac{\xi_i}{\eta_2}\right), v_2(\xi_i)\right)\right|\right. \\
&\quad \left. + |\omega_1| \sum_{j=1}^{n} |\delta_j|_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi} \left|f\left(\theta_j, u_1(\theta_j), v_1\left(\frac{\theta_j}{\eta_1}\right)\right) - f\left(\theta_j, x_2(\theta_j), y_2\left(\frac{\theta_j}{\eta_1}\right)\right)\right|\right] \\
&\quad + {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi} \left|f\left(\tau, u_1(\tau), v_1\left(\frac{\tau}{\eta_1}\right)\right) - f\left(\tau, u_2(\tau), v_2\left(\frac{\tau}{\eta_1}\right)\right)\right| \\
&\leqslant \frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)} \left[\sum_{i=1}^{m} |\mu_i|_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi} \left(\mathcal{K}_{1,g}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\xi_i,a)\,\|u_1-u_2\|_{\mathbb{W}}\right.\right. \\
&\quad \left. + \mathcal{K}_{2,g}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\xi_i,a)\,\|v_1-v_2\|_{\mathbb{W}}\right) + |\omega_1| \sum_{j=1}^{n} |\delta_j|_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi} \left(\mathcal{L}_{1,f}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\theta_j,a)\,\|u_1-u_2\|_{\mathbb{W}}\right. \\
&\quad \left.\left. + \mathcal{L}_{2,f}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\theta_j,a)\,\|v_1-v_2\|_{\mathbb{W}}\right)\right] + {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left(\mathcal{L}_{1,f}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)\,\|u_1-u_2\|_{\mathbb{W}}\right. \\
&\quad \left. + \mathcal{L}_{2,f}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)\,\|v_1-v_2\|_{\mathbb{W}}\right) \\
&\leqslant \left[\mathcal{L}_{1,f}\Omega_1^{\gamma_1} + \mathcal{K}_{1,g}\Omega_2^{\gamma_1}\right]\|u_1-u_2\|_{\mathbb{W}} + \left[\mathcal{L}_{2,f}\Omega_1^{\gamma_2} + \mathcal{K}_{2,g}\Omega_2^{\gamma_2}\right]\|v_1-v_2\|_{\mathbb{W}}.
\end{aligned} \tag{4.8}
$$

In the same way, one has

$$
\begin{aligned}
&\left|{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau,a)\left(\mathcal{F}_2(u_1,v_1)(\tau) - \mathcal{F}_2(u_2,v_2)(\tau)\right)\right| \\
&\quad \leqslant \left[\mathcal{L}_{1,f}\Xi_2^{\gamma_1} + \mathcal{K}_{1,g}\Xi_1^{\gamma_1}\right]\|u_1-u_2\|_{\mathbb{W}} + \left[\mathcal{L}_{2,f}\Xi_2^{\gamma_2} + \mathcal{K}_{2,g}\Xi_1^{\gamma_2}\right]\|v_1-v_2\|_{\mathbb{W}}.
\end{aligned} \tag{4.9}
$$

Applying the inequalities (4.8)-(4.9), it implies

$$
\begin{aligned}
\|\mathcal{F}(u_1,v_1)(\tau) - \mathcal{F}(u_2,v_2)(\tau)\|_{\mathbb{W}} &\leqslant \left[(\Omega_1^{\gamma_1}+\Xi_2^{\gamma_1})\mathcal{L}_{1,f} + (\Omega_2^{\gamma_1}+\Xi_1^{\gamma_1})\mathcal{K}_{1,g}\right]\|u_1-u_2\|_{\mathbb{W}} \\
&\quad + \left[(\Omega_1^{\gamma_2}+\Xi_2^{\gamma_2})\mathcal{L}_{2,f} + (\Omega_2^{\gamma_2}+\Xi_1^{\gamma_2})\mathcal{K}_{2,g}\right]\|v_1-v_2\|_{\mathbb{W}} \\
&\leqslant \left[\|u_1-u_2\|_{\mathbb{W}} + \|v_1-v_2\|_{\mathbb{W}}\right] \sum_{i=1}^{2}\left[(\Omega_1^{\gamma_i}+\Xi_2^{\gamma_i})\mathcal{L}_{i,f} + (\Omega_2^{\gamma_i}+\Xi_1^{\gamma_i})\mathcal{K}_{i,g}\right].
\end{aligned}
$$

Under the condition (4.4), it follows that $\mathcal{F}$ satisfies the contraction property. Consequently, by [17], the operator $\mathcal{F}$ has a unique fixed point. Thus, the coupled system (1.3) has a unique solution. □

Next, we prove the existence result for the coupled system (1.3) based on a fixed point theory of Krasnoselśkii's type [23].

Theorem 4.2. *Suppose that* $f, g \in \mathcal{C}([a,b]\times\mathbb{R}^2, \mathbb{R})$ *verifies* $(\mathcal{H}_1)$ *in Theorem 4.1 and the following condition is true:*

$(\mathcal{H}_2)$ *there exists two positive functions* $\Upsilon_1, \Upsilon_2 \in \mathcal{C}([a,b], \mathbb{R}^+)$ *so that*

$$
|f(\tau,u,v)| \leqslant \Upsilon_1(\tau), \qquad |g(\tau,u,v)| \leqslant \Upsilon_2(\tau), \qquad (\tau,u,v) \in [a,b]\times\mathbb{R}^2.
$$

Therefore, the coupled system (1.3) *has at least one solution, given that*

$$
\sum_{i=1}^{2}\left[(\Delta_1^{\gamma_i}+\Xi_2^{\gamma_i})\mathcal{L}_{i,f} + (\Omega_2^{\gamma_i}+\Theta_1^{\gamma_i})\mathcal{K}_{i,g}\right] < 1. \tag{4.10}
$$

Proof. Define a closed and bounded ball $\mathcal{B}_{r_2} := \{(u,v) \in \mathbb{W} \times \mathbb{W} : \|(u,v)\|_{\mathbb{W}} \leqslant r_2\}$, where

$$r_2 \geqslant \Upsilon^*(\Omega_1^{\gamma_1} + \Omega_2^{\gamma_1} + \Xi_1^{\gamma_2} + \Xi_2^{\gamma_2}) > 0,$$

when $\Upsilon^* = \max\{\|\Upsilon_1\|, \|\Upsilon_2\|\}$,

$$\|\Upsilon_1\|_{\mathbb{W}} = \sup_{\tau \in [a,b]} \left| {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\Upsilon_1(\tau) \right| \quad \text{and} \quad \|\Upsilon_2\|_{\mathbb{W}} = \sup_{\tau \in [a,b]} \left| {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau,a)\Upsilon_2(\tau) \right|. \tag{4.11}$$

Next, $\mathcal{F}(u,v)(\tau)$, which is provided by (4.1), will be separated into

$$\mathcal{F}(u,v)(\tau) = \mathcal{F}_1(u,v)(\tau) + \mathcal{F}_2(u,v)(\tau) = \mathcal{F}_{1,1}(u,v)(\tau) + \mathcal{F}_{1,2}(u,v)(\tau) + \mathcal{F}_{2,1}(u,v)(\tau) + \mathcal{F}_{2,2}(u,v)(\tau),$$

where

$$\begin{aligned}
\mathcal{F}_{1,1}(u,v)(\tau) :&= {}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi} f\left(\tau, u(\tau), v\left(\frac{\tau}{\eta_1}\right)\right),\\
\mathcal{F}_{1,2}(u,v)(\tau) :&= \frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)} \left[\sum_{i=1}^{m} \mu_i\, {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi} g\left(\xi_i, u\left(\frac{\xi_i}{\eta_2}\right), v(\xi_i)\right) \right.\\
&\quad \left. + \omega_1 \sum_{j=1}^{n} \delta_j\, {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi} f\left(\theta_j, u(\theta_j), v\left(\frac{\theta_j}{\eta_1}\right)\right) \right],\\
\mathcal{F}_{2,1}(u,v)(\tau) :&= {}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi} g\left(\tau, u\left(\frac{\tau}{\eta_2}\right), v(\tau)\right),\\
\mathcal{F}_{2,2}(u,v)(\tau) :&= \frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\rho^{\frac{\gamma_2}{k}-1}\Gamma_k(\gamma_2)} \left[\sum_{j=1}^{n} \delta_j\; {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi} f\left(\theta_j, u(\theta_j), v\left(\frac{\theta_j}{\eta_1}\right)\right) \right.\\
&\quad \left. + \omega_2 \sum_{i=1}^{m} \mu_i\; {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi} g\left(\xi_i, u\left(\frac{\xi_i}{\eta_2}\right), v(\xi_i)\right) \right].
\end{aligned}$$

Firstly, we show that $\mathcal{F} \subset \mathcal{B}_{r_2}$. By applying the assumption $(\mathcal{H}_2)$, the properties (4.11), and Lemma 2.5, for (u,v), $(\bar{u},\bar{v}) \in \mathcal{B}_{r_2}$ with $\tau \in [a,b]$, it follows that

$$\begin{aligned}
&\left| {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\big(\mathcal{F}_{1,1}(u,v)(\tau) + \mathcal{F}_{1,2}(\bar{u},\bar{v})(\tau)\big) \right|\\
&\leqslant \frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)} \left[\sum_{i=1}^{m} |\mu_i|_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi} \left| g\left(\xi_i, \bar{u}\left(\frac{\xi_i}{\eta_2}\right), \bar{v}(\xi_i)\right) \right| \right.\\
&\quad \left. + |\omega_1| \sum_{j=1}^{n} |\delta_j|_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi} \left| f\Big(\theta_j, \bar{u}(\theta_j), \bar{v}\Big(\frac{\theta_j}{\eta_1}\Big)\Big) \right| \right] + {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi} \left| f\Big(\tau, u(\tau), v\Big(\frac{\tau}{\eta_1}\Big)\Big) \right|\\
&\leqslant \left(\frac{|\omega_1|\,\Gamma_k(\gamma_1)}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)} \sum_{j=1}^{n} \frac{|\delta_j|\, {}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1-k+\alpha_1+\lambda_j}{k}}(\theta_j,a)}{\Gamma_k(\gamma_1+\alpha_1+\lambda_j)} + \frac{\Gamma_k(\gamma_1){}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1}{k}}(b,a)}{\rho^{\frac{\alpha_1}{k}}\Gamma_k(\gamma_1+\alpha_1)} \right) \|\Upsilon_1\|_{\mathbb{W}}\\
&\quad + \frac{\Gamma_k(\gamma_1)}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)} \sum_{i=1}^{m} \frac{|\mu_i|\, {}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1-k+\alpha_2+\sigma_i}{k}}(\xi_i,a)}{\Gamma_k(\gamma_1+\alpha_2+\sigma_i)} \|\Upsilon_2\|_{\mathbb{W}} \leqslant \Upsilon^*(\Omega_1^{\gamma_1} + \Omega_2^{\gamma_1}).
\end{aligned}$$

In the same ways, one has the following result

$$\left| {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau,a)\big(\mathcal{F}_{2,1}(u,v)(\tau) + \mathcal{F}_{2,2}(\bar{u},\bar{v})(\tau)\big) \right| \leqslant \Upsilon^*(\Xi_1^{\gamma_2} + \Xi_2^{\gamma_2}).$$

This yields that

$$\|(\mathcal{F}_{1,1}+\mathcal{F}_{2,1})(u,v)(\tau)\|_{\mathbb{W}}+\|(\mathcal{F}_{1,2}+\mathcal{F}_{2,2})(\bar{u},\bar{v})(\tau)\|_{\mathbb{W}}\leqslant \Upsilon^*(\Omega_1^{\gamma_1}+\Omega_2^{\gamma_1}+\Xi_1^{\gamma_2}+\Xi_2^{\gamma_2})\leqslant r_2.$$

Then $(\mathcal{F}_{1,1}+\mathcal{F}_{2,1})(u,v)+(\mathcal{F}_{1,2}+\mathcal{F}_{2,2})(\bar{u},\bar{v})\in\mathcal{B}_{r_2}$. Next, we show $(\mathcal{F}_{1,2},\mathcal{F}_{2,2})$ is a contraction. By using the assumption $(\mathcal{H}_1)$ and Lemma 2.5, for $(u_1,v_1),(u_2,v_2)\in\mathcal{B}_{r_2}$, it follows

$$\begin{aligned}
&\left|{}_k^\rho\Psi_\psi^{1-\frac{\gamma_1}{k}}(\tau,a)\left(\mathcal{F}_{1,2}(u_1,v_1)(\tau)-\mathcal{F}_{1,2}(u_2,v_2)(\tau)\right)\right|\\
&\leqslant\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left[\sum_{i=1}^m|\mu_i|_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}\left|g\left(\xi_i,u_1\left(\frac{\xi_i}{\eta_2}\right),v_1(\xi_i)\right)-g\left(\xi_i,u_2\left(\frac{\xi_i}{\eta_2}\right),v_2(\xi_i)\right)\right|\right.\\
&\quad\left.+|\omega_1|\sum_{j=1}^n|\delta_j|_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}\left|f\left(\theta_j,u_1(\theta_j),v_1\left(\frac{\theta_j}{\eta_1}\right)\right)-f\left(\theta_j,u_2(\theta_j),v_2\left(\frac{\theta_j}{\eta_1}\right)\right)\right|\right]\\
&\leqslant\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left[\sum_{i=1}^m|\mu_i|_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}\left(\mathcal{K}_{1,g}{}_k^\rho\Psi_\psi^{\frac{\gamma_1}{k}-1}(\xi_i,a)\,\|u_1-u_2\|_{\mathbb{W}}\right.\right.\\
&\quad\left.+\mathcal{K}_{2,g}{}_k^\rho\Psi_\psi^{\frac{\gamma_2}{k}-1}(\xi_i,a)\,\|v_1-v_2\|_{\mathbb{W}}\right)+|\omega_1|\sum_{j=1}^n|\delta_j|_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}\left(\mathcal{L}_{1,f}{}_k^\rho\Psi_\psi^{\frac{\gamma_1}{k}-1}(\theta_j,a)\,\|u_1-u_2\|_{\mathbb{W}}\right.\\
&\quad\left.\left.+\mathcal{L}_{2,f}{}_k^\rho\Psi_\psi^{\frac{\gamma_2}{k}-1}(\theta_j,a)\,\|v_1-v_2\|_{\mathbb{W}}\right)\right]\\
&=\left[\mathcal{L}_{1,f}\Delta_1^{\gamma_1}+\mathcal{K}_{1,g}\Omega_2^{\gamma_1}\right]\|u_1-u_2\|_{\mathbb{W}}+\left[\mathcal{L}_{2,f}\Delta_1^{\gamma_2}+\mathcal{K}_{2,g}\Omega_2^{\gamma_2}\right]\|v_1-v_2\|_{\mathbb{W}},
\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}
&\left|{}_k^\rho\Psi_\psi^{1-\frac{\gamma_2}{k}}(\tau,a)\left(\mathcal{F}_{2,2}(u_1,v_1)(\tau)-\mathcal{F}_{2,2}(u_2,v_2)(\tau)\right)\right|\\
&\quad\leqslant\left[\mathcal{L}_{1,f}\Xi_2^{\gamma_1}+\mathcal{K}_{1,g}\Theta_1^{\gamma_1}\right]\|u_1-u_2\|_{\mathbb{W}}+\left[\mathcal{L}_{2,f}\Xi_2^{\gamma_2}+\mathcal{K}_{2,g}\Theta_1^{\gamma_2}\right]\|v_1-v_2\|_{\mathbb{W}}.
\end{aligned}\tag{4.13}$$

