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Existence and stability of a mild solution to an abstract sequential time-fractional diffusion equation of pantograph type

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Abstract

This article studies a sequential time-fractional diffusion equation characterized by the Caputo derivative. We establish the existence of a mild solution using the resolvent operator and Schaefer's fixed point technique. Moreover, we explore Ulam-Hyers and Ulam-Hyers-Rassias stabilities through nonlinear methods. The study also presents examples of applications of these techniques, such as application to a partial Caputo time-fractional diffusion equation.

Keywords: Caputo fractional derivative (CFD), pantograph problem, time-fractional diffusion equation (TFDE), Ulam-Hyers-Rassias stability (UHR-S), resolvent operator (R-operator).

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1. Introduction

Fixed point theorems play an essential role in studying the qualitative properties of partial, ordinary,

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and fractional differential equations. For more details on fractional calculus and its applications, refer to [1–3, 15, 20, 21, 28–31, 33, 34, 38, 40, 42, 45, 46, 48]. Formerly, researchers employed fractional calculus to investigate the existence results and stability of sequential fractional differential equations (SFDE). Ahmed and Ntouyas [4] investigated the existence and uniqueness of solutions for Caputo-Hadamard SFDE of order neutral functional. Ahmed et al. [5] studied SFDE with semi-periodic and integro-multipoint conditions. Al-Khateeb et al. [7], demonstrated UH stability and uniqueness for nonlinear SFDE with integral boundary conditions. Aqlan et al. [9] evolved the existence theory for SFDE with anti-periodic conditions. Salem and Almaghamsi [44] investigated the solvability of SFDE at resonance, while Zada et al. [51] demonstrated the existence and stability analysis of a coupled system of nonlinear Caputo SFDE with integral conditions. In addition to previous work on SFDE, other scholars have investigated pantograph FDE. Ali et al. [6] proposed the stability analysis of the initial problem of implicit pantograph-type FDE with impulsive conditions, Balachandran et al. [12] discussed in depth with examples the existence of solutions of nonlinear pantograph FDE. Derfel and Iserles [14] focused on two FDE with complex functions and addressed qualitative theorems of solutions and their asymptotic action. We also refer the reader to the important research of Iserl [22, 23] that deals with the exact and discretized stability of the pantograph problem. Khan et al. [10] conducted a significant study on a nonlinear multi-term problem of pantograph FDE. Vivek et al. [49] also conducted research on the existence and UH-S results for a neutral pantograph problem with complex order. In [8], the authors used the numerical collocation techniques to solve the pantograph FDE:

$$\mathcal{D}^{i\alpha}\zeta(\mathtt{w}) = \sigma(\mathtt{w}) + \sum_{r=0}^{m} \ell_r(\mathtt{w})\, y(p_r\mathtt{w}),\ a < w < b,\ n-1 \leqslant m\alpha \leqslant n,$$

under the initial conditions $\mathcal{D}^{i}\zeta(c) = \rho_i$, $a \leqslant c \leqslant b$, $i = 0, 1, \ldots, n-1$, where $0 < p_r < 1$ and ρ_i is suitable constant. In [19], the authors treated a pantograph SFDE,

$$\left(\mathcal{D}^{\alpha} + \kappa\mathcal{D}^{\delta}\right)\zeta(\mathtt{w}) = \phi(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{D}^{\delta}\zeta(\eta\mathtt{w})),\ \mathtt{w} \in [0, \chi], \tag{1.1}$$

with

$$\zeta(0) = \sigma(\zeta),\ \zeta(\chi) = \theta,\ \theta \in \mathbb{R}. \tag{1.2}$$

where $1 < \alpha \leqslant 2$, $0 < \delta \leqslant 1$, $0 < \eta < 1$, $\kappa \in \mathbb{R}^{+}$, $\mathcal{D}^{\alpha}$ and $\mathcal{D}^{\delta}$ denote the CFD, and ϕ and σ are continuous functions. The authors explored the issue defined by equations uniqueness of (1.1)-(1.2), as well as its UH and UHR stability characteristics. Pantograph equations have a significant impact on pure and practical mathematics, as well as physics. In acknowledgment of their significance, many researchers have generalized these equations in numerous directions and investigated the availability of numerical and theoretical solutions to such problems. In addition, other authors used various fractional derivative operators to study the existence and UH stability of these generalized pantograph equations.

The theory of nonlocal problem with abstract FDE has progressed significantly over the previous two decades, as evidenced by the references [11, 16–18, 24, 27, 32, 36] and the broader relevant literature. In [50], the authors treated the TFDE with the equation

$$\frac{\partial^{\alpha}\zeta(x,\mathtt{w})}{\partial\mathtt{w}^{\alpha}} - \frac{\partial^{2}\zeta(x,\mathtt{w})}{\partial x^{2}} = -Q(x)\zeta(x,\mathtt{w}),\ \ 0 \leqslant x \leqslant 1,\ \ 0 \leqslant \mathtt{w} \leqslant \chi,$$

and the initial condition $\zeta(x, 0) = \Phi(x)$, and

$$\frac{\partial\zeta(0,\mathtt{w})}{\partial x} - k\zeta(0,\mathtt{w}) = 0,\ \ \frac{\partial\zeta(1,\mathtt{w})}{\partial x} + k\zeta(1,\mathtt{w}) = \mu(\mathtt{w}),\ \ 0 \leqslant \mathtt{w} \leqslant \chi,$$

with $Q(x) = Q(1-x)$, $Q(x) \geqslant 0$ is a symmetric potential. In [47], the authors investigated the existence of TFDE with the equation

$$\frac{\partial^{\alpha}\zeta(x,\mathtt{w})}{\partial\mathtt{w}^{\alpha}} - \frac{\partial^{2}\zeta(x,\mathtt{w})}{\partial x^{2}} = Q(x,\mathtt{w}),\ \ 0 \leqslant \mathtt{w} \leqslant \chi,\ \ 0 \leqslant x \leqslant 1,$$

supplemented by the initial constrain $\zeta(x,0) = \Phi(x)$, and the conditions of Dirichlet type $\zeta(0,w) = 0$, $\zeta(1,w) = 0$, $0 \leqslant w \leqslant \chi$, with $\mathcal{D}^{\alpha}$ denotes the Caputo time-fractional derivative. Let us define the operator $\mathcal{B}$ by $\mathcal{B}\zeta(x) = \zeta''(x)$ with domain $D(\mathcal{B}) = \{\zeta \in Z : \zeta(0) = \zeta(1) = 0\}$, then the abstract TFDE becomes

$$D^{\alpha}\zeta(w) - \mathcal{B}\zeta(w) = Q(w,\zeta), \quad 0 \leqslant w \leqslant \chi,$$

and the corresponding sequential abstract TFDE is given by

$$D^{\beta}\left(D^{\alpha}\zeta(w) - \mathcal{B}\zeta(w)\right) = Q(w,\zeta), \quad 0 \leqslant w \leqslant \chi,$$

where $0 < \alpha < 1$, $0 < \beta < 1$, and the closed linear unbounded operator $\mathcal{B}$ defined on $D(\mathcal{B}) \subset Z$, where Z is a Banach space.