From the inequalities (4.12)-(4.13), we obtain the result as follows

$$\begin{aligned}
&\|(\mathcal{F}_{1,2}+\mathcal{F}_{2,2})(u_1,v_1)(\tau)-(\mathcal{F}_{1,2}+\mathcal{F}_{2,2})(u_2,v_2)(\tau)\|_{\mathbb{W}}\\
&\quad\leqslant\left[(\Delta_1^{\gamma_1}+\Xi_2^{\gamma_1})\mathcal{L}_{1,f}+(\Omega_2^{\gamma_1}+\Theta_1^{\gamma_1})\mathcal{K}_{1,g}\right]\|u_1-u_2\|_{\mathbb{W}}\\
&\qquad+\left[(\Delta_1^{\gamma_2}+\Xi_2^{\gamma_2})\mathcal{L}_{2,f}+(\Omega_2^{\gamma_2}+\Theta_1^{\gamma_2})\mathcal{K}_{2,g}\right]\|v_1-v_2\|_{\mathbb{W}}\\
&\quad\leqslant\left[\|u_1-u_2\|_{\mathbb{W}}+\|v_1-v_2\|_{\mathbb{W}}\right]\sum_{i=1}^2\left[(\Delta_1^{\gamma_i}+\Xi_2^{\gamma_i})\mathcal{L}_{i,f}+(\Omega_2^{\gamma_i}+\Theta_1^{\gamma_i})\mathcal{K}_{i,g}\right].
\end{aligned}\tag{4.14}$$

From (4.14) with (4.10), which yields that $(\mathcal{F}_{1,2}+\mathcal{F}_{2,2})$ is a contraction. Applying the continuity property of f and g, which means that $(\mathcal{F}_{1,1},\mathcal{F}_{2,1})$ is continuous and uniformly bounded on $\mathcal{B}_{r_2}$ as

$$\|(\mathcal{F}_{1,1},\mathcal{F}_{2,1})(u,v)\|_{\mathbb{W}}\leqslant(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})\|\Upsilon_1\|_{\mathbb{W}}+(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})\|\Upsilon_2\|_{\mathbb{W}}.$$

Finally, we prove that the set $(\mathcal{F}_{1,1}+\mathcal{F}_{2,1})\mathcal{B}_{r_2}$ is equi-continuous. For any $a\leqslant\tau_1\leqslant\tau_2\leqslant b$ and $(u,v)\in\mathcal{B}_{r_2}$, it follows that

$$\begin{aligned}
&\left|{}_k^\rho\Psi_\psi^{1-\frac{\gamma_1}{k}}(\tau_2,a)\mathcal{F}_{1,1}(u,v)(\tau_2)-{}_k^\rho\Psi_\psi^{1-\frac{\gamma_1}{k}}(\tau_1,a)\mathcal{F}_{1,1}(u,v)(\tau_1)\right|\\
&\leqslant\left|{}_k^\rho\Psi_\psi^{1-\frac{\gamma_1}{k}}(\tau_2,a)_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau_2,u(\tau_2),v\left(\frac{\tau_2}{\eta_1}\right)\right)-{}_k^\rho\Psi_\psi^{1-\frac{\gamma_1}{k}}(\tau_1,a)_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau_1,u(\tau_1),v\left(\frac{\tau_1}{\eta_1}\right)\right)\right|\\
&\leqslant\left|{}_k^\rho\Psi_\psi^{1-\frac{\gamma_1}{k}}(\tau_2,a)_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau_1,u(\tau_1),v\left(\frac{\tau_1}{\eta_1}\right)\right)-{}_k^\rho\Psi_\psi^{1-\frac{\gamma_1}{k}}(\tau_1,a)_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau_1,u(\tau_1),v\left(\frac{\tau_1}{\eta_1}\right)\right)\right|
\end{aligned}\tag{4.15}$$

$$+\left|{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau_2,a)\,{}_{k}\mathcal{J}_{\tau_1}^{\alpha_1,\rho;\psi}f\left(\tau_2,u(\tau_2),v\left(\frac{\tau_2}{\eta_1}\right)\right)\right|$$
$$\leqslant \frac{\|\Upsilon_1\|_{\mathbb{W}}\Gamma_k(\gamma_1)}{\rho^{\frac{\alpha_1}{k}}\Gamma_k(\alpha_1+\gamma_1)}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1+\gamma_1-k}{k}}(\tau_1,a)\left|{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau_2,a)-{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau_1,a)\right|+{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1+\gamma_1-k}{k}}(\tau_2,\tau_1){}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau_2,a)\right],$$

and

$$\begin{aligned}&\left|{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau_2,a)\mathcal{F}_{2,1}(u,v)(\tau_2)-{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau_1,a)\mathcal{F}_{2,1}(u,v)(\tau_1)\right|\\&\quad\leqslant \frac{\|\Upsilon_2\|_{\mathbb{W}}\Gamma_k(\gamma_2)}{\rho^{\frac{\alpha_2}{k}}\Gamma_k(\alpha_2+\gamma_2)}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_2+\gamma_2-k}{k}}(\tau_1,a)\left|{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau_2,a)-{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau_1,a)\right|\right.\\&\qquad\left.+{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_2+\gamma_2-k}{k}}(\tau_2,\tau_1){}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau_2,a)\right].\end{aligned} \tag{4.16}$$

Since (4.15) and (4.16) are an independent of $(u,v)\in\mathcal{B}_{r_2}$ and (4.15) and (4.16) tend to zero as $\tau_2\to\tau_1$. Then $\|(\mathcal{F}_{1,1}+\mathcal{F}_{2,1})(u,v)(\tau_2)-(\mathcal{F}_{1,1}+\mathcal{F}_{2,1})(u,v)(\tau_1)\|_{\mathbb{W}}\to 0$ as $\tau_2\to\tau_1$. So, $(\mathcal{F}_{1,1},\mathcal{F}_{2,1})$ is equi-continuous, which means that $(\mathcal{F}_{1,1},\mathcal{F}_{2,1})$ is compact on $\mathcal{B}_{r_2}$ by utilizing the Arzelá-Ascoli theorem. As a consequence, all assumptions of [23] are fulfilled. Therefore, the coupled system (1.3) has at least one solution. □

4.2. *Ulam-Hyers-Mittag-Leffler stability and its generalization*

Now, we analyze the stability for the coupled system (1.3) by means of the integral description of its solution provided by $u(\tau)=\mathcal{F}_1(u,v)(\tau)$, $v(\tau)=\mathcal{F}_2(u,v)(\tau)$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are provided by (4.2) and (4.3), respectively. Before analyzing, we define the definitions of UHML stability for the coupled system (1.3).

Definition 4.3. The coupled system (1.3) is considered *UHML* stable, if there are two constants $\mathfrak{C}_f>0$, $\mathfrak{C}_g>0$ so that, for every $\epsilon>0$, and for any solution $(u^*,v^*)\in\mathbb{W}\times\mathbb{W}$ of

$$\left|{}^{H}_{a,k}\mathfrak{D}^{\alpha_1,\beta,\rho;\psi}u^*(\tau)-f\left(\tau,u^*(\tau),v^*\left(\frac{\tau}{\eta_1}\right)\right)\right|\leqslant\epsilon,\quad \left|{}^{H}_{a,k}\mathfrak{D}^{\alpha_2,\beta,\rho;\psi}v^*(\tau)-g\left(\tau,u^*\left(\frac{\tau}{\eta_2}\right),v^*(\tau)\right)\right|\leqslant\epsilon, \tag{4.17}$$

there is a solution $(u,v)\in\mathbb{W}\times\mathbb{W}$ of (1.3) verifying

$$|(u,v)-(u^*,v^*)|\leqslant(\mathfrak{C}_f+\mathfrak{C}_g)\epsilon\mathbb{E}_{k,\alpha_1+\alpha_2,k}\left(\kappa_{f,g}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\right),\quad \kappa_{f,g}\geqslant 0,\quad \tau\in[a,b].$$

Definition 4.4. The coupled system (1.3) is considered generalized UHML stable, if there are two functions $\zeta_f\in\mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$, $\zeta_g\in\mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$ via $\zeta_f(0)=0=\zeta_g(0)$, such that for every $\epsilon>0$, and for any $(u^*,v^*)\in\mathbb{W}\times\mathbb{W}$ of

$$\left|{}^{H}_{a,k}\mathfrak{D}^{\alpha_1,\beta,\rho;\psi}u^*(\tau)-f\left(\tau,u^*(\tau),v^*\left(\frac{\tau}{\eta_1}\right)\right)\right|\leqslant\zeta_f(\tau),$$
$$\left|{}^{H}_{a,k}\mathfrak{D}^{\alpha_2,\beta,\rho;\psi}v^*(\tau)-g\left(\tau,u^*\left(\frac{\tau}{\eta_2}\right),v^*(\tau)\right)\right|\leqslant\zeta_g(\tau),$$

there exists a solution $(u,v)\in\mathbb{W}\times\mathbb{W}$ of (1.3) verifying

$$|(u,v)-(u^*,v^*)|\leqslant(\zeta_f(\epsilon)+\zeta_g(\epsilon))\mathbb{E}_{k,\alpha_1+\alpha_2,k}\left(\kappa_{f,g}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\right),\quad \kappa_{f,g}\geqslant 0,\quad \tau\in[a,b].$$

Definition 4.5. The proposed system (1.3) is considered UHRML stable with respect to a function $\chi(\tau)$, if there are two constants $\mathfrak{C}_{f_\chi}>0$, $\mathfrak{C}_{g_\chi}>0$ so that, for every $\epsilon>0$, and for any solution $(u^*,v^*)\in\mathbb{W}\times\mathbb{W}$ of

$$\left|{}^{H}_{a,k}\mathfrak{D}^{\alpha_1,\beta,\rho;\psi}u^*(\tau)-f\left(\tau,u^*(\tau),v^*\left(\frac{\tau}{\eta_1}\right)\right)\right|\leqslant\epsilon\chi(\tau), \tag{4.18}$$

$$\left|{}^{H}_{a,k}\mathfrak{D}^{\alpha_2,\beta,\rho;\psi}v^*(\tau) - g\left(\tau, u^*\left(\frac{\tau}{\eta_2}\right), v^*(\tau)\right)\right| \leqslant \epsilon\chi(\tau), \tag{4.19}$$

there exists a solution $(u, v) \in \mathbb{W} \times \mathbb{W}$ of the proposed system (1.3) verifying

$$|(u,v) - (u^*,v^*)| \leqslant (\mathfrak{C}_{f_\chi} + \mathfrak{C}_{g_\chi})\epsilon\chi(\tau)\mathbb{E}_{k,\alpha_1+\alpha_2,k}\left(\kappa_{f_\chi,g_\chi}\Psi_\psi^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\right), \quad \tau \in [a,b], \quad \kappa_{f_\chi,g_\chi} \geqslant 0.$$

Definition 4.6. The coupled system (1.3) is considered generalized UHRML stable with respect to a function $\chi(\tau)$ such that for every $\epsilon > 0$, and for any solution $(u^*, v^*) \in \mathbb{W} \times \mathbb{W}$ of

$$\left|{}^{H}_{a,k}\mathfrak{D}^{\alpha_1,\beta,\rho;\psi}u^*(\tau) - f\left(\tau, u^*(\tau), v^*\left(\frac{\tau}{\eta_1}\right)\right)\right| \leqslant \chi(\tau),$$
$$\left|{}^{H}_{a,k}\mathfrak{D}^{\alpha_2,\beta,\rho;\psi}v^*(\tau) - g\left(\tau, u^*\left(\frac{\tau}{\eta_2}\right), v^*(\tau)\right)\right| \leqslant \chi(\tau),$$

there exists a solution $(u, v) \in \mathbb{W} \times \mathbb{W}$ of (1.3) verifying

$$|(u,v) - (u^*,v^*)| \leqslant (\mathfrak{C}_{f_\chi} + \mathfrak{C}_{g_\chi})\chi(\tau)\mathbb{E}_{k,\alpha_1+\alpha_2,k}\left(\kappa_{f_\chi,g_\chi}\Psi_\psi^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\right), \quad \mathfrak{C}_{f_\chi}, \mathfrak{C}_{g_\chi} > 0,\ \kappa_{f_\chi,g_\chi} \geqslant 0.$$

Remark 4.7. Let $(u^*, v^*) \in \mathbb{W} \times \mathbb{W}$ is the solution of (4.17) if and only if there exist $w_{u^*,v^*} \in \mathbb{W}$ (depends on (u^*, v^*)) so that

(i) $|w_{u^*,v^*}(\tau)| \leqslant \epsilon$;

(ii) ${}^{H}_{a,k}\mathfrak{D}^{\alpha_1,\beta,\rho;\psi}u^*(\tau) = f\left(\tau, u^*(\tau), v^*\left(\frac{\tau}{\eta_1}\right)\right) + w_{u^*,v^*}(\tau)$; and

(iii) ${}^{H}_{a,k}\mathfrak{D}^{\alpha_2,\beta,\rho;\psi}v^*(\tau) = g\left(\tau, u^*\left(\frac{\tau}{\eta_2}\right), v^*(\tau)\right) + w_{u^*,v^*}(\tau)$, $\tau \in [a,b]$.