The study of abstract fractional differential equations plays an important role in applied mathematics. For example, in [43], Saeedian et al. studied the memory effect of an SIR epedimic model using the Caputo fractional derivative. They proved that this effect plays an essential role in the spreading of diseases. The mathematical form of this model is:

$$\begin{cases} D^{\alpha}S(w) = \Lambda - \mu S - \dfrac{\beta SI}{1+\alpha_1 S + \alpha_2 I + \alpha_3 SI}, \\ D^{\alpha}I(w) = \dfrac{\beta SI}{1+\alpha_1 S + \alpha_2 I + \alpha_3 SI} - (\mu + d + r)I, \\ D^{\alpha}R(w) = rI - \mu R, \end{cases} \tag{1.3}$$

where $S(w)$, $I(w)$, and $R(w)$ represent the numbers of susceptible, infective, and recovered individuals at time w, respectively. Let $\zeta(w) = (S(w), I(w), R(w))^T$, then system (1.3) can be reformed as following abstract fractional differential equation:

$$D^{\alpha}\zeta(w) - \mathcal{B}\zeta(w) = Q(w,\zeta(w)),$$

where

$$\mathcal{B} = \begin{pmatrix} -\mu & 0 & 0 \\ 0 & -(\mu+d+r) & 0 \\ 0 & r & -\mu \end{pmatrix}, \quad Q(w,\zeta(w)) = \begin{pmatrix} \Lambda - \dfrac{\beta SI}{1+\alpha_1 S + \alpha_2 I + \alpha_3 SI} \\ \dfrac{\beta SI}{1+\alpha_1 S + \alpha_2 I + \alpha_3 SI} \\ 0 \end{pmatrix}.$$

Motivate by the papers [8, 19, 47, 50], we will study an abstract generalized problem, the sequential equation, pantograph equation, and time-fractional diffusion equation. The study of abstract sequential pantograph time-fractional diffusion equations is motivated by their ability to model complex processes that exhibit both memory effects and proportional delays, which are common in fields such as physics, biology, and control theory. Traditional diffusion equations often fail to capture anomalous diffusion and hereditary behavior observed in real-world systems. By incorporating time-fractional derivatives and pantograph-type delays, these models provide a more accurate and flexible framework. In our work, we treat the existence of mild solution, UH and UHR stability properties of abstract sequential pantograph TFDE with boundary conditions

$$\begin{cases} \mathcal{D}^{\delta}\left(\mathcal{D}^{\alpha} - \mathcal{B}\right)\zeta(w) = \mathcal{G}_{\lambda}(w, \zeta(w), \zeta(\eta w), \mathcal{J}^{\sigma}\zeta(\eta w)), \quad w \in [0,\chi], \\ \zeta(0) = \zeta_0, \quad \zeta(\chi) = \mathcal{J}^{\alpha}_{\chi}(\mathcal{B}\zeta), \end{cases} \tag{1.4}$$

where $\mathcal{D}^{\alpha}$, $\mathcal{D}^{\delta}$ are two CFD and $\mathcal{J}^{\sigma}$ the Sturm-Liouville integral, $0 < \alpha < 1$, $0 < \delta, \sigma < 1$, $0 < \eta < 1$, $\mathcal{B}$ is a closed linear unbounded operator with $D(\mathcal{B}) \subset Z$, Z is a Banach space, $D(\mathcal{B})$ is the domain of the mapping $\mathcal{B}$ equipped with the graph norm $\|\zeta\|_{D(\mathcal{B})} = \|\zeta\| + \|\mathcal{B}\zeta\|$ and $\|\zeta\|_{C([0,\chi),Z)} = \max_{w\in[0,\chi]} \|\zeta(w)\|$ and $G_{\lambda} \in C([0,\chi], Z)$ depending on a parameter $\lambda \geqslant 0$ with $G_{\lambda} : [0,\chi] \times Z^3 \to Z$ is a continuous function.

Here, the major goal is to study the mild solutions of abstract sequential pantograph TFDE (1.4) by using the Schaefer theorem and R-operator. Furthermore, we investigate UH and UHR stabilities for the proposed problem (1.4).

This article is organized as follows. In Section 2, we present the basic preliminaries and results that are related to our analysis. In Section 3, we investigate the existence of mild solutions and prove theorems of UH-S and UHR-S for the problem (1.4). Section 4 focuses the case where $\mathcal{B} = \kappa$, for some $\kappa \in \mathbb{R}$, and presents the corresponding existence results. Finally, Section 5 studies an example involving a partial TFDE with CFD.

2. Preliminaries

This part gives some relevant definitions and results.

Definition 2.1 ([26])**.** Consider $\delta \in \mathbb{R}$ with $m-1 < \delta < m$, such that m is a positive integer. The CFD of δ-order for h in $C^m(0,\infty)$ is defined as

$$\mathcal{D}^{\delta}\left[\zeta(\mathtt{w})\right] = \frac{1}{\Gamma(m-\alpha)}\int_a^{\mathtt{w}} (\mathtt{w}-\nu)^{m-\delta-1}\zeta^{(m)}(\nu)d\nu = \mathcal{J}^{m-\delta}\zeta^{(m)}(\mathtt{w}), \quad \mathtt{w} > 0.$$

Definition 2.2 ([26])**.** The Riemann-Liouville integral of δ-order, $\delta > 0$ of h, is define by

$$\mathcal{J}_{\mathtt{w}}^{\delta}\left[\zeta(\mathtt{w})\right] = \frac{1}{\Gamma(\delta)}\int_0^{\mathtt{w}} (\mathtt{w}-\nu)^{\delta-1}\zeta(\nu)d\nu, \ \mathtt{w} > 0,$$

where $\Gamma(\delta) = \int_0^{\infty} e^{-x}x^{\delta-1}dx$.

In this part, we shall discuss the following lemmas. For additional details, please see the reference [26].

Lemma 2.3 ([19])**.** *Let* $\beta, \delta > 0$, $\zeta \in L^1([a,b])$. *Then* $\mathcal{J}^{\beta}\mathcal{J}^{\delta}\zeta(\mathtt{w}) = \mathcal{J}^{\beta+\delta}\zeta(\mathtt{w})$, $\mathcal{D}^{\delta}\mathcal{J}^{\delta}\zeta(\mathtt{w}) = \zeta(\mathtt{w})$, *where* $\mathtt{w} \in [0,\chi]$.

Lemma 2.4 ([19])**.** *Let* $\delta > \beta > 0$, $\zeta \in L^1([a,b])$. *Then* $\mathcal{D}^{\beta}\mathcal{J}^{\delta}\zeta(\mathtt{w}) = \mathcal{J}^{\delta-\beta}\zeta(\mathtt{w})$, *where* $\mathtt{w} \in [0,\chi]$.

In all this work, we suppose the integral equation

$$\zeta(\mathtt{w}) = \frac{1}{\Gamma(\alpha)}\int_0^{\mathtt{w}} (\mathtt{w}-\nu)^{\alpha-1}\mathcal{B}\zeta(\nu)d\nu, \ \mathtt{w} \geqslant 0, \tag{2.1}$$

has a corresponding R-operators $(\mathcal{R}(\mathtt{w}))_{\mathtt{w}\geqslant 0}$ on Z.

Definition 2.5 ([41])**.** Let $\mathcal{R}(\mathtt{w})_{\mathtt{w}\geqslant 0}$ denotes a family of bounded linear mappings, and $\mathcal{R}(\mathtt{w})_{\mathtt{w}\geqslant 0}$ acting on the Banach space Z is called a R-operator for the equation (2.1) if the next conditions are verified:

a) $\mathcal{R}(\cdot)x \in C([0,\infty), Z)$ and $\mathcal{R}(0)x = x$, $\forall x \in Z$.
b) $\mathcal{R}(\mathtt{w})D(\mathcal{B}) \subset D(\mathcal{B})$, $\mathcal{B}\mathcal{R}(\mathtt{w})x = \mathcal{R}(\mathtt{w})\mathcal{B}x$, $\forall x \in D(\mathcal{B})$, and $\mathtt{w} \geqslant 0$.
c) $\mathcal{R}(\mathtt{w})x = x + \frac{1}{\Gamma(\alpha)}\int_0^{\mathtt{w}} (\mathtt{w}-x)^{\alpha-1}\mathcal{B}\mathcal{R}(\nu)xd\nu$, where $\mathtt{w} \geqslant 0$ and $\forall x \in D(\mathcal{B})$.

Definition 2.6 ([41])**.** A R-operator $(\mathcal{R}(\mathtt{w}))_{\mathtt{w}\geqslant 0}$ for (2.1) is differentiable if $\mathcal{R}(\mathtt{w})x \in W^{1,1}_{loc}(\mathbb{R}^+, Z)$ for each $x \in D(\mathcal{B})$ and there is $\varphi_A \in L^1_{loc}(\mathbb{R}^+)$, where $\|\mathcal{R}'(\mathtt{w})\| \leqslant \varphi_{\mathcal{B}}\|x\|_{D(\mathcal{B})}$ for every $x \in D(\mathcal{B})$.

Definition 2.7 ([41])**.** Consider $\nu \in C([0,\chi], Z)$ to be a mild solution of the integral equation

$$\zeta(\mathtt{w}) = \frac{\mathcal{B}}{\Gamma(\delta)}\int_0^{\mathtt{w}} (\mathtt{w}-\nu)^{\delta-1}\zeta(\nu)d\nu + H(\mathtt{w}), \quad \mathtt{w} \in [0,\chi], \tag{2.2}$$

if $\int_0^{\mathtt{w}} (\mathtt{w}-\nu)^{\delta-1}\nu(\nu)d\nu \in D(\mathcal{B})$, $\forall \mathtt{w} \in [0,\chi]$, $\zeta(\mathtt{w}) \in L^1([0,\chi], Z)$ and

$$\nu(\mathtt{w}) = \frac{\mathcal{B}}{\Gamma(\delta)}\int_0^{\mathtt{w}} (\mathtt{w}-\nu)^{\delta-1}\nu(\nu)d\nu + H(\mathtt{w}), \quad \mathtt{w} \in [0,\chi].$$

The next result follows from [39, 41].

Lemma 2.8 ([41]). *Suppose the conditions of Definition 2.5 are satisfied, so we have the following properties.*

i) *Suppose $(\mathcal{R}(\mathtt{w}))_{\mathtt{w}\geqslant 0}$ is analytic, $\zeta \in C^{\delta}([0,\chi], Z)$ for some $\delta \in (0,1)$, then*

$$\zeta(\mathtt{w}) = \mathcal{R}(\mathtt{w})\zeta(\mathtt{w}) + \int_0^{\mathtt{w}} \mathcal{R}'(\mathtt{w}-\nu)\left(\zeta(\nu)-\zeta(\mathtt{w})\right)d\nu, \quad \mathtt{w} \in [0,\chi],$$

is a mild solution of (2.2) on $[0,\chi]$.

ii) *Assume $(\mathcal{R}(\mathtt{w}))_{\mathtt{w}\geqslant 0}$ is differentiable, $\zeta \in C([0,\chi], D(\mathcal{B}))$, then $h : [0,\chi] \to Z$ define by*

$$\zeta(\mathtt{w}) = \int_0^{\mathtt{w}} \mathcal{R}'(\mathtt{w}-\nu)\zeta(\nu)d\nu + \zeta(\mathtt{w}), \quad \mathtt{w} \in [0,\chi],$$

is a mild solution of (2.2) on $[0,\chi]$.

Now, we present the UH-S and UHR-S stability for the abstract sequential pantograph TFDE (see [13, 19, 25]).

Definition 2.9 ([35]). The abstract sequential pantograph TFDE (1.4) is UH-S, if $\exists \gamma_{\mathcal{G}_\lambda}$: $\forall \mu > 0$ and $\forall \nu \in C([0,\chi], Z)$ solution of the following inequality:

$$\left|\mathcal{D}^{\delta}\left(\mathcal{D}^{\alpha}-\mathcal{B}\right)\zeta(\mathtt{w}) - \mathcal{G}_\lambda(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right| \leqslant \mu, \quad \mathtt{w} \in [0,\chi], \tag{2.3}$$

$\exists \zeta \in C([0,\chi], Z)$ solution of the abstract sequential pantograph TFDEs (1.4) with

$$|\nu(\mathtt{w}) - \zeta(\mathtt{w})| \leqslant \gamma_{\mathcal{G}_\lambda}\mu, \ \mathtt{w} \in [0,\chi].$$

Definition 2.10 ([35]). The abstract sequential pantograph TFDE (1.4) is said to be generalized UH-S if $\exists g_{\mathcal{G}\lambda} \in (\mathbb{R}^+, \mathbb{R}_+)$ with $g(0) = 0$, such that $\forall \nu \in C([0,\chi], Z)$ solution of the inequality (2.3), $\exists \zeta \in C([0,\chi], Z)$ solution of the abstract sequential pantograph TFDE (1.4) with the following property:

$$|\nu(\mathtt{w}) - \zeta(\mathtt{w})| \leqslant g_{\mathcal{G}_\lambda}(\mu), \quad \mathtt{w} \in [0,\chi].$$

Definition 2.11 ([37]). The abstract sequential pantograph TFDE (1.4) is UHR-S concerning $\psi \in C([0,\chi], \mathbb{R}^+)$ if $\exists \theta_{\mathcal{G}_\lambda} \geqslant 0$, where $\forall \mu > 0$ and $\forall \nu \in C([0,\chi], Z)$ solutions of the following inequality:

$$\left|\mathcal{D}^{\delta}\left(\mathcal{D}^{\alpha}-\mathcal{B}\right)\zeta(\mathtt{w}) - \mathcal{G}_\lambda(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right| \leqslant \mu\psi(\mathtt{w}), \quad \mathtt{w} \in [0,\chi], \tag{2.4}$$

so, $\exists \zeta \in C([0,\chi], Z)$ solution of the abstract sequential pantograph TFDE (1.4) with

$$|\nu(\mathtt{w}) - \zeta(\mathtt{w})| \leqslant \theta_{\mathcal{G}_\lambda}\mu\psi(\mathtt{w}), \quad \mathtt{w} \in [0,\chi].$$

Definition 2.12 ([37]). The abstract sequential pantograph TFDE (1.4) is generalized UHR-S concerning $\psi \in C([0,\chi], \mathbb{R}^+)$ if $\exists \theta_{\mathcal{G}_\lambda,\psi} \geqslant$ such that $\forall \nu \in C([0,\chi], Z)$ solution of the following inequality:

$$\left|\mathcal{D}^{\delta}\left(\mathcal{D}^{\alpha}-\mathcal{B}\right)\zeta(\mathtt{w}) - \mathcal{G}_\lambda(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right| \leqslant \psi(\mathtt{w}), \quad \mathtt{w} \in [0,\chi],$$

so, $\exists \zeta \in C([0,\chi], Z)$ solution of the abstract sequential pantograph TFDE (1.4) with

$$|\nu(\mathtt{w}) - \zeta(\mathtt{w})| \leqslant \theta_{\mathcal{G}_\lambda,\psi}\psi(\mathtt{w}), \quad \mathtt{w} \in [0,\chi].$$

3. Main results

The first part of this section focuses on the existence of mild solutions for the abstract sequential pantograph TFDE (1.4). Throughout this paper, the next hypotheses are assumed.

(H1) The operator $\mathcal{B} : D(\mathcal{B}) \subset Z \to Z$ is closed linear.

(H2) It is assumed that R-operator $\mathcal{R}(\mathtt{w})$, for $\mathtt{w} \geqslant 0$, is differentiable and $\exists \varphi_{\mathcal{B}} \in L^1_{loc}([0,\infty); \mathbb{R}^+)$, satisfying the following inequality: $\|\mathcal{R}'(\mathtt{w})\zeta\| \leqslant \varphi_{\mathcal{B}}(\mathtt{w}) \|\zeta\|_{D(\mathcal{B})}$, $\forall \mathtt{w} > 0$.