Remark 4.8. Let $(u^*, v^*) \in \mathbb{W} \times \mathbb{W}$ is the solution of (4.18)-(4.19) if and only if there exist $z_{u^*,v^*} \in \mathbb{W}$ (depends on (u^*, v^*)) so that

(i) $|z_{u^*,v^*}(\tau)| \leqslant \epsilon\chi(\tau)$;

(ii) ${}^{H}_{a,k}\mathfrak{D}^{\alpha_1,\beta,\rho;\psi}u^*(\tau) = f\left(\tau, u^*(\tau), v^*\left(\frac{\tau}{\eta_1}\right)\right) + z_{u^*,v^*}(\tau)$; and

(iii) ${}^{H}_{a,k}\mathfrak{D}^{\alpha_2,\beta,\rho;\psi}v^*(\tau) = g\left(\tau, u^*\left(\frac{\tau}{\eta_2}\right), v^*(\tau)\right) + z_{u^*,v^*}(\tau)$, $\tau \in [a,b]$.

Lemma 4.9. *Let* $\alpha_l \in (0,1)$, $l = 1,2$, $\beta \in [0,1]$, $\rho \in (0,1]$, *and* $k \in \mathbb{R}^+$. *If* $(u^*, v^*) \in \mathbb{W} \times \mathbb{W}$ *satisfies the inequalities* (4.17), *then* (u^*, v^*) *satisfies the following inequalities:*

$$\left\|u^*(\tau) - \mathcal{M}_{u^*,v^*}(\tau) - {}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau, u^*(\tau), v^*\left(\frac{\tau}{\eta_1}\right)\right)\right\|_{\mathbb{W}} \leqslant \Lambda_1\epsilon,$$
$$\left\|v^*(\tau) - \mathcal{N}_{u^*,v^*}(\tau) - {}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}g\left(\tau, u^*\left(\frac{\tau}{\eta_2}\right), v^*(\tau)\right)\right\|_{\mathbb{W}} \leqslant \Lambda_2\epsilon,$$

where

$$\begin{aligned}\mathcal{M}_{u^*,v^*}(\tau) = \frac{{}^{\rho}_{k}\Psi_\psi^{\frac{\gamma_1}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}&\left[\sum_{i=1}^{m}\mu_i\ {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}g\left(\xi_i, u^*\left(\frac{\xi_i}{\eta_2}\right), v^*(\xi_i)\right)\right.\\ &\left.+\omega_1\sum_{j=1}^{n}\delta_j\ {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}f\left(\theta_j, u^*(\theta_j), v^*\left(\frac{\theta_j}{\eta_1}\right)\right)\right],\end{aligned} \tag{4.20}$$

$$\begin{aligned}\mathcal{N}_{u^*,v^*}(\tau) = \frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\rho^{\frac{\gamma_2}{k}-1}\Gamma_k(\gamma_2)} &\left[\sum_{j=1}^{n}\delta_j\ {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi} f\left(\theta_j, u^*(\theta_j), v^*\left(\frac{\theta_j}{\eta_1}\right)\right)\right.\\ &\left.+\omega_2\sum_{i=1}^{m}\mu_i\ {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi} g\left(\xi_i, u^*\left(\frac{\xi_i}{\eta_2}\right), v^*(\xi_i)\right)\right],\end{aligned} \tag{4.21}$$

and $\Lambda_1 = \Omega_1^k + \Omega_2^k$ *and* $\Lambda_2 = \Xi_1^k + \Xi_2^k$.

Proof. Let (u^*, v^*) be a solution of (4.17). Applying (ii) in Remark 4.7, we obtain

$$\begin{cases} {}^{H}_{a,k}\mathfrak{D}^{\alpha_1,\beta,\rho;\psi} u^*(\tau) = f\left(\tau, u^*(\tau), v^*\left(\frac{\tau}{\eta_1}\right)\right) + w_{u^*,v^*}(\tau), \quad \eta_1 > 1,\\ {}^{H}_{a,k}\mathfrak{D}^{\alpha_2,\beta,\rho;\psi} v^*(\tau) = g\left(\tau, u^*\left(\frac{\tau}{\eta_2}\right), v^*(\tau)\right) + w_{u^*,v^*}(\tau), \quad \eta_2 > 1,\\ \lim\limits_{\tau\to a} {}_{a,k}\mathcal{J}^{k-\gamma_1,\rho;\psi} u^*(\tau) = \sum\limits_{i=1}^{m}\mu_i\ {}_{a,k}\mathcal{J}^{\sigma_i;\psi} v^*(\xi_i), \quad \xi_i \in (a,b], \quad i = 1,2,\ldots,m,\\ \lim\limits_{\tau\to a} {}_{a,k}\mathcal{J}^{k-\gamma_2,\rho;\psi} v^*(\tau) = \sum\limits_{j=1}^{n}\delta_j\ {}_{a,k}\mathcal{J}^{\lambda_j,\rho;\psi} u^*(\theta_j), \quad \theta_j \in (a,b], \quad j = 1,2,\ldots,n. \end{cases} \tag{4.22}$$

Applying the result in Lemma 2.6, a solution of (4.22) is given by

$$\begin{aligned} u^*(\tau) &= {}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi} w_{u^*,v^*}(\tau) + \mathcal{M}_{u^*,v^*}(\tau) + {}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi} f\left(\tau, u^*(\tau), v^*\left(\frac{\tau}{\eta_1}\right)\right)\\ &\quad + \frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left(\sum_{i=1}^{m}\mu_i\ {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi} w_{u^*,v^*}(\xi_i) + \omega_1\sum_{j=1}^{n}\delta_j\ {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi} w_{u^*,v^*}(\theta_j)\right),\\ v^*(\tau) &= {}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi} w_{u^*,v^*}(\tau) + \mathcal{N}_{u^*,v^*}(\tau) + {}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi} g\left(\tau, u^*\left(\frac{\tau}{\eta_2}\right), v^*(\tau)\right)\\ &\quad + \frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\Gamma_k\rho^{\frac{\gamma_2}{k}-1}(\gamma_2)}\left(\sum_{j=1}^{n}\delta_j\ {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi} w_{u^*,v^*}(\theta_j) + \omega_2\sum_{i=1}^{m}\mu_i\ {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi} w_{u^*,v^*}(\xi_i)\right), \end{aligned}$$

where $\mathcal{M}_{u^*,v^*}(\tau)$ and $\mathcal{N}_{u^*,v^*}(\tau)$ are denoted by (4.20)-(4.21). Applying Remark 4.7 (i) with Theorem 4.1 and Lemma 2.5, implies that

$$\begin{aligned} &\left|{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\left(u^*(\tau) - \mathcal{M}_{u^*,v^*}(\tau) - {}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi} f\left(\tau, u^*(\tau), v^*\left(\frac{\tau}{\eta_1}\right)\right)\right)\right|\\ &\leqslant \frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left[\sum_{i=1}^{m}|\mu_i|{}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}\left|w_{u^*,v^*}(\xi_i)\right| + |\omega_1|\sum_{j=1}^{n}\left|\delta_j\right|{}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}\left|w_{u^*,v^*}(\theta_j)\right|\right]\\ &\quad + {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a){}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left|w_{u^*,v^*}(b)\right|\\ &\leqslant \epsilon\left\{\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left[\sum_{i=1}^{m}|\mu_i|{}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}(1)(\xi_i) + |\omega_1|\sum_{j=1}^{n}\left|\delta_j\right|{}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}(1)(\theta_j)\right]\right.\\ &\quad \left. + {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(b,a){}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}(1)(b)\right\}\\ &= \epsilon\left\{\frac{\Gamma_k(k)}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1-k+\alpha_2}{k}}\Gamma_k(\gamma_1)}\sum_{i=1}^{m}\frac{|\mu_i|\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{k-k+\alpha_2+\sigma_i}{k}}(\xi_i,a)}{\rho^{\frac{\sigma_i}{k}}\Gamma_k(k+\alpha_2+\sigma_i)}\right. \end{aligned}$$

$$+\frac{|\omega_1|\Gamma_k(k)}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1-k+\alpha_1}{k}}\Gamma_k(\gamma_1)}\sum_{j=1}^{n}\frac{|\delta_j|\,{}_k^\rho\Psi_\psi^{\frac{k-k+\alpha_1+\lambda_j}{k}}(\theta_j,a)}{\rho^{\frac{\lambda_j}{k}}\Gamma_k(k+\alpha_1+\lambda_j)}+\frac{\Gamma_k(\ell){}_k^\rho\Psi_\psi^{\frac{\ell+\alpha_1-\gamma_1}{k}}(b,a)}{\rho^{\frac{\alpha_1}{k}}\Gamma_k(\ell+\alpha_1)}\Bigg\}=(\Omega_2^k+\Omega_1^k)\epsilon=\Lambda_1\epsilon.$$

Similarly, we have

$$\begin{aligned}
&\left|{}_k^\rho\Psi_\psi^{1-\frac{\gamma_2}{k}}(\tau,a)\left(y^*(\tau)-\mathcal{N}_{u^*,v^*}(\tau)-{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}g\left(\tau,u^*\left(\frac{\tau}{\eta_2}\right),v^*(\tau)\right)\right)\right|\\
&\leqslant\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_2}{k}-1}\Gamma_k(\gamma_2)}\left[\sum_{j=1}^{n}|\delta_j|{}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}\left|w_{u^*,v^*}(\theta_j)\right|+|\omega_2|\sum_{i=1}^{m}|\mu_i|{}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}\left|w_{u^*,v^*}(\xi_i)\right|\right]\\
&\quad+{}_k^\rho\Psi_\psi^{1-\frac{\gamma_2}{k}}(\tau,a){}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}\left|w_{u^*,v^*}(b)\right|\\
&\leqslant\epsilon\Bigg\{\frac{\Gamma_k(k)}{|1-\omega_1\omega_2|\,\rho^{\frac{\gamma_2-k+\alpha_1}{k}}\Gamma_k(\gamma_2)}\sum_{j=1}^{n}\frac{|\delta_j|{}_k^\rho\Psi_\psi^{\frac{k-k+\alpha_1+\lambda_j}{k}}(\theta_j,a)}{\rho^{\frac{\lambda_j}{k}}\Gamma_k(\ell+\alpha_1+\lambda_j)}\\
&\quad+\frac{|\omega_2|\Gamma_k(k)}{|1-\omega_1\omega_2|\,\rho^{\frac{\gamma_2-k+\alpha_2}{k}}\Gamma_k(\gamma_2)}\sum_{i=1}^{m}\frac{|\mu_i|{}_k^\rho\Psi_\psi^{\frac{k-k+\alpha_2+\sigma_i}{k}}(\xi_i,a)}{\rho^{\frac{\sigma_i}{k}}\Gamma_k(k+\alpha_2+\sigma_i)}+\frac{\Gamma_k(k){}_k^\rho\Psi_\psi^{\frac{k+\alpha_2-\gamma_2}{k}}(b,a)}{\rho^{\frac{\alpha_2}{k}}\Gamma_k(k+\alpha_2)}\Bigg\}\\
&=\epsilon\Bigg\{\frac{1}{|1-\omega_1\omega_2|\Gamma_k(\gamma_2)}\left[\sum_{j=1}^{n}|\delta_j|k^{-\frac{\alpha_1+\lambda_j}{k}}{}_k^\rho\Psi_\psi^{\frac{\alpha_1+\lambda_j}{k}}(\theta_j,a)\mathbb{E}_{\alpha_1+\alpha_2,\frac{\alpha_1+\lambda_j}{k}+1}\left(\left(k^{-1}{}_k^\rho\Psi_\psi(\theta_j,a)\right)^{\alpha_1+\alpha_2}\right)\right.\\
&\quad\left.+|\omega_2|\sum_{i=1}^{m}|\mu_i|k^{-\frac{\alpha_2+\sigma_i}{k}}{}_k^\rho\Psi_\psi^{\frac{\alpha_2+\sigma_i}{k}}(\xi_i,a)\mathbb{E}_{\alpha_1+\alpha_2,\frac{\alpha_2+\sigma_i}{k}+1}\left(\left(k^{-1}{}_k^\rho\Psi_\psi(\xi_i,a\right)^{\alpha_1+\alpha_2}\right)\right]\\
&\quad+k^{-\frac{\alpha_2}{k}}{}_k^\rho\Psi_\psi^{\frac{k+\alpha_2-\gamma_2}{k}}(b,a)\mathbb{E}_{\alpha_1+\alpha_2,\frac{\alpha_2}{k}+1}\left(\left(k^{-1}{}_k^\rho\Psi_\psi(b,a)\right)^{\alpha_1+\alpha_2}\right)\Bigg\}=(\Xi_1^k+\Xi_2^k)\epsilon=\Lambda_2\epsilon.
\end{aligned}$$

The results are obtained. ☐

Theorem 4.10. *From all conditions in Theorem 4.1, the coupled system (1.3) is UHML stable and then its generalization stable.*