Lemma 3.1. *Let* $H \in C([0,\chi], Z)$ *and consider the problem*

$$\begin{cases} \mathcal{D}^{\delta}(\mathcal{D}^{\alpha} - \mathcal{B})\zeta(\mathtt{w}) = H(\mathtt{w}), & \mathtt{w} \in [0,\chi], \\ \zeta(0) = \zeta_0, \ \zeta(\chi) = \mathcal{J}^{\alpha}_{\chi}(\mathcal{B}\zeta), & \delta \in \mathbb{R}. \end{cases} \tag{3.1}$$

Then the problem (3.1) *is equivalent to*

$$\begin{aligned} \zeta(\mathtt{w}) = & \frac{1}{\Gamma(\alpha)} \int_0^{\mathtt{w}} (\mathtt{w}-x)^{\alpha-1} \left(\frac{1}{\Gamma(\delta)} \int_0^x (x-\nu)^{\delta-1} H(\nu) d\nu \right) dx \\ & - \frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}} \left(\frac{1}{\Gamma(\alpha)} \int_0^{\chi} (\chi-x)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_0^x (x-\nu)^{\delta-1} H(\nu) d\nu \right) dx \right) \\ & + \left(1 - \frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\right) \zeta_0 + \frac{1}{\Gamma(\alpha)} \int_0^{\mathtt{w}} (\mathtt{w}-x)^{\alpha-1} \mathcal{B}\zeta(x) dx. \end{aligned} \tag{3.2}$$

Remark 3.2. The equation (3.2) can also be transformed to the the integral equation

$$\zeta(\mathtt{w}) = \frac{1}{\Gamma(\alpha)} \int_0^{\mathtt{w}} (\mathtt{w}-x)^{\alpha-1} \mathcal{B}\zeta(x) dx + \hat{h}(\mathtt{w}), \quad \mathtt{w} \geqslant 0,$$

where

$$\hat{h}(\mathtt{w}) = \mathcal{J}^{\alpha}_{\mathtt{w}} \left(\mathcal{J}^{\delta}_{\mathtt{w}} H(\mathtt{w}) \right) - \frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}} \mathcal{J}^{\alpha}_{\chi} \left(\mathcal{J}^{\delta}_{\mathtt{w}} H(\mathtt{w}) \right) + \left(1 - \frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\right) \zeta_0.$$

We use the Lemma 3.1, so the problem (1.4) is equivalent to

$$\begin{aligned} \zeta(\mathtt{w}) = & \mathcal{J}^{\alpha}_{\mathtt{w}} (\mathcal{B}\zeta(\mathtt{w})) + \mathcal{J}^{\alpha}_{\mathtt{w}} \left(\mathcal{J}^{\delta}_{\mathtt{w}} (\mathcal{G}_{\lambda}(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))) \right) \\ & - \frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}} \mathcal{J}^{\alpha}_{\chi} \left(\mathcal{J}^{\delta}_{\mathtt{w}} (\mathcal{G}_{\lambda}(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))) \right) + \left(1 - \frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\right) \zeta_0, \end{aligned}$$

where $\mathtt{w} \in [0,\chi]$.

Definition 3.3. A function $\zeta \in C([0,\chi], Z)$ is mild solution of (1.4), on $[0,\chi]$ if $\int_0^{\mathtt{w}} (\mathtt{w}-x)^{\alpha-1} \zeta(x) dx \in D(\mathcal{B})$ for all $\mathtt{w} \in [0,\chi]$ and

$$\begin{aligned} \zeta(\mathtt{w}) = & \mathcal{B} \left(\mathcal{J}^{\alpha}_{\mathtt{w}} (\zeta(\mathtt{w})) \right) + \mathcal{J}^{\alpha}_{\mathtt{w}} \left(\mathcal{J}^{\delta}_{\mathtt{w}} (\mathcal{G}_{\lambda}(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))) \right) \\ & - \frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}} \mathcal{J}^{\alpha}_{\chi} \left(\mathcal{J}^{\delta}_{\mathtt{w}} (\mathcal{G}_{\lambda}(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))) \right) + \left(1 - \frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\right) \zeta_0, \quad \mathtt{w} \in [0,\chi]. \end{aligned}$$

3.1. Existence results

Here, we establish the existence theorem of the proposed system (1.4) by employing technique of Schaefer.

Theorem 3.4. *Suppose that* (H1) and (H2) *hold true and* $\exists \mathcal{M} \geqslant 0$ *such that*

$$|\mathcal{G}_{\lambda}(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))| \leqslant \mathcal{M}, \quad \mathtt{w} \in [0,\chi], \ 0 < \eta < 1.$$

Then, the problem (1.4) *owns at least one mild solution on* $[0,\chi]$.

Proof. We assume that there exists an R-operator $\mathcal{R}(\mathtt{w})$, with $\mathtt{w} \geqslant 0$, which is differentiable. Additionally, it is assumed that the function $\mathcal{G}_\lambda$ is continuous in the space Z and since the Remark 3.2 and the Lemma 2.8 property 3, let the operator $\Lambda : C([0,\chi], Z) \to C([0,\chi], Z)$, for $\lambda \geqslant 0$, defined by

$$\begin{aligned}(\Lambda\zeta)(\mathtt{w}) = &\int_0^{\mathtt{w}} \mathcal{R}'(\mathtt{w}-\nu)\left(\mathcal{J}_\nu^\alpha\left(\mathcal{J}_\nu^\delta\left(\mathcal{G}_\lambda(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^\sigma\zeta(\eta\nu))\right)\right)\right.\\ &\left.-\frac{\nu^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_\nu^\delta\left(\mathcal{G}_\lambda(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^\sigma\zeta(\eta\nu))\right)\right)+\left(1-\frac{\nu^\alpha}{\chi^\alpha}\right)\zeta_0\right)d\nu\\ &+\mathcal{J}_{\mathtt{w}}^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta\left(\mathcal{G}_\lambda(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^\sigma\zeta(\eta\mathtt{w}))\right)\right)\\ &-\frac{\mathtt{w}^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta\left(\mathcal{G}_\lambda(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^\sigma\zeta(\eta\mathtt{w}))\right)\right)+\left(1-\frac{\mathtt{w}^\alpha}{\chi^\alpha}\right)\zeta_0.\end{aligned} \tag{3.3}$$

Now, we prove Λ defined by (3.3) is completely continuous. Since $\mathcal{G}_\lambda$ is continuous, then Λ is continuous. Let

$$B_r = \{\zeta \in C([0,\chi], Z), \|\zeta\| \leqslant r, r \geqslant 0\},$$

be a bounded ball in $C([0,\chi], Z)$. Then for $\mathtt{w} \in [0,\chi]$, we have