Proof. Suppose that $\epsilon>0$, $(u^*,v^*)\in\mathbb{W}\times\mathbb{W}$ is a solution of (4.17), and $(u,v)\in\mathbb{W}\times\mathbb{W}$ is the unique solution of (1.3). From the result in Lemma 2.6, it yields

$$u(\tau)=\mathcal{M}_{u,v}(\tau)+{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau,u(\tau),v\left(\frac{\tau}{\eta_1}\right)\right),\qquad v(\tau)=\mathcal{N}_{u,v}(\tau)+{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}g\left(\tau,u\left(\frac{\tau}{\eta_2}\right),v(\tau)\right).$$

It is easy to see that $\mathcal{M}_{u,v}(\tau)=\mathcal{M}_{u^*,v^*}(\tau)$ and $\mathcal{N}_{u,v}(\tau)=\mathcal{N}_{u^*,v^*}(\tau)$. Then, $|\mathcal{M}_{u,v}(\tau)-\mathcal{M}_{u^*,v^*}(\tau)|=0$ and $|\mathcal{N}_{u,v}(\tau)-\mathcal{N}_{u^*,v^*}(\tau)|=0$. Now, using Lemma 4.9 and $|u-v|\leqslant|u|+|v|$, for each $u,v\in\mathbb{R}$, $\tau\in[a,b]$, it yields that

$$\begin{aligned}
&\left|{}_k^\rho\Psi_\psi^{1-\frac{\gamma_1}{k}}(\tau,a)\left(u(\tau)-u^*(\tau)\right)\right|\\
&\leqslant\left|{}_k^\rho\Psi_\psi^{1-\frac{\gamma_1}{k}}(\tau,a)\left(\mathcal{M}_{u,v}(\tau)-\mathcal{M}_{u^*,v^*}(\tau)\right)\right|\\
&\quad+{}_k^\rho\Psi_\psi^{1-\frac{\gamma_1}{k}}(\tau,a){}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left|f\left(\tau,u(\tau),v\left(\frac{\tau}{\eta_1}\right)\right)-f\left(\tau,u^*(\tau),v^*\left(\frac{\tau}{\eta_1}\right)\right)\right|\\
&\quad+\left|{}_k^\rho\Psi_\psi^{1-\frac{\gamma_1}{k}}(\tau,a)\left[u^*(\tau)-\mathcal{M}_{u^*,v^*}(\tau)-{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau,u^*(\tau),v^*\left(\frac{\tau}{\eta_1}\right)\right)\right]\right|
\end{aligned}$$

$$\leqslant {}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left(\mathcal{L}_{1,f}\,{}_{k}^{\rho}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)\,\|u-u^*\|_{\mathbb{W}}+\mathcal{L}_{2,f}\,{}_{k}^{\rho}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)\,\|v-v^*\|_{\mathbb{W}}\right)+\Lambda_1\epsilon$$
$$\leqslant \mathcal{L}_{1,f}\frac{\Gamma_k(\gamma_1)\,{}_{k}^{\rho}\Psi_{\psi}^{\frac{\alpha_1}{k}}(b,a)}{\rho^{\frac{\alpha_1}{k}}\Gamma_k(\gamma_1+\alpha_1)}\,\|u-u^*\|_{\mathbb{W}}+\mathcal{L}_{2,f}\,{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(b,a)\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left(\|v-v^*\|_{\mathbb{W}}\right)+\Lambda_1\epsilon.$$

Then, we have

$$\|u-u^*\|_{\mathbb{W}}\leqslant \frac{\Lambda_1\epsilon}{1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})}+\frac{\mathcal{L}_{2,f}\,{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(b,a)}{1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})}\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left(\|v-v^*\|_{\mathbb{W}}\right). \tag{4.23}$$

In the similar method, we obtain that

$$\left|{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau,a)\,(v(\tau)-v^*(\tau))\right|$$
$$\leqslant \left|{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau,a)\,(\mathcal{N}_{u,v}(\tau)-\mathcal{N}_{u^*,v^*}(\tau))\right|$$
$$+\,{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau,a)\left|{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}g\left(\tau,u\left(\frac{\tau}{\eta_1}\right),v(\tau)\right)-{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}g\left(\tau,u^*\left(\frac{\tau}{\eta_1}\right),v^*(\tau)\right)\right|$$
$$+\left|{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau,a)\left[v^*(\tau)-\mathcal{N}_{u^*,v^*}(\tau)-{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}g\left(\tau,u^*\left(\frac{\tau}{\eta_1}\right),v^*(\tau)\right)\right]\right|$$
$$\leqslant {}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau,a)\,{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}\left(\mathcal{K}_{1,g}\,{}_{k}^{\rho}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)\,\|u-u^*\|_{\mathbb{W}}+\mathcal{K}_{2,g}\,{}_{k}^{\rho}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)\,\|v-v^*\|_{\mathbb{W}}\right)+\Lambda_2\epsilon$$
$$\leqslant \mathcal{K}_{1,g}\,{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(b,a)\,{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}\left(\|u-u^*\|_{\mathbb{W}}\right)+\mathcal{K}_{2,g}\frac{\Gamma_k(\gamma_2)\,{}_{k}^{\rho}\Psi_{\psi}^{\frac{\alpha_2}{k}}(b,a)}{\rho^{\frac{\alpha_2}{k}}\Gamma_k(\gamma_2+\alpha_2)}\,\|v-v^*\|_{\mathbb{W}}+\Lambda_2\epsilon,$$

which implies that

$$\|v-v^*\|_{\mathbb{W}}\leqslant \frac{\Lambda_2\epsilon}{1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})}+\frac{\mathcal{K}_{1,g}\,{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(b,a)}{1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})}\,{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}\left(\|u-u^*\|_{\mathbb{W}}\right). \tag{4.24}$$

By using Theorem 3.1 and Corollary 3.3 into (4.23)-(4.24), it yields that

$$\|u-u^*\|_{\mathbb{W}}\leqslant \left[\frac{\epsilon}{1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})}\left(\Lambda_1+\frac{\mathcal{L}_{2,f}\Lambda_2\,{}_{k}^{\rho}\Psi_{\psi}^{1+\frac{\alpha_1-\gamma_1}{k}}(b,a)}{[1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})]\rho^{\frac{\alpha_1}{k}}\Gamma_k(\alpha_1+k)}\right)\right]$$
$$\times\,\mathbb{E}_{k,\alpha_1+\alpha_2,k}\left(\frac{(\rho^{\frac{\alpha_1+\alpha_2}{k}})^{-1}\mathcal{L}_{2,f}\mathcal{K}_{1,g}\,{}_{k}^{\rho}\Psi_{\psi}^{2-\frac{\gamma_1+\gamma_2}{k}}(b,a)}{[1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})][1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})]}\,{}_{k}^{\rho}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\right),$$
$$\|v-v^*\|_{\mathbb{W}}\leqslant \left[\frac{\epsilon}{1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})}\left(\Lambda_2+\frac{\mathcal{K}_{1,g}\Lambda_1\,{}_{k}^{\rho}\Psi_{\psi}^{1+\frac{\alpha_2-\gamma_2}{k}}(b,a)}{[1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})]\rho^{\frac{\alpha_2}{k}}\Gamma_k(\alpha_2+k)}\right)\right]$$
$$\times\,\mathbb{E}_{k,\alpha_1+\alpha_2,k}\left(\frac{(\rho^{\frac{\alpha_1+\alpha_2}{k}})^{-1}\mathcal{L}_{2,f}\mathcal{K}_{1,g}\,{}_{k}^{\rho}\Psi_{\psi}^{2-\frac{\gamma_1+\gamma_2}{k}}(b,a)}{[1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})][1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})]}\,{}_{k}^{\rho}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\right).$$

By setting

$$\mathfrak{C}_f:=\frac{1}{1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})}\left(\Lambda_1+\frac{\mathcal{L}_{2,f}\Lambda_2\,{}_{k}^{\rho}\Psi_{\psi}^{1+\frac{\alpha_1-\gamma_1}{k}}(b,a)}{[1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})]\rho^{\frac{\alpha_1}{k}}\Gamma_k(\alpha_1+k)}\right),$$

$$\mathfrak{C}_g := \frac{1}{1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})}\left(\Lambda_2+\frac{\mathcal{K}_{1,g}\Lambda_1{}^{\rho}_{k}\Psi_{\psi}^{1+\frac{\alpha_2-\gamma_2}{k}}(b,a)}{[1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})]\rho^{\frac{\alpha_2}{k}}\Gamma_k(\alpha_2+k)}\right),$$

$$\kappa_{f,g} := \frac{(\rho^{\frac{\alpha_1+\alpha_2}{k}})^{-1}\mathcal{L}_{2,f}\mathcal{K}_{1,g}{}^{\rho}_{k}\Psi_{\psi}^{2-\frac{\gamma_1+\gamma_2}{k}}(b,a)}{[1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})][1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})]},$$

it implies

$$\|(u,v)-(u^*,v^*)\|_{\mathbb{W}} \leqslant (\mathfrak{C}_f+\mathfrak{C}_g)\epsilon\mathbb{E}_{k,\alpha_1+\alpha_2,k}\left(\kappa_{f,g}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\right).$$

Hence, the coupled system (1.3) is UHML stable. Additionally, taking $\zeta_f(\epsilon)=\mathfrak{C}_f\epsilon$ and $\zeta_g(\epsilon)=\mathfrak{C}_g\epsilon$ under $\zeta_f(0)=\zeta_g(0)=0$, it implies

$$\|(u,v)-(u^*,v^*)\|_{\mathbb{W}} \leqslant \Big(\zeta_f(\epsilon)+\zeta_g(\epsilon)\Big)\mathbb{E}_{k,\alpha_1+\alpha_2,k}\left(\kappa_{f,g}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\right).$$

Therefore, the coupled system (1.3) is generalized UHML stable. □

Next, we give the following assumption.

($\mathcal{H}_3$) Suppose that $\alpha_1+\alpha_2>0$ and $\chi\in\mathcal{C}([a,b],\mathbb{R})$ be a non-decreasing. There exists $\mathfrak{C}_\chi\in\mathbb{R}^+$, for any $\tau\in[a,b]$, such that,

$$_{a,k}\mathcal{J}^{\alpha_1+\alpha_2,\rho;\psi}(\chi(\tau)) \leqslant \mathfrak{C}_\chi\chi(\tau). \tag{4.25}$$

Lemma 4.11. *Assume that $\alpha_l\in(0,1)$, $l=1,2$, $\beta\in[0,1]$, $\rho\in(0,1]$, and $k\in\mathbb{R}^+$. If $(u^*,v^*)\in\mathbb{W}\times\mathbb{W}$ satisfies the inequalities (4.18)-(4.19), then (u^*,v^*) satisfies*

$$\left\|u^*(\tau)-\mathcal{M}_{u^*,v^*}(\tau)-{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau,u^*(\tau),v^*\left(\frac{\tau}{\eta_1}\right)\right)\right\|_{\mathbb{W}} \leqslant \Lambda_3\mathfrak{C}_\chi\epsilon\chi(\tau),$$
$$\left\|v^*(\tau)-\mathcal{N}_{u^*,v^*}(\tau)-{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}g\left(\tau,u^*\left(\frac{\tau}{\eta_2}\right),v^*(\tau)\right)\right\|_{\mathbb{W}} \leqslant \Lambda_4\mathfrak{C}_\chi\epsilon\chi(\tau),$$

where $\mathcal{M}_{u^*,v^*}(\tau)$ and $\mathcal{N}_{u^*,v^*}(\tau)$ are given by (4.20) and (4.21), respectively,

$$\Lambda_3 = \frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left(\sum_{i=1}^{m}|\mu_i|+|\omega_1|\sum_{j=1}^{n}|\delta_j|\right)+{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(b,a),$$
$$\Lambda_4 = \frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_2}{k}-1}\Gamma_k(\gamma_2)}\left(\sum_{j=1}^{n}|\delta_j|+|\omega_2|\sum_{i=1}^{m}|\mu_i|\right)+{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(b,a).$$

Proof. Let (u^*,v^*) is a solution of (4.18)-(4.19). By using Remark 4.8, it follows that

$$\begin{cases} {}^{H}_{a,k}\mathfrak{D}^{\alpha_1,\beta,\rho;\psi}u^*(\tau)=f\left(\tau,u^*(\tau),v^*\left(\frac{\tau}{\eta_1}\right)\right)+z_{u^*,v^*}(\tau), & \tau\in(a,b], \quad \eta_1>1,\\ {}^{H}_{a,k}\mathfrak{D}^{\alpha_2,\beta,\rho;\psi}v^*(\tau)=g\left(\tau,u^*\left(\frac{\tau}{\eta_2}\right),v^*(\tau)\right)+z_{u^*,v^*}(\tau), & \tau\in(a,b], \quad \eta_2>1,\\ \lim\limits_{\tau\to a}{}_{a,k}\mathcal{J}^{k-\gamma_1,\rho;\psi}u^*(\tau)=\sum\limits_{i=1}^{m}\mu_i\;{}_{a,k}\mathcal{J}^{\sigma_i;\psi}v^*(\xi_i), & \xi_i\in(a,b], \quad i=1,2,\ldots,m,\\ \lim\limits_{\tau\to a}{}_{a,k}\mathcal{J}^{k-\gamma_2,\rho;\psi}v^*(\tau)=\sum\limits_{j=1}^{n}\delta_j\;{}_{a,k}\mathcal{J}^{\lambda_j,\rho;\psi}u^*(\theta_j), & \theta_j\in(a,b], \quad j=1,2,\ldots,n. \end{cases} \tag{4.26}$$