$$\begin{aligned}|(\Lambda\zeta)(\mathtt{w})| \leqslant &\int_0^{\mathtt{w}} \left|\mathcal{R}'(\mathtt{w}-\nu)\right|\left(\mathcal{J}_\nu^\alpha\left(\mathcal{J}_\nu^\delta\left(|\mathcal{G}_\lambda(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^\sigma\zeta(\eta\nu))|\right)\right)\right.\\ &\left.+\frac{\nu^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_\nu^\delta\left(|\mathcal{G}_\lambda(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^\sigma\zeta(\eta\nu))|\right)\right)+\left|\left(1-\frac{\nu^\alpha}{\chi^\alpha}\right)\zeta_0\right|\right)d\nu\\ &+\mathcal{J}_{\mathtt{w}}^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta\left(|\mathcal{G}_\lambda(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^\sigma\zeta(\eta\mathtt{w}))|\right)\right)\\ &+\frac{\mathtt{w}^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta\left(|\mathcal{G}_\lambda(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^\sigma\zeta(\eta\mathtt{w}))|\right)\right)+\left|\left(1-\frac{\mathtt{w}^\alpha}{\chi^\alpha}\right)\zeta_0\right|\\ \leqslant &\,\mathcal{M}\int_0^{\mathtt{w}}\left|\mathcal{R}'(\mathtt{w}-\nu)\right|\left(\mathcal{J}_\nu^\alpha\left(\mathcal{J}_\nu^\delta(1)\right)+\frac{\nu^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_\nu^\delta(1)\right)\right)d\nu+\int_0^{\mathtt{w}}\left|\mathcal{R}'(\mathtt{w}-\nu)\right|\left|\zeta_0\left(1-\frac{\nu^\alpha}{\chi^\alpha}\right)\right|d\nu\\ &+\mathcal{M}\left[\mathcal{J}_{\mathtt{w}}^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta(1)\right)+\frac{\mathtt{w}^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta(1)\right)\right]+\left|\left(1-\frac{\mathtt{w}^\alpha}{\chi^\alpha}\right)\zeta_0\right|\\ \leqslant &\,\mathcal{M}\|\varphi_{\mathcal{B}}\|_{L^1}\left(\mathcal{J}_{\mathtt{w}}^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta(1)\right)+\frac{\mathtt{w}^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta(1)\right)\right)+2\|\varphi_{\mathcal{B}}\|_{L^1}\|\zeta_0\|\\ &+\mathcal{M}\left[\mathcal{J}_{\mathtt{w}}^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta(1)\right)+\frac{\mathtt{w}^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta(1)\right)\right]+2\|\zeta_0\|\\ \leqslant &\,\mathcal{M}(\|\varphi_{\mathcal{B}}\|_{L^1}+1)\left(\mathcal{J}_{\mathtt{w}}^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta(1)\right)+\frac{\mathtt{w}^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_{\mathtt{w}}^\delta(1)\right)\right)+2(\|\varphi_{\mathcal{B}}\|_{L^1}+1)\|\zeta_0\|\\ \leqslant &\,(\|\varphi_{\mathcal{B}}\|_{L^1}+1)\left(2\frac{\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\mathcal{M}+2\|\zeta_0\|\right),\end{aligned}$$

which consequently implies that

$$\|\Lambda\zeta(\mathtt{w})\| \leqslant (\|\varphi_{\mathcal{B}}\|_{L^1}+1)\left(2\frac{\chi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\mathcal{M}+2\|\zeta_0\|\right). \tag{3.4}$$

The next step is to investigate that $\Lambda\zeta$ is equicontinuous set within the space $C([0,\chi], Z)$. Let $\tau_1, \tau_2 \in [0,\chi]$ with $\tau_1 < \tau_2$ and $\zeta \in B_r$. Then, we have

$$\begin{aligned}|(\Lambda\zeta)(\tau_1)-(\Lambda\zeta)(\tau_2)| \leqslant &\left|\int_0^{\tau_1}\mathcal{R}'(\tau_1-\nu)\left(\mathcal{J}_\nu^\alpha\left(\mathcal{J}_x^\delta\left(\mathcal{G}_\lambda(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^\sigma\zeta(\eta\nu))\right)\right)\right.\right.\\ &\left.-\frac{\nu^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_x^\delta\left(\mathcal{G}_\lambda(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^\sigma\zeta(\eta\nu))\right)\right)+\left(1-\frac{\nu^\alpha}{\chi^\alpha}\right)\zeta_0\right)d\nu\end{aligned}$$

$$
\begin{aligned}
&-\int_0^{\tau_2} S'(\tau_2-\nu)\left(\mathcal{J}_\nu^\alpha\left(\mathcal{J}_x^\delta\left(\mathcal{G}_\lambda(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^\sigma\zeta(\eta\nu)\right)\right)\right)\\
&-\frac{\nu^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_x^\delta\left(\mathcal{G}_\lambda(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^\sigma\zeta(\eta\nu))\right)\right)+\left(1-\frac{\nu^\alpha}{\chi^\alpha}\right)\zeta_0\Big)\,d\nu\Bigg|\\
&+\Bigg|\mathcal{J}_{\tau_1}^\alpha\left(\mathcal{J}_x^\beta\left(\mathcal{G}_\lambda(w,\zeta(w),\zeta(\eta w),\mathcal{J}^\sigma\zeta(\eta w))\right)\right)+\frac{\tau_1^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_x^\delta\left(\mathcal{G}_\lambda(w,\zeta(w),\zeta(\eta w),\mathcal{J}^\sigma\zeta(\eta w))\right)\right)\\
&-\mathcal{J}_{\tau_2}^\alpha\left(\mathcal{J}_x^\delta\left(\mathcal{G}_\lambda(w,\zeta(w),\zeta(\eta w),\mathcal{J}^\sigma\zeta(\eta w))\right)\right)-\frac{\tau_2^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_x^\delta\left(\mathcal{G}_\lambda(w,\zeta(w),\zeta(\eta w),\mathcal{J}^\sigma\zeta(\eta w))\right)\right)\Bigg|\\
&+\left|\left(1-\frac{\tau_1^\alpha}{\chi^\alpha}\right)\zeta_0-\left(1-\frac{\tau_2^\alpha}{\chi^\alpha}\right)\zeta_0\right|.
\end{aligned}
$$

As the difference $\tau_2-\tau_1$ approaches 0, the right-hand side of the preceding inequality also approaches to zero, independently of the choice of h from the bounded set B_r. Consequently, by the Arzel-Ascoli theorem, we have $\Lambda: C([0,\chi],Z)\to C([0,\chi],Z)$ is completely continuous. Finally, let us define $V=\{\zeta\in C([0,\chi],Z):\zeta=\gamma\Lambda\zeta,\ \ 0<\gamma<1\}$ and prove that V is bounded. For $\zeta\in V$ and $w\in[0,\chi]$. By using (3.4), we get,

$$
\|\zeta\|=\|\gamma\Lambda\zeta\|\leqslant(\|\varphi_{\mathcal{B}}\|_{L^1}+1)\left(2\frac{\chi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\mathcal{M}+2\|\zeta_0\|\right).
$$

Therefore, V is bounded. As a consequence to the well known Schaefer fixed point technique, we infer that the system (1.4) owns at least one solution on $[0,\chi]$. □

3.2. Stability

In the second part, we discuss UH-S and UHR-S of the problem (1.4). We assume following.

(H3) For $\lambda\geqslant 0$, $\mathcal{G}_\lambda:[0,\chi]\times Z^3\to Z$ is completely continuous and $\exists L_\lambda>0$ such that: $\forall w\in[0,\chi], z_i,x_i\in Z$, $i=1-3$, we have

$$
\|\mathcal{G}_\lambda(w,z_1,z_2,z_3)-\mathcal{G}_\lambda(w,x_1,x_2,x_3)\|\leqslant L_\lambda\left(\|z_1-x_1\|+\|z_2-x_2\|+\|z_3-x_3\|\right).
$$

(H4) There exist $\lambda_0\geqslant 0,\forall\lambda\geqslant\lambda_0$,

$$
\frac{2L_\lambda\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^\sigma}{\Gamma(\sigma+1)}+2\right)\|\varphi_{\mathcal{B}}\|_{L^1}<1-\frac{2L_\lambda\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^\sigma}{\Gamma(\sigma+1)}+2\right). \tag{3.5}
$$

Theorem 3.5. *Let* (H1)-(H4) *are satisfied. For* $0<\delta,\alpha,\sigma<1$, *then the abstract sequential pantograph TFDE* (1.4) *is UH-S and consequently, generalized UH-S.*