From Lemma 2.6, a solution of (4.26) is

$$u^*(\tau)={}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}z_{u^*,v^*}(\tau)+\mathcal{M}_{u^*,v^*}(\tau)+{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau,u^*(\tau),v^*\left(\frac{\tau}{\eta_1}\right)\right)$$

$$+\frac{{}_{k}^{\rho}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left(\sum_{i=1}^{m}\mu_i\ {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}z_{u^*,\nu^*}(\xi_i)+\omega_1\sum_{j=1}^{n}\delta_j\ {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}z_{u^*,\nu^*}(\theta_j)\right),$$

$$\nu^*(\tau)={}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}z_{u^*,\nu^*}(\tau)+\mathcal{N}_{u^*,\nu^*}(\tau)+{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}g\left(\tau,u^*\left(\frac{\tau}{\eta_2}\right),\nu^*(\tau)\right)$$

$$+\frac{{}_{k}^{\rho}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)}{(1-\omega_1\omega_2)\rho^{\frac{\gamma_2}{k}-1}\Gamma_k(\gamma_2)}\left(\sum_{j=1}^{n}\delta_j\ {}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}z_{u^*,\nu^*}(\theta_j)+\omega_2\sum_{i=1}^{m}\mu_i\ {}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}z_{u^*,\nu^*}(\xi_i)\right),$$

where $\mathcal{M}_{u^*,\nu^*}(\tau)$ and $\mathcal{N}_{u^*,\nu^*}(\tau)$ are given by (4.20)-(4.21). Now, we apply Theorem 4.1, Lemma 2.5, and Remark 4.8, we obtain that

$$\left|{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\left[u^*(\tau)-\mathcal{M}_{u^*,\nu^*}(\tau)-{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau,u^*(\tau),\nu^*\left(\frac{\tau}{\eta_1}\right)\right)\right]\right|$$
$$\leqslant\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left(\sum_{i=1}^{m}|\mu_i|{}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}\,|z_{u^*,\nu^*}(s)|\,(\xi_i)\right.$$
$$\left.+|\omega_1|\sum_{j=1}^{n}|\delta_j|{}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}\,|z_{u^*,\nu^*}(s)|\,(\theta_j)\right)+{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a){}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\,|z_{u^*,\nu^*}(s)|\,(b)$$
$$\leqslant\epsilon\left\{\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left(\sum_{i=1}^{m}|\mu_i|{}_{a,k}\mathcal{J}^{\alpha_2+\sigma_i,\rho;\psi}\,(\chi(\xi_i))+|\omega_1|\sum_{j=1}^{n}|\delta_j|{}_{a,k}\mathcal{J}^{\alpha_1+\lambda_j,\rho;\psi}\,(\chi(\theta_j))\right)\right.$$
$$\left.+{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(b,a){}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\,(\chi(b))\right\}.$$

Using $(\mathcal{H}_3)$, we obtain

$$\left|{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\left[u^*(\tau)-\mathcal{M}_{u^*,\nu^*}(\tau)-{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau,u^*(\tau),\nu^*\left(\frac{\tau}{\eta_1}\right)\right)\right]\right|$$
$$\leqslant\left[\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_1}{k}-1}\Gamma_k(\gamma_1)}\left(\sum_{i=1}^{m}|\mu_i|+|\omega_1|\sum_{j=1}^{n}|\delta_j|\right)+{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(b,a)\right]\mathfrak{C}_\chi\epsilon\chi(\tau)=\Lambda_3\mathfrak{C}_\chi\epsilon\chi(\tau).$$

Similarly,

$$\left|{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(\tau,a)\left[\nu^*(\tau)-\mathcal{N}_{u^*,\nu^*}(\tau)-{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}g\left(\tau,u^*\left(\frac{\tau}{\eta_2}\right),\nu^*(\tau)\right)\right]\right|$$
$$\leqslant\left[\frac{1}{|1-\omega_1\omega_2|\rho^{\frac{\gamma_2}{k}-1}\Gamma_k(\gamma_1)}\left(\sum_{j=1}^{n}|\delta_j|+|\omega_2|\sum_{i=1}^{m}|\mu_i|\right)+{}_{k}^{\rho}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(b,a)\right]\mathfrak{C}_\chi\epsilon\chi(\tau)=\Lambda_4\mathfrak{C}_\chi\epsilon\chi(\tau).$$

The proof is finished. □

Theorem 4.12. *From all conditions in Theorem 4.1, the coupled system (1.3) is UHRML stable and consequently generalized UHRML stable.*

Proof. Assume that $\epsilon>0$, $(u^*,\nu^*)\in\mathbb{W}\times\mathbb{W}$ is a solution of (4.18)-(4.19), and $(u,\nu)\in\mathbb{W}\times\mathbb{W}$ is a unique solution of (1.3). Applying Lemma 2.6, it implies

$$x(\tau)=\mathcal{M}_{u,\nu}(\tau)+{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}f\left(\tau,u(\tau),\nu\left(\frac{\tau}{\eta_1}\right)\right),$$
$$\nu(\tau)=\mathcal{N}_{u,\nu}(\tau)+{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}g\left(\tau,u\left(\frac{\tau}{\eta_2}\right),\nu(\tau)\right).$$

Clearly, $\mathcal{M}_{u,\nu}(\tau)=\mathcal{M}_{u^*,\nu^*}(\tau)$ and $\mathcal{N}_{u,\nu}(\tau)=\mathcal{N}_{u^*,\nu^*}(\tau)$, then $|\mathcal{M}_{\nu,u}(\tau)-\mathcal{M}_{u^*,\nu^*}(\tau)|=0$ and $|\mathcal{N}_{u,\nu}(\tau)-$

$\mathcal{N}_{u^*,v^*}(\tau)| = 0$. Using Lemma 4.11 and $|u-v| \leqslant |u|+|v|$, for any $u, v \in \mathbb{R}$, $\tau \in [a,b]$, one has

$$
\begin{aligned}
&\left| {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\,[u(\tau)-u^*(\tau)] \right| \\
&\leqslant \left| {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\,(\mathcal{M}_{u,v}(\tau)-\mathcal{M}_{u^*,v^*}(\tau)) \right| \\
&\quad + {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left| f\left(\tau,u(\tau),v\left(\frac{\tau}{\eta_1}\right)\right) - f\left(\tau,u^*(\tau),v^*\left(\frac{\tau}{\eta_1}\right)\right)\right| \\
&\quad + \left| {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\left[u^*(\tau)-\mathcal{M}_{u^*,v^*}(\tau) - {}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi} f\left(\tau,u^*(\tau),v^*\left(\frac{\tau}{\eta_1}\right)\right)\right]\right| \\
&\leqslant {}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(\tau,a)\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left(\mathcal{L}_{1,f}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_1}{k}-1}(\tau,a)\,\|u-u^*\|_{\mathbb{W}} + \mathcal{L}_{2,f}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma_2}{k}-1}(\tau,a)\,\|v-v^*\|_{\mathbb{W}}\right) + \Lambda_3\mathfrak{C}_{\chi}\epsilon\chi(\tau) \\
&\leqslant \mathcal{L}_{1,f}\frac{\Gamma_k(\gamma_1)\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1}{k}}(b,a)}{\rho^{\frac{\alpha_1}{k}}\Gamma_k(\gamma_1+\alpha_1)}\|u-u^*\|_{\mathbb{W}} + \mathcal{L}_{2,f}\,{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(b,a)\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left(\|v-v^*\|_{\mathbb{W}}\right) + \Lambda_3\mathfrak{C}_{\chi}\epsilon\chi(\tau).
\end{aligned}
$$

Then,

$$
\|u-u^*\|_{\mathbb{W}} \leqslant \frac{\Lambda_3\mathfrak{C}_{\chi}\epsilon\chi(\tau)}{1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})} + \frac{\mathcal{L}_{2,f}\,{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1}{k}}(b,a)}{1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})}\,{}_{a,k}\mathcal{J}^{\alpha_1,\rho;\psi}\left(\|v-v^*\|_{\mathbb{W}}\right). \tag{4.27}
$$

In the similar procedure, we have

$$
\|v-v^*\|_{\mathbb{W}} \leqslant \frac{\Lambda_4\mathfrak{C}_{\chi}\epsilon\chi(\tau)}{1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})} + \frac{\mathcal{K}_{1,g}\,{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_2}{k}}(b,a)}{1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})}\,{}_{a,k}\mathcal{J}^{\alpha_2,\rho;\psi}\left(\|u-u^*\|_{\mathbb{W}}\right). \tag{4.28}
$$

By utilizing Theorem 3.1 and Corollary 3.3 with (4.27)-(4.28), it yields that

$$
\begin{aligned}
\|u-u^*\|_{\mathbb{W}} \leqslant{}& \left[\frac{\mathfrak{C}_{\chi}\epsilon\chi(\tau)}{1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})}\left(\Lambda_3 + \frac{\mathcal{L}_{2,f}\Lambda_4\,{}^{\rho}_{k}\Psi_{\psi}^{1+\frac{\alpha_1-\gamma_1}{k}}(b,a)}{[1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})]\rho^{\frac{\alpha_1}{k}}\Gamma_k(\alpha_1+k)}\right)\right] \\
&\times \mathbb{E}_{k,\alpha_1+\alpha_2,k}\left(\frac{(\rho^{\frac{\alpha_1+\alpha_2}{k}})^{-1}\mathcal{L}_{2,f}\mathcal{K}_{1,g}\,{}^{\rho}_{k}\Psi_{\psi}^{2-\frac{\gamma_1+\gamma_2}{k}}(b,a)}{[1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})][1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})]}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\right), \\
\|v-v^*\|_{\mathbb{W}} \leqslant{}& \left[\frac{\mathfrak{C}_{\chi}\epsilon\chi(\tau)}{1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})}\left(\Lambda_4 + \frac{\mathcal{K}_{1,g}\Lambda_3\,{}^{\rho}_{k}\Psi_{\psi}^{1+\frac{\alpha_2-\gamma_2}{k}}(b,a)}{[1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})]\rho^{\frac{\alpha_2}{k}}\Gamma_k(\alpha_2+k)}\right)\right] \\
&\times \mathbb{E}_{k,\alpha_1+\alpha_2,k}\left(\frac{(\rho^{\frac{\alpha_1+\alpha_2}{k}})^{-1}\mathcal{L}_{2,f}\mathcal{K}_{1,g}\,{}^{\rho}_{k}\Psi_{\psi}^{2-\frac{\gamma_1+\gamma_2}{k}}(b,a)}{[1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})][1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})]}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\right).
\end{aligned}
$$

Thus,

$$
\|(u,v)-(u^*,v^*)\|_{\mathbb{W}} \leqslant (\mathfrak{C}_{f_\chi}+\mathfrak{C}_{g_\chi})\epsilon\chi(\tau)\mathbb{E}_{k,\alpha_1+\alpha_2,k}\left(\kappa_{f_\chi,g_\chi}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\right),
$$

where

$$
\begin{aligned}
\mathfrak{C}_{f_\chi} &:= \frac{\mathfrak{C}_{\chi}}{1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})}\left(\Lambda_3 + \frac{\mathcal{L}_{2,f}\Lambda_4\,{}^{\rho}_{k}\Psi_{\psi}^{1+\frac{\alpha_1-\gamma_1}{k}}(b,a)}{[1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})]\rho^{\frac{\alpha_1}{k}}\Gamma_k(\alpha_1+k)}\right), \\
\mathfrak{C}_{g_\chi} &:= \frac{\mathfrak{C}_{\chi}}{1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})}\left(\Lambda_4 + \frac{\mathcal{K}_{1,g}\Lambda_3\,{}^{\rho}_{k}\Psi_{\psi}^{1+\frac{\alpha_2-\gamma_2}{k}}(b,a)}{[1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})]\rho^{\frac{\alpha_2}{k}}\Gamma_k(\alpha_2+k)}\right),
\end{aligned}
$$

$$\kappa_{f_\chi,g_\chi} := \frac{(\rho^{\frac{\alpha_1+\alpha_2}{k}})^{-1}\mathcal{L}_{2,f}\mathcal{K}_{1,g}{}^{\rho}_{k}\Psi_{\psi}^{1-\frac{\gamma_1+\gamma_2}{k}}(b,a)}{[1-\mathcal{L}_{1,f}(\Omega_1^{\gamma_1}-\Delta_1^{\gamma_1})][1-\mathcal{K}_{2,g}(\Xi_1^{\gamma_2}-\Theta_1^{\gamma_2})]}.$$

Hence, the coupled system (1.3) is UHRML stable. Additionally, taking $\epsilon = 1$, yields

$$\|(u,v)-(u^*,v^*)\|_{\mathbb{W}} \leqslant (\mathfrak{C}_{f_\chi}+\mathfrak{C}_{g_\chi})\chi(\tau)\mathbb{E}_{k,\alpha_1+\alpha_2,k}\Big(\kappa_{f_\chi,g_\chi}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)\Big).$$

Therefore, the coupled system (1.3) is generalized UHRML stable. □

5. Numerical examples

This part provides two numerical simulations to demonstrate the illustration and practical applicability of the main findings.