Proof. Assume that $\tilde{\zeta}\in C([0,\chi],Z)$ satisfying:

$$
\left|\mathcal{D}^\delta(\mathcal{D}^\alpha-A)\tilde{\zeta}(w)-\mathcal{G}_\lambda(w,\tilde{\zeta}(w),\tilde{\zeta}(\eta w),\mathcal{J}^\sigma\tilde{\zeta}(\eta w))\right|\leqslant\mu,\quad w\in[0,\chi], \tag{3.6}
$$

and let $\zeta\in C([0,\chi],Z)$ the unique solution of the system

$$
\begin{cases}\mathcal{D}^\beta(\mathcal{D}^\alpha-\mathcal{B})\zeta(w)=\mathcal{G}_\lambda(w,\zeta(w),\zeta(\eta w),\mathcal{J}^\sigma\zeta(\eta w)),\quad w\in[0,\chi],\\ \zeta(0)=\tilde{\zeta}(0),\zeta(\chi)=\tilde{\zeta}(\chi).\end{cases} \tag{3.7}
$$

By the Lemma 3.1, we can write

$$
\begin{aligned}
\zeta(w)=&\int_0^w\mathcal{R}'(w-\nu)\left(\mathcal{J}_\nu^\alpha\left(\mathcal{J}_x^\beta\mathcal{G}_\lambda(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^\sigma\zeta(\eta\nu))\right)\right.\\
&-\frac{\nu^\alpha}{\chi^\alpha}\mathcal{J}_\chi^\alpha\left(\mathcal{J}_x^\beta\mathcal{G}_\lambda(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^\sigma\zeta(\eta\nu))\right)+\left(1-\frac{\nu^\alpha}{\chi^\alpha}\right)\zeta_0\Big)\,d\nu
\end{aligned}
$$

$$+\mathcal{J}_{\mathtt{w}}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\beta}\mathcal{G}_{\lambda}(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right)-\frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\beta}\mathcal{G}_{\lambda}(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right)+\left(1-\frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\right)\zeta_{0}.$$

By Definition 2.9 of UH-S and by integration of (3.6), we have

$$\begin{aligned}
&\Bigg|\tilde{\zeta}(\mathtt{w})-\int_{0}^{\mathtt{w}}\mathcal{R}'(\mathtt{w}-\nu)\Big(\mathcal{J}_{\nu}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right)\right)\\
&\quad-\frac{\nu^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right)\right)+\left(1-\frac{\nu^{\alpha}}{\chi^{\alpha}}\right)\zeta_{0}\Big)\,d\nu\\
&\quad-\mathcal{J}_{\mathtt{w}}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right)\right)\\
&\quad+\frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right)\right)-\left(1-\frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\right)\zeta_{0}\Bigg|\leqslant\frac{\mu}{\Gamma(\alpha+1)}\chi^{\alpha}.
\end{aligned}\tag{3.8}$$

If $\tilde{\zeta}(0)=\zeta(0)$ and $\tilde{\zeta}(\chi)=\zeta(\chi)$, then we have

$$\begin{aligned}
\tilde{\zeta}(\mathtt{w})-\zeta(\mathtt{w})=\ &\tilde{\zeta}(\mathtt{w})-\int_{0}^{\mathtt{w}}\mathcal{R}'(\mathtt{w}-\nu)\Big(\mathcal{J}_{\nu}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^{\sigma}\zeta(\eta\nu))\right)\right)\\
&-\frac{\nu^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^{\sigma}\zeta(\eta\nu))\right)\right)+\left(1-\frac{\nu^{\alpha}}{\chi^{\alpha}}\right)\zeta_{0}\Big)\,d\nu\\
&-\mathcal{J}_{\mathtt{w}}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right)\right)\\
&+\frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right)\right)-\left(1-\frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\right)\zeta_{0}\\
&+\int_{0}^{\mathtt{w}}\mathcal{R}'(\mathtt{w}-\nu)\Big(\mathcal{J}_{\nu}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))-\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))\right)\right)\\
&+\frac{\nu^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))-\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))\right)\right)\Big)\,d\nu\\
&+\mathcal{J}_{\mathtt{w}}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\tilde{\zeta}(\mathtt{w}),\tilde{\zeta}(\eta\mathtt{w}),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\mathtt{w}))-\mathcal{G}_{\lambda}(\mathtt{w},\tilde{\zeta}(\mathtt{w}),\tilde{\zeta}(\eta\mathtt{w}),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\mathtt{w}))\right)\right)\\
&-\frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\tilde{\zeta}(\mathtt{w}),\tilde{\zeta}(\eta\mathtt{w}),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\mathtt{w}))-\mathcal{G}_{\lambda}(\mathtt{w},\tilde{\zeta}(\mathtt{w}),\tilde{\zeta}(\eta\mathtt{w}),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\mathtt{w}))\right)\right),
\end{aligned}$$

so

$$\begin{aligned}
\tilde{\zeta}(\mathtt{w})-\zeta(\mathtt{w})=\ &\tilde{\zeta}(\mathtt{w})-\int_{0}^{\mathtt{w}}\mathcal{R}'(\mathtt{w}-\nu)\Big(\mathcal{J}_{\nu}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))\right)\right)\\
&-\frac{\nu^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))\right)\right)+\left(1-\frac{\nu^{\alpha}}{\chi^{\alpha}}\right)\zeta_{0}\Big)\,d\nu\\
&-\mathcal{J}_{\mathtt{w}}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\tilde{\zeta}(\mathtt{w}),\tilde{\zeta}(\eta\mathtt{w}),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\mathtt{w}))\right)\right)\\
&+\frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\tilde{\zeta}(\mathtt{w}),\tilde{\zeta}(\eta\mathtt{w}),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\mathtt{w}))\right)\right)-\left(1-\frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\right)\zeta_{0}\\
&+\int_{0}^{\mathtt{w}}\mathcal{R}'(\mathtt{w}-\nu)\Big(\mathcal{J}_{\nu}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))-\mathcal{G}_{\lambda}(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^{\sigma}\zeta(\eta\nu))\right)\right)\\
&+\frac{\nu^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))-\mathcal{G}_{\lambda}(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^{\sigma}\zeta(\eta\nu))\right)\right)\Big)\,d\nu\\
&+\mathcal{J}_{\mathtt{w}}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\tilde{\zeta}(\mathtt{w}),\tilde{\zeta}(\eta\mathtt{w}),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\mathtt{w}))-\mathcal{G}_{\lambda}(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right)\right)\\
&-\frac{\mathtt{w}^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\mathtt{w},\tilde{\zeta}(\mathtt{w}),\tilde{\zeta}(\eta\mathtt{w}),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\mathtt{w}))-\mathcal{G}_{\lambda}(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^{\sigma}\zeta(\eta\mathtt{w}))\right)\right),
\end{aligned}\tag{3.9}$$

and on the other hand

$$\left|\tilde{\zeta}(\mathtt{w})-\zeta(\mathtt{w})\right|=\Bigg|\tilde{\zeta}(\mathtt{w})-\int_{0}^{\mathtt{w}}\mathcal{R}'(\mathtt{w}-\nu)\left(\mathcal{J}_{\nu}^{\alpha}\left(\mathcal{J}_{\mathrm{x}}^{\delta}\left(\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))\right)\right)\right.$$