Example 5.1. Given a nonlinear coupled system of Ambartsumian equations under the nonlocal conditions in the context of the (k,ψ)-HPFDO:

$$\begin{cases} {}^{H}_{0,\frac{9}{10}}\mathfrak{D}^{\frac{\sqrt{\pi}}{2},\frac{\sqrt{\pi}}{4},\frac{\pi}{4};\psi}u(\tau) = f\left(\tau,u(\tau),v\left(\frac{\tau}{2}\right)\right), \quad \tau\in(0,1],\\ {}^{H}_{0,\frac{9}{10}}\mathfrak{D}^{\frac{\sqrt{\pi}}{3},\frac{\sqrt{\pi}}{4},\frac{\pi}{4};\psi}v(\tau) = g\left(\tau,u\left(\frac{\tau}{3}\right),v(\tau)\right), \quad \tau\in(0,1],\\ \lim\limits_{\tau\to 0}{}_{0,\frac{9}{10}}\mathcal{J}^{\frac{1}{2}-\gamma_1,\frac{\pi}{4};\psi}u(\tau) = \sum_{i=1}^{2}\left(\frac{1}{2^i}\right){}_{0,\frac{9}{10}}\mathcal{J}^{\frac{1}{1+4i},\frac{\pi}{4};\psi}v\left(\frac{2i-1}{2i+1}\right),\\ \lim\limits_{\tau\to 0}{}_{0,\frac{9}{10}}\mathcal{J}^{\frac{1}{2}-\gamma_2,\frac{\pi}{4};\psi}v(\tau) = \sum_{j=1}^{3}\left(\frac{1}{4^j}\right){}_{0,\frac{9}{10}}\mathcal{J}^{\frac{j}{3},\frac{\pi}{4};\psi}u\left(\frac{3j-1}{4j-1}\right). \end{cases} \tag{5.1}$$

Under the coupled system (5.1), we achieve $\alpha_l = \sqrt{\pi}/(l+1)$, $\eta_l = l+1$, $l=1,2$, $\beta = \sqrt{\pi}/4$, $\rho = \pi/4$, $k = 9/10$, $a = 0$, $b = 1$, $\mu_i = 1/2^i$, $\sigma_i = 1/(1+4i)$, $\xi_i = (2i-1)/(2i+1)$, $i=1,2$, $\delta_j = 1/4^j$, $\lambda_j = j/3$, $\theta_j = (3j-1)/(4j-1)$, $j=1,2,3$, and $\psi(\tau) = -\exp(-2\tau)$. Then, $\alpha_1/k \approx 0.984697 < 1$, $\alpha_2/k \approx 0.656464 < 1$, $\gamma_1 \approx 0.892330$, $\gamma_2 \approx 0.727821$, $\omega_1 \approx 0.627556$, $\omega_2 \approx 0.298084$, $\Delta_1^k \approx 0.166568$, $\Delta_1^{\gamma_1} \approx 0.168784$, $\Delta_1^{\gamma_2} \approx 0.229276$, $\Theta_1^k \approx 0.161019$, $\Theta_1^{\gamma_1} \approx 0.163049$, $\Theta_1^{\gamma_2} \approx 0.217541$, $\Omega_1^k \approx 1.109342$, $\Omega_1^{\gamma_1} \approx 1.120760$, $\Omega_1^{\gamma_2} \approx 1.425547$, $\Omega_2^k \approx 0.661227$, $\Omega_2^{\gamma_1} \approx 0.669564$, $\Omega_2^{\gamma_2} \approx 0.893336$, $\Xi_1^k \approx 1.108528$, $\Xi_1^{\gamma_1} \approx 1.117850$, $\Xi_1^{\gamma_2} \approx 1.362094$, $\Xi_2^k \approx 0.216834$, $\Xi_2^{\gamma_1} \approx 0.219818$, and $\Xi_1^{\gamma_2} \approx 0.298465$. Now, the nonlinear functions are given as matching for the proposed system (5.1):

$$f(\tau,u(\tau),v(\tau)) = e^{-2\tau} + \frac{2e^{-3\tau}}{\cos(2\pi\tau)+5}\left(\frac{|u(\tau)|}{|u(\tau)|+3}+\frac{|v(\frac{\tau}{2})|}{|v(\frac{\tau}{2})|+5}\right), \tag{5.2}$$

$$g(\tau,u(\tau),v(\tau)) = \sin(3\pi\tau) + \frac{1-\tau}{3\sin(\pi\tau)+6}\left(\frac{3|u(\frac{\tau}{3})|}{|u(\frac{\tau}{3})|+4}+\frac{|v(\tau)|}{|v(\tau)|+2}\right). \tag{5.3}$$

Using (5.2)-(5.3), for any u_i, $v_i \in \mathbb{R}$, $i=1,2$, and $\tau\in[0,1]$, we achieve

$$|f(\tau,u_1,v_1)-f(\tau,u_2,v_2)| \leqslant \frac{1}{9}|u_1-u_2| + \frac{1}{15}|v_1-v_2|, \quad |g(\tau,u_1,v_1)-g(\tau,u_2,v_2)| \leqslant \frac{1}{8}|u_1-u_2| + \frac{1}{12}|v_1-v_2|.$$

From condition $(\mathcal{H}_1)$ in Theorem 4.1, we obtain the constants $\mathcal{L}_{1,f} = 1/9$, $\mathcal{L}_{2,f} = 1/15$, $\mathcal{K}_{1,g} = 1/8$, and $\mathcal{K}_{2,g} = 1/12$. Then, $\sum_{i=1}^{2}[(\Omega_1^{\gamma_i}+\Xi_2^{\gamma_i})\mathcal{L}_{i,f} + (\Omega_2^{\gamma_i}+\Xi_1^{\gamma_i})\mathcal{K}_{i,g}] \approx 0.675255 < 1$, we obtain that Theorem 4.1 is satisfied. Hence, the coupled system (5.1) has a unique solution on $[0,1]$. Next, by condition $(\mathcal{H}_2)$ in Theorem 4.2 with (5.2)-(5.3), we achieve

$$|f(\tau,u(\tau),v(\tau))| \leqslant e^{-2\tau} + \frac{4e^{-3\tau}}{\cos(2\pi\tau)+5} \quad \text{and} \quad |g(\tau,u(\tau),v(\tau))| \leqslant 1 + \frac{4(1-\tau)}{3\sin(\pi\tau)+6}.$$

Then,

$$\Upsilon_1(\tau) = e^{-2\tau} + \frac{2e^{-3\tau}}{\cos(2\pi\tau)+5} \quad \text{and} \quad \Upsilon_2(\tau) = 1 + \frac{2(1-\tau)}{3\sin(\pi\tau)+6}.$$

By direct computation, we have $\sum_{i=1}^{2}\left[(\Delta_1^{\gamma_i}+\Xi_2^{\gamma_i})\mathcal{L}_{i,f}+(\Omega_2^{\gamma_i}+\Theta_1^{\gamma_i})\mathcal{K}_{i,g}\right]\approx 0.274999<1$. Hence, by Theorem 4.2, the coupled system (5.1) has at least one solution on $[0,1]$. Furthermore, we can compute all constants $\Lambda_1\approx 1.770569$ and $\Lambda_2\approx 1.325362$. Applying above constants, we achieve $\mathfrak{C}_f\approx 2.073791>0$, $\mathfrak{C}_g\approx 1.570461>0$, $\kappa_{f,g}\approx 0.008236>0$. Hence, by all conditions in Theorem 4.10, the coupled system (5.1) is UHML stable on $[0,1]$. Taking $\zeta_f(\epsilon)=\mathfrak{C}_f\epsilon$ and $\zeta_g(\epsilon)=\mathfrak{C}_g\epsilon$ with $\zeta_f(0)=\zeta_g(0)=0$, the coupled system (5.1) is generalized UHML stable on $[0,1]$. In addition, by taking $\chi(\tau)={}_k^{\rho}\Psi_{\psi}^{\frac{2}{k}-1}(\tau,a)$ in (4.25), then

$$
{}_{a,k}\mathfrak{J}^{\alpha_1+\alpha_2,\rho;\psi}\left(\chi(\tau)\right)=\frac{\Gamma_k(2)\,{}_k^{\rho}\Psi_{\psi}^{\frac{2+\alpha_1+\alpha_2}{k}-1}(\tau,a)}{\rho^{\frac{\alpha_1+\alpha_2}{k}}\Gamma_k(2+\alpha_1+\alpha_2)}=\frac{\Gamma_k(2)\,{}_k^{\rho}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(\tau,a)}{\rho^{\frac{\alpha_1+\alpha_2}{k}}\Gamma_k(2+\alpha_1+\alpha_2)}\chi(\tau).
$$

This yields that $\mathfrak{C}_\chi:=(\Gamma_k(2)\,{}_k^{\rho}\Psi_{\psi}^{\frac{\alpha_1+\alpha_2}{k}}(1,0))/(\rho^{\frac{\alpha_1+\alpha_2}{k}}\Gamma_k(2+\alpha_1+\alpha_2))\approx 0.235640$. From all previous constants, we can compute the values $\Lambda_3\approx 1.934753$, $\Lambda_4\approx 1.298036$, $\mathfrak{C}_{f_\chi}\approx 0.531477>0$, $\mathfrak{C}_{g_\chi}\approx 0.368739>0$, and $\kappa_{f_\chi,g_\chi}\approx 0.010420>0$. Therefore, by all conclusions in Theorem 4.12, the coupled system (5.1) is UHRML stable on $[0,1]$. If we take $\epsilon=1$, the coupled system (5.1) is generalized UHRML stable on $[0,1]$.

Next, we use the numerical technique in [35] to show the graphical approximations of the given Example 5.2.

Example 5.2. Given a nonlinear coupled initial value system of Ambartsumian equations in the sense of the (k,ψ)-Caputo-PFDO:

$$
\begin{cases} {}_{0,k}^{C}\mathfrak{D}^{\alpha_1,\rho;\psi}u(\tau)=f\left(\tau,u(\tau),v\left(\frac{\tau}{\eta_2}\right)\right), \quad {}_{0,k}^{C}\mathfrak{D}^{\alpha_2,\rho;\psi}v(\tau)=g\left(\tau,u\left(\frac{\tau}{\eta_1}\right),v(\tau)\right), \\ u(0)=0, \quad v(0)=0, \end{cases} \tag{5.4}
$$

where $f\left(\tau,u(\tau),v\left(\tau/\eta_2\right)\right)$ and $g\left(\tau,u\left(\tau/\eta_1\right),v(\tau)\right)$ are provided by (5.2)-(5.3), respectively.

To achieve the numerical approximation of the proposed problem (5.4), we apply (2.7) in Lemma 2.7. Then, the operators ${}_{0,k}^{C}\mathfrak{D}^{\alpha_1,\rho;\psi}u(\tau)$ and ${}_{0,k}^{C}\mathfrak{D}^{\alpha_2,\rho;\psi}v(\tau)$ can be replaced as follows:

$$
\begin{aligned} {}_{0,k}^{C}\mathfrak{D}^{\alpha_1,\rho;\psi}u(\tau)\approx{}& A_N^{\alpha_1}(\psi(\tau)-\psi(0))^{1-\frac{\alpha_1}{k}}\left[(1-\rho)u(\tau)+k\rho\frac{u'(\tau)}{\psi'(\tau)}\right] \\ &-e^{\frac{\rho-1}{k\rho}\psi(\tau)}\sum_{i=1}^{N}B_{N,i}^{\alpha_1}(\psi(\tau)-\psi(0))^{1-\frac{\alpha_1}{k}-i}\mathcal{V}_i(\tau), \end{aligned} \tag{5.5}
$$

$$
\begin{aligned} {}_{0,k}^{C}\mathfrak{D}^{\alpha_2,\rho;\psi}v(\tau)\approx{}& A_N^{\alpha_2}(\psi(\tau)-\psi(0))^{1-\frac{\alpha_2}{k}}\left[(1-\rho)v(\tau)+k\rho\frac{v'(\tau)}{\psi'(\tau)}\right] \\ &-e^{\frac{\rho-1}{k\rho}\psi(\tau)}\sum_{i=1}^{N}B_{N,i}^{\alpha_2}(\psi(\tau)-\psi(0))^{1-\frac{\alpha_2}{k}-i}\mathcal{U}_i(\tau), \end{aligned} \tag{5.6}
$$

where

$$
\begin{aligned} A_N^{\alpha_\ell}&=\frac{1}{\rho^{1-\frac{\alpha_\ell}{k}}\Gamma_k(2k-\alpha_\ell)}\sum_{i=0}^{N}\frac{\Gamma\left(i+\frac{\alpha_\ell}{k}-1\right)}{i!\Gamma\left(\frac{\alpha_\ell}{k}-1\right)}, \quad \ell=1,2, \\ B_{N,i}^{\alpha_\ell}&=\frac{\Gamma\left(i+\frac{\alpha_\ell}{k}-1\right)}{\rho^{1-\frac{\alpha_\ell}{k}}(i-1)!\Gamma_k(2k-\alpha_\ell)\Gamma\left(\frac{\alpha_\ell}{k}-1\right)}, \quad \ell=1,2, \quad i=1,2,\ldots,N, \end{aligned}
$$

and $\mathcal{V}_i(\tau)$, $\mathcal{U}_i(\tau)$ are given in Lemma 2.7. Then

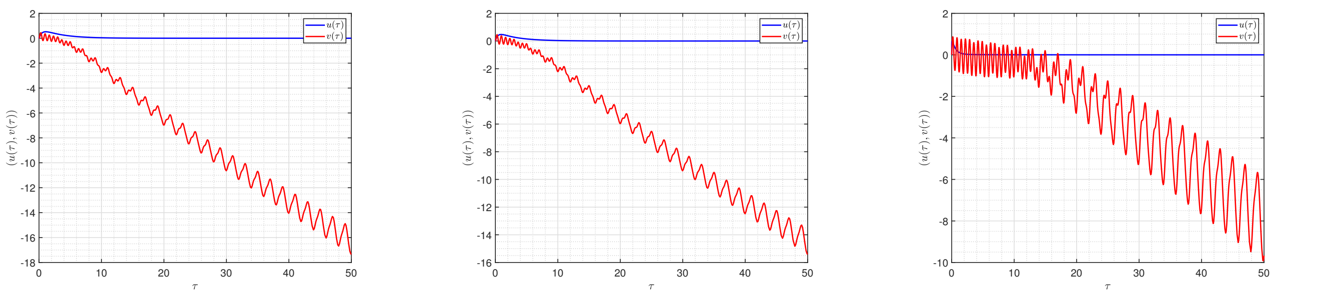
$$
\begin{cases} \mathcal{V}_i'(\tau)=(\psi(\tau)-\psi(0))^{i-1}\psi'(\tau)e^{\frac{1-\rho}{k\rho}\psi(\tau)}\left[(1-\rho)u(\tau)+k\rho\frac{u'(\tau)}{\psi'(\tau)}\right], \quad i=1,2,\ldots,N, \\ \mathcal{U}_i'(\tau)=(\psi(\tau)-\psi(0))^{i-1}\psi'(\tau)e^{\frac{1-\rho}{k\rho}\psi(\tau)}\left[(1-\rho)v(\tau)+k\rho\frac{v'(\tau)}{\psi'(\tau)}\right], \quad i=1,2,\ldots,N, \\ \mathcal{V}_i(0)=0, \quad \mathcal{U}_i(0)=0. \end{cases} \tag{5.7}
$$

By applying (5.5)-(5.7), then the problem (5.4) can be written as

$$
\begin{cases}
A_N^{\alpha_1}(\psi(\tau)-\psi(0))^{1-\frac{\alpha_1}{k}}\left[(1-\rho)u(\tau)+k\rho\frac{u'(\tau)}{\psi'(\tau)}\right] \\
\quad -e^{\frac{\rho-1}{k\rho}\psi(\tau)}\sum\limits_{i=1}^{N}B_{N,i}^{\alpha_1}(\psi(\tau)-\psi(0))^{1-\frac{\alpha_1}{k}-i}\mathcal{V}_i(\tau)=f\left(\tau,u(\tau),v\left(\frac{\tau}{\eta_2}\right)\right), \\
A_N^{\alpha_2}(\psi(\tau)-\psi(0))^{1-\frac{\alpha_2}{k}}\left[(1-\rho)v(\tau)+k\rho\frac{v'(\tau)}{\psi'(\tau)}\right] \\
\quad -e^{\frac{\rho-1}{k\rho}\psi(\tau)}\sum\limits_{i=1}^{N}B_{N,i}^{\alpha_2}(\psi(\tau)-\psi(0))^{1-\frac{\alpha_2}{k}-i}\mathcal{U}_i(\tau)=g\left(\tau,u\left(\frac{\tau}{\eta_1}\right),v(\tau)\right), \\
\mathcal{V}_i'(\tau)=(\psi(\tau)-\psi(0))^{i-1}\psi'(\tau)e^{\frac{1-\rho}{k\rho}\psi(\tau)}\left[(1-\rho)u(\tau)+k\rho\frac{u'(\tau)}{\psi'(\tau)}\right], \quad i=1,2,\ldots,N, \\
\mathcal{U}_i'(\tau)=(\psi(\tau)-\psi(0))^{i-1}\psi'(\tau)e^{\frac{1-\rho}{k\rho}\psi(\tau)}\left[(1-\rho)v(\tau)+k\rho\frac{v'(\tau)}{\psi'(\tau)}\right], \quad i=1,2,\ldots,N, \\
u(0)=0,\ \ v(0)=0,\ \ \mathcal{V}_i(0)=0,\ \ \mathcal{U}_i(0)=0.
\end{cases} \tag{5.8}
$$

Using a built-in function ode45 in MATLAB, the approximate solution of the coupled system (5.8) can be shown with various parameters $k = 0.9, 1.0, 10$ and $\psi(\tau) = \tau, \tau^{3/2}, -e^{-\tau}, \log(\tau+1)$, and $\ln\tau$. For the graphical simulations, we set $\alpha_l = \sqrt{\pi}/(l+1)$, $\eta_l = l+1$, $l = 1,2$, $\rho = \pi/4$, $a = 0$, $b = 50$, and $N = 5$. The graphical simulations of the approximate solution are divided them into seven cases.

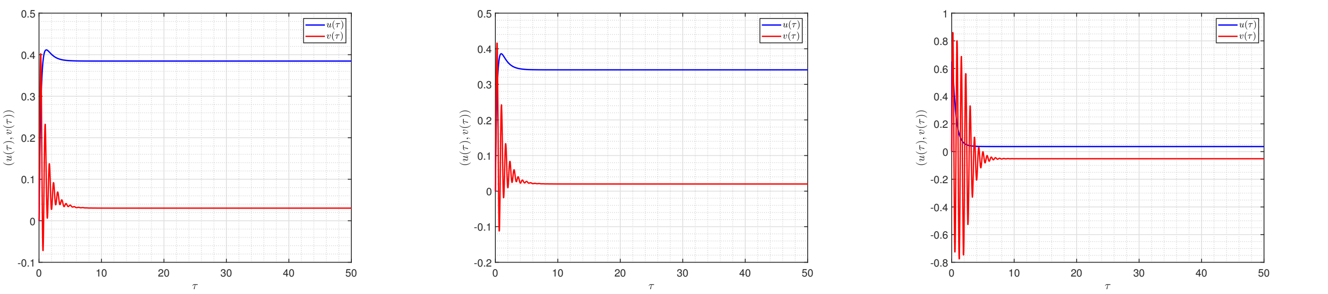
Case 1. Figure 1 displays the dynamics of the approximate solution via the fixed $\psi(\tau) = \tau$ and vary values $k = 0.9, 1.0, 10$. As k grows from 0.9 to 10, the solution $u(\tau)$ converges to a steady state while $v(\tau)$ oscillates more, see Figures 1a-1c.



(a) $k = 0.9$ and $\psi(\tau) = \tau$. **(b)** $k = 1.0$ and $\psi(\tau) = \tau$. **(c)** $k = 10$ and $\psi(\tau) = \tau$.

Figure 1: The graphical simulations of the approximate solutions via $\alpha_1 = \sqrt{\pi}/2$, $\alpha_2 = \sqrt{\pi}/3$, $\rho = \pi/4$, $N = 5$, $k = 0.9, 1.0, 10$, and $\psi(\tau) = \tau$ for the coupled system (5.8).

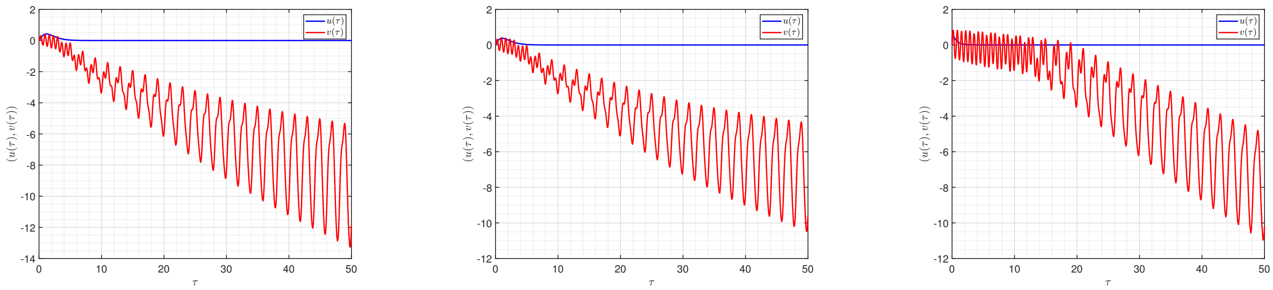
Case 2. Figure 2 displays the dynamics of the approximate solution via the fixed $\psi(\tau) = -e^{-\tau}$ and vary values $k = 0.9, 1.0, 10$. As k grows from 0.9 to 10, the solution $u(\tau)$ converges to a steady state, grows $v(\tau)$ oscillates before settling into a steady state, see Figures 2a-2c.



(a) $k = 0.9$ and $\psi(\tau) = -e^{-\tau}$. **(b)** $k = 1.0$ and $\psi(\tau) = -e^{-\tau}$. **(c)** $k = 10$ and $\psi(\tau) = -e^{-\tau}$.

Figure 2: The graphical simulations of the approximate solutions via $\alpha_1 = \sqrt{\pi}/2$, $\alpha_2 = \sqrt{\pi}/3$, $\rho = \pi/4$, $N = 5$, $k = 0.9, 1.0, 10$, and $\psi(\tau) = -e^{-\tau}$ for the coupled system (5.8).

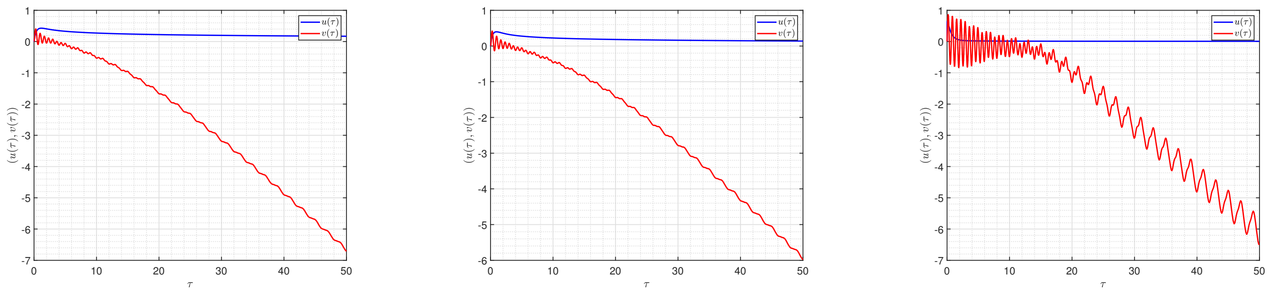
Case 3. Figure 3 displays the dynamics of the approximate solution via the fixed $\psi(\tau) = \tau^{3/2}$ and vary values $k = 0.9, 1.0, 10$. As k grows from 0.9 to 10, the solution $u(\tau)$ converges to a steady state while $v(\tau)$ oscillates more, see Figures 3a-3c.



(a) $k = 0.9$ and $\psi(\tau) = \tau^{3/2}$. **(b)** $k = 1.0$ and $\psi(\tau) = \tau^{3/2}$. **(c)** $k = 10$ and $\psi(\tau) = \tau^{3/2}$.

Figure 3: The graphical simulations of the approximate solutions via $\alpha_1 = \sqrt{\pi}/2$, $\alpha_2 = \sqrt{\pi}/3$, $\rho = \pi/4$, $N = 5$, $k = 0.9, 1.0, 10$, and $\psi(\tau) = \tau^{3/2}$ for the coupled system (5.8).

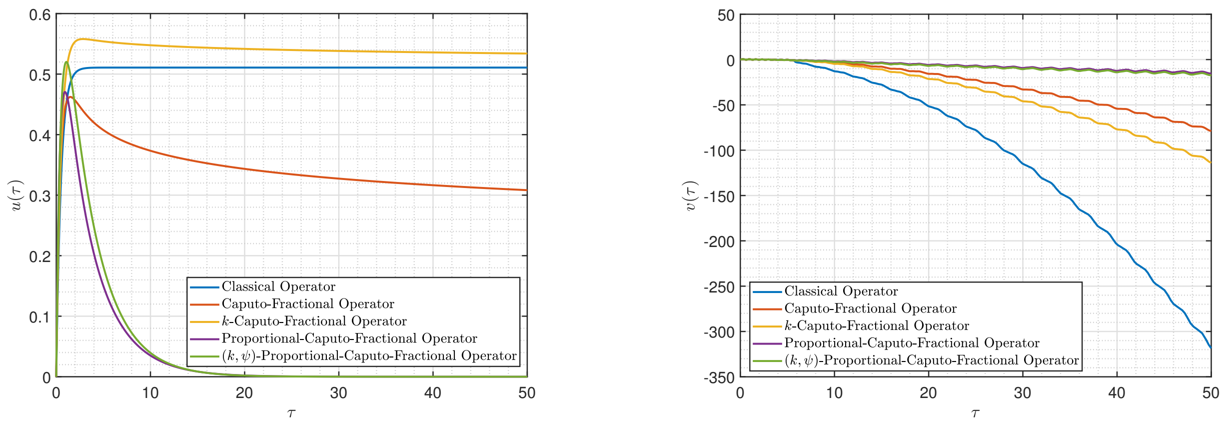
Case 4. Figure 4 displays the dynamics of the approximate solution via the fixed $\psi(\tau) = \log(\tau + 1)$ and vary values $k = 0.9, 1.0, 10$. As k grows from 0.9 to 10, the solution $u(\tau)$ converges to a steady state while $v(\tau)$ oscillates more, see Figures 4a-4c.



(a) $k = 0.9$ and $\psi(\tau) = \log(\tau + 1)$. **(b)** $k = 1.0$ and $\psi(\tau) = \log(\tau + 1)$. **(c)** $k = 10$ and $\psi(\tau) = \log(\tau + 1)$.

Figure 4: The graphical simulations of the approximate solutions via $\alpha_1 = \sqrt{\pi}/2$, $\alpha_2 = \sqrt{\pi}/3$, $\rho = \pi/4$, $N = 5$, $k = 0.9, 1.0, 10$, and $\psi(\tau) = \log(\tau + 1)$ for the coupled system (5.8).

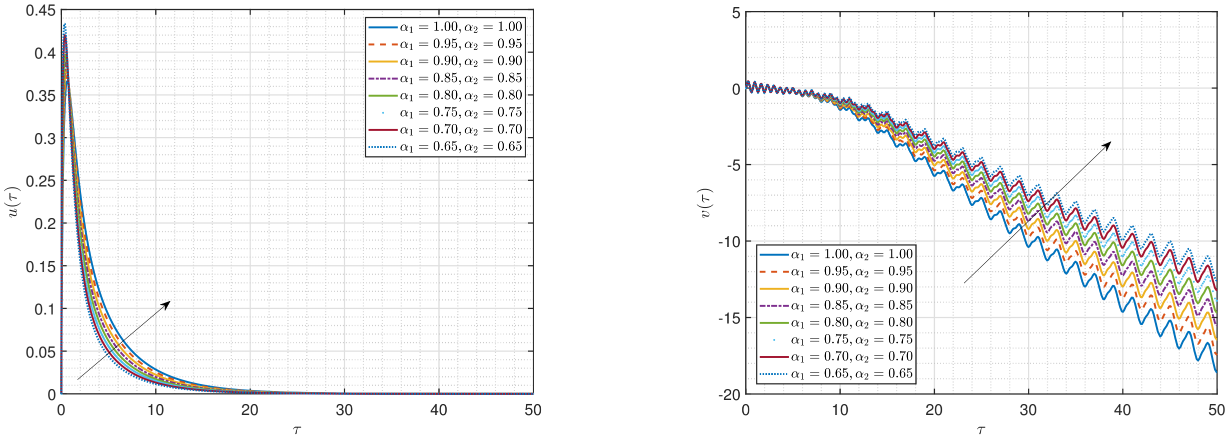
Case 5. We compared the approximate solution of the coupled system (5.8) with the classical-order ($\alpha_1 = 1, \alpha_2 = 1, \beta = 1, k = 1, \rho = 1$), fractional-order ($\alpha_1 = \sqrt{\pi}/2, \alpha_2 = \sqrt{\pi}/3, \beta = 1, k = 1, \rho = 1$), proportional fractional-order ($\alpha_1 = \sqrt{\pi}/2, \alpha_2 = \sqrt{\pi}/3, \beta = 1, k = 1, \rho = 0.95$), and k-fractional-order ($\alpha_1 = \sqrt{\pi}/2, \alpha_2 = \sqrt{\pi}/3, \beta = 1, k = 0.9, \rho = 1$), as shown in Figure 5.