$$
\begin{aligned}
&-\frac{\nu^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{x}^{\delta}\left(\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))\right)\right)+\left(1-\frac{\nu^{\alpha}}{\chi^{\alpha}}\right)\zeta_0\Big)\,d\nu\\
&-\mathcal{J}_{w}^{\alpha}\left(\mathcal{J}_{x}^{\delta}\left(\mathcal{G}_{\lambda}(w,\tilde{\zeta}(w),\tilde{\zeta}(\eta w),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta w))\right)\right)\\
&+\frac{w^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{x}^{\delta}\left(\mathcal{G}_{\lambda}(w,\tilde{\zeta}(w),\tilde{\zeta}(\eta w),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta w))\right)\right)-\left(1-\frac{w^{\alpha}}{\chi^{\alpha}}\right)\zeta_0\Big|\\
&+\int_0^{w}\left|\mathcal{R}'(w-\nu)\right|\Big(\mathcal{J}_{\nu}^{\alpha}\left(\mathcal{J}_{x}^{\delta}\left(\left|\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))-\mathcal{G}_{\lambda}(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^{\sigma}\zeta(\eta\nu))\right|\right)\right)\\
&+\frac{\nu^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{x}^{\delta}\left(\left|\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))-\mathcal{G}_{\lambda}(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^{\sigma}\zeta(\eta\nu)))\right|\right)\right)\Big)\,d\nu\\
&+\mathcal{J}_{w}^{\alpha}\left(\mathcal{J}_{x}^{\delta}\left(\left|\mathcal{G}_{\lambda}(w,\tilde{\zeta}(w),\tilde{\zeta}(\eta w),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta w))-\mathcal{G}_{\lambda}(w,\zeta(w),\zeta(\eta w),\mathcal{J}^{\sigma}\zeta(\eta w))\right|\right)\right)\\
&+\frac{w^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{x}^{\delta}\left(\left|\mathcal{G}_{\lambda}(w,\tilde{\zeta}(w),\tilde{\zeta}(\eta w),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta w))-\mathcal{G}_{\lambda}(w,\zeta(w),\zeta(\eta w),\mathcal{J}^{\sigma}\zeta(\eta w))\right|\right)\right),
\end{aligned}
$$

then

$$
\begin{aligned}
\left|\tilde{\zeta}(w)-\zeta(w)\right| \leqslant{}& \frac{\mu}{\Gamma(\alpha+1)}\chi^{\alpha}+L_{\lambda}\left(\|\tilde{\zeta}-\zeta\|+\|\tilde{\zeta}-\zeta\|+\|\mathcal{J}^{\sigma}\tilde{\zeta}-\mathcal{J}^{\sigma}\zeta\|\right)\\
&\times\int_0^{w}\mathcal{R}'(w-\nu)\left(\mathcal{J}_{\nu}^{\alpha}\left(\mathcal{J}_{x}^{\delta}(1)\right)+\mathcal{J}_{\nu}^{\alpha}\left(\mathcal{J}_{x}^{\delta}(1)\right)\right)d\nu\\
&+L_{\lambda}\left(\|\tilde{\zeta}-\zeta\|+\|\tilde{\zeta}-\zeta\|+\|\mathcal{J}^{\sigma}\tilde{\zeta}-\mathcal{J}^{\sigma}\zeta\|\right)\left(\mathcal{J}_{w}^{\alpha}\left(\mathcal{J}_{x}^{\delta}(1)\right)+\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{x}^{\delta}(1)\right)\right).
\end{aligned}
$$

Since

$$
\begin{aligned}
\left|\mathcal{J}^{\sigma}\tilde{\zeta}(w)-\mathcal{J}^{\sigma}\zeta(w)\right| &= \left|\frac{1}{\Gamma(\sigma)}\int_0^{w}(w-\nu)^{\sigma-1}\tilde{\zeta}(\nu)d\nu-\frac{1}{\Gamma(\sigma)}\int_0^{w}(w-\nu)^{\sigma-1}\zeta(\nu)d\nu\right|\\
&\leqslant \frac{1}{\Gamma(\sigma)}\int_0^{w}(w-\nu)^{\sigma-1}\left|\tilde{\zeta}(\nu)-\zeta(\nu)\right|d\nu \leqslant \frac{\chi^{\sigma}}{\Gamma(\sigma+1)}\left|\tilde{\zeta}(\nu)-\zeta(\nu)\right|,
\end{aligned}
\tag{3.10}
$$

then, due to (3.8) and (3.10), we get

$$
\begin{aligned}
\left|\tilde{\zeta}(w)-\zeta(w)\right| &\leqslant \frac{\mu}{\Gamma(\alpha+1)}\chi^{\alpha}+\frac{2L_{\lambda}\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^{\sigma}}{\Gamma(\sigma+1)}+2\right)\|\tilde{\zeta}-\zeta\|\\
&\quad+\|\varphi_{\mathcal{B}}\|_{L^1}\frac{2L_{\lambda}\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^{\sigma}}{\Gamma(\sigma+1)}+2\right)\|\tilde{\zeta}-\zeta\|\\
&\leqslant \frac{\mu}{\Gamma(\alpha+1)}\chi^{\alpha}+(1+\|\varphi_{\mathcal{B}}\|_{L^1})\frac{2L_{\lambda}\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^{\sigma}}{\Gamma(\sigma+1)}+2\right)\|\tilde{\zeta}-\zeta\|.
\end{aligned}
$$

Hence,

$$
\left(1-(1+\|\varphi_{\mathcal{B}}\|_{L^1})\frac{2L_{\lambda}\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^{\sigma}}{\Gamma(\sigma+1)}+2\right)\right)\|\tilde{\zeta}-\zeta\| \leqslant \frac{\mu}{\Gamma(\alpha+1)}\chi^{\alpha}.
$$

Consequently, we have

$$
\|\tilde{\zeta}-\zeta\| \leqslant \frac{\chi^{\alpha}}{\left(1-(1+\|\varphi_{\mathcal{B}}\|_{L^1})\frac{2L_{\lambda}\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^{\sigma}}{\Gamma(\sigma+1)}+2\right)\right)\Gamma(\alpha+1)}\mu=\beta_{\mathcal{G}_{\lambda}}\mu.
$$

Consequently, the abstract sequential pantograph TFDE (1.4) is shown to be UH-S. Furthermore, by considering $g_{\mathcal{G}\lambda}(\mu)=\beta\mathcal{G}\lambda\mu$ with $g\mathcal{G}_{\lambda}(0)=0$, we establish that the abstract sequential pantograph TFDE (1.4) is also generalized UH-S.

We will now proceed to study the UHR-S of the abstract sequential pantograph TFDE (1.4). □

Theorem 3.6. *Assume* (H1)-(H4) *and*

(H5) $\exists \psi \in C([0,\chi], \mathbb{R}_+)$ *and* $\exists \theta_\psi > 0$ *such that,* $\forall \mathtt{w} \in [0,\chi]$, $\frac{1}{\Gamma(\alpha)} \int_0^{\mathtt{w}} (\mathtt{w}-\nu)^{\alpha-1} \psi(\nu) d\nu \leqslant \theta_\psi \psi(\mathtt{w})$ *is satisfied.*
Then the abstract sequential pantograph TFDE (1.4) *is UHR-S.*

Proof. Let $\tilde{\zeta} \in C([0,\chi], Z)$ be a solution of the inequality (2.4), i.e.,

$$\left|\mathcal{D}^\delta \left(\mathcal{D}^\alpha - \mathcal{B}\right) \tilde{\zeta}(\mathtt{w}) - \mathcal{G}_\lambda(\mathtt{w}, \tilde{\zeta}(\mathtt{w}), \tilde{\zeta}(\eta\mathtt{w}), \mathcal{J}^\sigma \tilde{\zeta}(\eta\mathtt{w}))\right| \leqslant \mu\psi(\mathtt{w}), \quad \mathtt{w} \in [0,\chi]. \tag{3.11}$$