(a) The approximate solution $u(\tau)$. **(b)** The approximate solution $v(\tau)$.

Figure 5: Comparing of the approximate solutions for classical-order, fractional-order, proportional fractional-order, and k-fractional-order via $\beta = 1$, $N = 5$, and $\psi(\tau) = \tau$ for the coupled system (5.8).

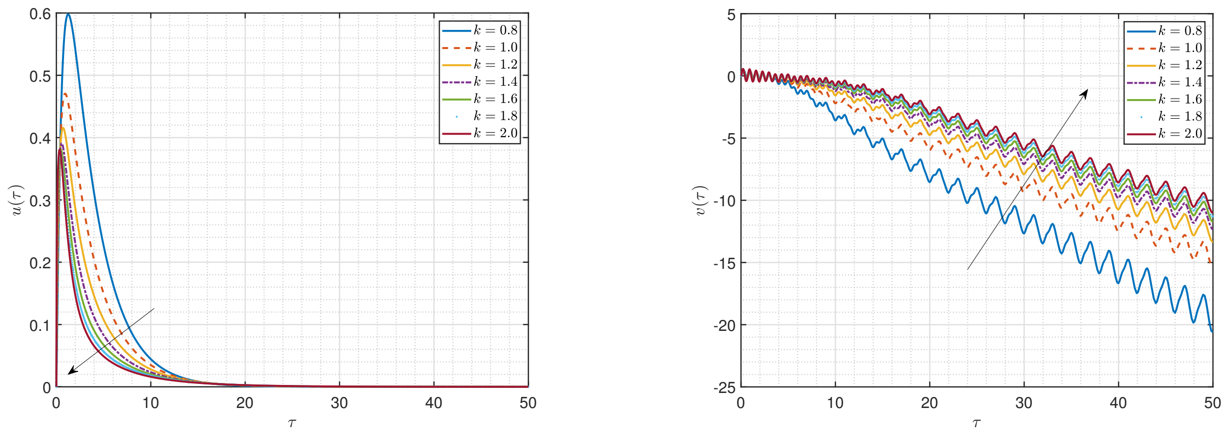
Case 6. Figure 6 displays the dynamics of the approximate solution via the fixed $\psi(\tau) = \tau$, $\rho = \pi/4$, $k = 1.5$, and vary values $\alpha_1 = 0.65, 0.70, 0.75, \ldots, 1.00$, $\alpha_2 = 0.65, 0.70, 0.75, \ldots, 1.00$. As α_1 and α_2 grow from 0.65 to 1.00, the solution $u(\tau)$ converges to a steady state while $v(\tau)$ oscillates more, see Figures 6a-6b.



(a) The approximate solution $u(\tau)$. **(b)** The approximate solution $v(\tau)$.

Figure 6: The graphical simulations of the approximate solutions via $\alpha_1 = 0.65, 0.70, 0.75, \ldots, 1.00$, $\alpha_2 = 0.65, 0.70, 0.75, \ldots, 1.00$, $\rho = \pi/4$, $k = 1.5$, $N = 5$, and $\psi(\tau) = \tau$ for the coupled system (5.8).

Case 7. Figure 7 displays the dynamics of the approximate solution via the fixed $\psi(\tau) = \tau$, $\alpha_1 = \sqrt{\pi}/2$, $\alpha_2 = \sqrt{\pi}/3$, $\rho = \pi/4$, and vary values $k = 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0$. As k grows from 0.8 to 2.0, the solution $u(\tau)$ converges to a steady state while $v(\tau)$ oscillates more, see Figures 7a-7b.



(a) The approximate solution $u(\tau)$. **(b)** The approximate solution $v(\tau)$.

Figure 7: The graphical simulations of the approximate solutions via $\alpha_1 = \sqrt{\pi}/2$, $\alpha_2 = \sqrt{\pi}/3$, $\rho = \pi/4$, $k = 0.8, 1.0, \ldots, 2.0$, $N = 5$, and $\psi(\tau) = \tau$ for the coupled system (5.8).

6. Conclusion

In a nutshell, we presented novel results that substantially contribute to the literature on (k, ψ)-HPFDO for BVPs of coupled Ambartsumian system. The famous fixed point theorems, such as Banach's and Krasnosel'skii's, were relied on to analyze the desired existence and uniqueness results. The Ulam stability in the context of UHML and UHRML were investigated to use the approximate solution to replace the exact solution. Moreover, numerical examinations illustrate the theoretical results obtained well, confirming the theoretical analysis results. The work presented here is more general than other coupled systems because the (k, ψ)-HPFDO can be reduced to various fractional and classical operators, as shown in Table 1. Our coupled system addresses a special case by determining the relevant parameters associated with the coupled system. If we set $\sigma_i = \lambda_j = 0$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we have the following multi-point nonlocal coupled system

$$\begin{cases} \lim\limits_{\tau \to a} {}_{a,k}\mathcal{J}^{k-\gamma_1,\rho;\psi} u(\tau) = \sum_{i=1}^{m} \mu_i v(\xi_i), & a < \xi_i \leqslant b, \quad i = 1, 2, \ldots, m, \\ \lim\limits_{\tau \to a} {}_{a,k}\mathcal{J}^{k-\gamma_2,\rho;\psi} v(\tau) = \sum_{j=1}^{n} \delta_j u(\theta_j), & a < \theta_j \leqslant b, \quad j = 1, 2, \ldots, n. \end{cases}$$

For future work, we intend to investigate the impulsive situations of the coupled system (1.3). In addition, various boundary conditions will be considered, such as the fractional-integro-differential condition. We will explore new analytical and numerical techniques to solve this operator.

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References

[1] W. Adel, H. Rezazadeh, M. Eslami, M. Mirzazadeh, *A numerical treatment of the delayed Ambartsumian equation over large interval*, J. Interdiscip. Math., **23** (2020), 1077–1091. 1
[2] S. Ahmad, A. Ullah, A. Akgül, M. De la Sen, *A study of fractional order Ambartsumian equation involving exponential decay kernel*, AIMS Math., **6** (2021), 9981–9997. 1
[3] A. A. Alatawi, M. Aljoufi, F. M. Alharbi, A. Ebaid, *Investigation of the surface brightness model in the Milky Way via homotopy perturbation method*, J. Appl. Math. Phys., **8** (2020), 434–442. 1
[4] R. M. S. Alyoubi, A. Ebaid, E. R. El-Zahar, M. D. Aljoufi, *A novel analytical treatment for the Ambartsumian delay differential equation with a variable coefficient*, AIMS Math., **9** (2024), 35743–35758. 1
[5] J. Alzabut, T. Abdeljawad, F. Jarad, W. Sudsutad, *A Gronwall inequality via the generalized proportional fractional derivative with applications*, J. Inequal. Appl., **2019** (2019), 12 pages. 1
[6] J. Alzabut, Y. Adjabi, W. Sudsutad, M.-U. Rehman, *New generalizations for Gronwall type inequalities involving a ψ-fractional operator and their applications*, AIMS Math., **6** (2021), 5053–5077. 1
[7] V. A. Ambartsumian, *On the theory of brightness fluctuations in the Milky Way*, C. R. (Doklady) Acad. Sci. URSS (N.S.), **44** (1944), 223–226. 1
[8] A. Atangana, *Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system*, Chaos Solitons Fractals, **102** (2017), 396–406. 1
[9] H. O. Bakodah, A. Ebaid, *Exact solution of Ambartsumian delay differential equation and comparison with Daftardar-Gejji and Jaffari approximate method*, Mathematics, **6** (2018), 10 pages. 1
[10] R. Bellman, *Stability theory of differential equations*, McGraw-Hill Book Co., New York-Toronto-London, (1953). 1
[11] R. Courant, *Differential and integral calculus. Vol. II*, John Wiley & Sons, New York, (1988). 3
[12] A. Ebaid, A. Al-Enazi, B. Z. Albalawi, M. D. Aljoufi, *Accurate approximate solution of Ambartsumian delay differential equation via decomposition method*, Math. Comput. Appl., **24** (2019), 9 pages. 1
[13] A. Ebaid, C. Cattani, A. S. Al Juhani, E. R. El-Zahar, *A novel exact solution for the fractional Ambartsumian equation*, Adv. Differ. Equ., **2021** (2021), 18 pages. 1
[14] N. Eghbali, V. Kalvandi, J. M. Rassias, *A fixed point approach to the Mittag-Leffler-Hyers-Ulam stability of a fractional integral equation*, Open Math., **14** (2016), 237–246. 1
[15] E. M. Elsayed, S. Manikandan, K. Kanagarajan, D. Vivek, *Existence and stability for Ambartsumian equation with Ξ-Hilfer generalized proportional fractional derivative*, Commun. Nonlinear Anal., **2** (2023), 1–17. 1, 1
[16] R. George, M. Houas, M. Ghaderi, S. Rezapour, S. K. Elagan, *On a coupled system of pantograph problem with three sequential fractional derivatives by using positive contraction-type inequalities*, Results Phys., **39** (2022), 9 pages. 1
[17] A. Granas, J. Dugundji, *Fixed point theory*, Springer-Verlag, New York, (2003). 4.1
[18] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publishing Co., River Edge, NJ, (2000). 1
[19] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224. 1
[20] D. Jiang, C. Bai, *On coupled Gronwall inequalities involving a ψ-fractional integral operator with its applications*, AIMS Math., **7** (2022), 7728–7741. 1, 1
[21] U. N. Katugampola, *New approach to generalized fractional integral*, Appl. Math. Comput., **218** (2011), 860–865. 1

[22] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B.V., Amsterdam, (2006). 1

[23] M. A. Krasnosel'skii, *Two remarks on the method of successive approximations*, Uspehi Mat. Nauk (N.S.), **10** (1955), 123–127. 4.1, 4.1

[24] K. D. Kucche, A. D. Mali, *On the nonlinear (k, Ψ)-Hilfer fractional differential equations*, Chaos Solitons Fractals, **152** (2021), 14 pages. 1, 2

[25] D. Kumar, J. Singh, D. Baleanu, S. Rathore, *Analysis of a fractional model of the Ambartsumian equation*, Eur. Phys. J. Plus, **133** (2018), 7 pages. 1

[26] E. L. Lima, *Real analysis (in Portuguese)*, Instituto de Matemática Pura e Aplicada, CNPq, Rio de Janeiro, (2004). 3

[27] S. Manikandan, S. Sivasundaram, D. Vivek, K. Kanagarajan, *Controllability and qualitative property results for Ambartsumian equation via Ξ-Hilfer generalized proportional fractional derivative on time scales*, Nonlinear Stud., **29** (2022), 1233–1255. 1

[28] S. K. Ntouyas, B. Ahmad, J. Tariboon, *Coupled systems of nonlinear proportional fractional differential equations of the Hilfer-type with multi-point and integro-multi-strip boundary conditions*, Foundations, **3** (2023), 241–259. 1

[29] M. D. Ortigueira, G. Bengochea, *A simple solution for the general fractional Ambartsumian equation*, Appl. Sci., **13** (2023), 15 pages. 1

[30] J. Patade, *Series solution of system of fractional order Ambartsumian equations: application in astronomy*, arXiv preprint arXiv:2008.04904, (2020), 5 pages. 1

[31] J. Patade, S. Bhalekar, *On analytical solution of Ambartsumian equation*, Nat. Acad. Sci. Lett., **40** (2017), 291–293. 1

[32] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300. 1

[33] A. Samadi, S. K. Ntouyas, B. Ahmad, J. Tariboon, *Investigation of a nonlinear coupled (k, ψ)-Hilfer fractional differential system with coupled (k, ψ)-Riemann-Liouville fractional integral boundary conditions*, Foundations, **2** (2022), 918–933. 1

[34] W. Sudsutad, J. Kongson, C. Thaiprayoon, *On generalized (k, ψ)-Hilfer proportional fractional operator and its applications to the higher-order Cauchy problem*, Bound. Value Probl., **2024** (2024), 32 pages. 1, 1, 2, 2.1, 2.2, 2.3, 2.4, 2.5, 2

[35] W. Sudsutad, C. Thaiprayoon, A. Aphithana, J. Kongson, W. Sae-dan, *Qualitative results and numerical approximations of the (k, ψ)-Caputo proportional fractional differential equations and applications to blood alcohol levels model*, AIMS Math., **9** (2024), 34013–34041. 1, 2, 2.7, 5

[36] S. Theswan, S. K. Ntouyas, J. Tariboon, *Coupled systems of ψ-Hilfer generalized proportional fractional nonlocal mixed boundary value problems*, AIMS Math., **8** (2023), 22009–22036. 1

[37] S. M. Ulam, *A collection of mathematical problems*, Interscience Publishers, New York-Londonk, (1960). 1

[38] R. Wong, *Asymptotic approximations of integrals*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (2001). 3

[39] H. Ye, J. Gao, Y. Ding, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl., **328** (2007), 1075–1081. 1