Let $\zeta \in C([0,\chi], Z)$ be a the unique solution of the problem (3.7), thanks to proof of last theorem, we have

$$\begin{aligned}
\zeta(\mathtt{w}) = & \int_0^{\mathtt{w}} \mathcal{R}'(\mathtt{w}-\nu) \left(\mathcal{J}_\nu^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\nu, \zeta(\nu), \zeta(\eta\nu), \mathcal{J}^\sigma \zeta(\eta\nu))\right)\right)\right. \\
& \left. - \frac{\nu^\alpha}{\chi^\alpha} \mathcal{J}_\chi^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\nu, \zeta(\nu), \zeta(\eta\nu), \mathcal{J}^\sigma \zeta(\eta\nu))\right)\right) + \left(1 - \frac{\nu^\alpha}{\chi^\alpha}\right) \zeta_0 \right) d\nu + \mathcal{J}_{\mathtt{w}}^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^\sigma \zeta(\eta\mathtt{w}))\right)\right) \\
& - \frac{\mathtt{w}^\alpha}{\chi^\alpha} \mathcal{J}_\chi^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^\sigma \zeta(\eta\mathtt{w}))\right)\right) + \left(1 - \frac{\mathtt{w}^\alpha}{\chi^\alpha}\right) \zeta_0.
\end{aligned}$$

Integrating (3.11), we get

$$\begin{aligned}
& \left|\tilde{\zeta}(\mathtt{w}) - \int_0^{\mathtt{w}} \mathcal{R}'(\mathtt{w}-\nu) \left(\mathcal{J}_\nu^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\nu, \tilde{\zeta}(\nu), \tilde{\zeta}(\eta\nu), \mathcal{J}^\sigma \tilde{\zeta}(\eta\nu))\right)\right)\right.\right. \\
& \left. - \frac{\nu^\alpha}{\chi^\alpha} \mathcal{J}_\chi^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\nu, \tilde{\zeta}(\nu), \tilde{\zeta}(\eta\nu), \mathcal{J}^\sigma \tilde{\zeta}(\eta\nu))\right)\right) + \left(1 - \frac{\nu^\alpha}{\chi^\alpha}\right) \zeta_0 \right) d\nu \\
& - \mathcal{J}_{\mathtt{w}}^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\mathtt{w}, \tilde{\zeta}(\mathtt{w}), \tilde{\zeta}(\eta\mathtt{w}), \mathcal{J}^\sigma \tilde{\zeta}(\eta\mathtt{w}))\right)\right) \\
& \left. + \frac{\mathtt{w}^\alpha}{\chi^\alpha} \mathcal{J}_\chi^\alpha \mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\mathtt{w}, \tilde{\zeta}(\mathtt{w}), \tilde{\zeta}(\eta\mathtt{w}), \mathcal{J}^\sigma \tilde{\zeta}(\eta\mathtt{w}))\right) - \left(1 - \frac{\mathtt{w}^\alpha}{\chi^\alpha}\right) \zeta_0 \right| \leqslant \frac{\mu}{\Gamma(\alpha)} \int_0^{\mathtt{w}} (\mathtt{w}-\nu)^{\alpha-1} \psi(\nu) d\nu.
\end{aligned} \tag{3.12}$$

Since the formula (3.9), we get

$$\begin{aligned}
\left|\tilde{\zeta}(\mathtt{w}) - \zeta(\mathtt{w})\right| \leqslant & \left|\tilde{\zeta}(\mathtt{w}) - \int_0^{\mathtt{w}} \mathcal{R}'(\mathtt{w}-\nu) \left(\mathcal{J}_\nu^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\nu, \zeta(\nu), \zeta(\eta\nu), \mathfrak{I}^\sigma \zeta(\eta\nu))\right)\right)\right.\right. \\
& \left. - \frac{\nu^\alpha}{\chi^\alpha} \mathcal{J}_\chi^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\nu, \zeta(\nu), \zeta(\eta\nu), \mathcal{J}^\sigma \zeta(\eta\nu))\right)\right) + \left(1 - \frac{\nu^\alpha}{\chi^\alpha}\right) \zeta_0 \right) d\nu \\
& - \mathcal{J}_{\mathtt{w}}^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^\sigma \zeta(\eta\mathtt{w}))\right)\right) \\
& + \frac{\mathtt{w}^\alpha}{\chi^\alpha} \mathcal{J}_\chi^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^\sigma \zeta(\eta\mathtt{w}))\right)\right) - \left(1 - \frac{\mathtt{w}^\alpha}{\chi^\alpha}\right) \zeta_0 \\
& + \int_0^{\mathtt{w}} \mathcal{R}'(\mathtt{w}-\nu) \left(\mathcal{J}_\nu^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\nu, \tilde{\zeta}(\nu), \tilde{\zeta}(\eta\nu), \mathcal{J}^\sigma \tilde{\zeta}(\eta\nu)) - \mathcal{G}_\lambda\left(s, \tilde{\zeta}(\nu), \tilde{\zeta}(\eta\nu), \mathcal{J}^\sigma \tilde{\zeta}(\eta\nu)\right)\right)\right)\right. \\
& \left. + \frac{\nu^\alpha}{\chi^\alpha} \mathcal{J}_\chi^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\nu, \tilde{\zeta}(\nu), \tilde{\zeta}(\eta\nu), \mathcal{J}^\sigma \tilde{\zeta}(\eta\nu)) - \mathcal{G}_\lambda(\nu, \tilde{\zeta}(\nu), \tilde{\zeta}(\eta\nu), \mathcal{J}^\sigma \tilde{\zeta}(\eta\nu))\right)\right)\right) d\nu \\
& + \mathcal{J}_{\mathtt{w}}^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\mathtt{w}, \tilde{\zeta}(\mathtt{w}), \tilde{\zeta}(\eta\mathtt{w}), \mathcal{J}^\sigma \tilde{\zeta}(\eta\mathtt{w})) - \mathcal{G}_\lambda(\mathtt{w}, \tilde{\zeta}(\mathtt{w}), \tilde{\zeta}(\eta\mathtt{w}), \mathcal{J}^\sigma \tilde{\zeta}(\eta\mathtt{w}))\right)\right) \\
& \left. - \frac{\mathtt{w}^\alpha}{\chi^\alpha} \mathcal{J}_\chi^\alpha \left(\mathcal{J}_x^\delta \left(\mathcal{G}_\lambda(\mathtt{w}, \tilde{\zeta}(\mathtt{w}), \tilde{\zeta}(\eta\mathtt{w}), \mathcal{J}^\sigma \tilde{\zeta}(\eta\mathtt{w})) - \mathcal{G}_\lambda(\mathtt{w}, \tilde{\zeta}(\mathtt{w}), \tilde{\zeta}(\eta\mathtt{w}), \mathcal{J}^\sigma \tilde{\zeta}(\eta\mathtt{w}))\right)\right)\right|.
\end{aligned}$$

So by (3.12), we obtain

$$\begin{aligned}
\left|\tilde{\zeta}(\mathtt{w}) - \zeta(\mathtt{w})\right| \leqslant & \frac{\mu}{\Gamma(\alpha)} \int_0^{\mathtt{w}} (\mathtt{w}-\nu)^{\alpha-1} \psi(\nu) d\nu \\
& + \left|\int_0^{\mathtt{w}} \mathcal{R}'(\mathtt{w}-\nu) \left(\mathcal{J}_\nu^\alpha \left(\mathcal{J}_x^\delta \mathcal{G}_\lambda(\nu, \tilde{\zeta}(\nu), \tilde{\zeta}(\eta\nu), \mathcal{J}^\sigma \tilde{\zeta}(\eta\nu)) - \mathcal{G}_\lambda(\nu, \zeta(\nu), \zeta(\eta\nu), \mathcal{J}^\sigma \zeta(\eta\nu))\right)\right)\right.
\end{aligned}$$

$$+\frac{\nu^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\chi}^{\delta}\mathcal{G}_{\lambda}(\nu,\tilde{\zeta}(\nu),\tilde{\zeta}(\eta\nu),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta\nu))-\mathcal{G}_{\lambda}(\nu,\zeta(\nu),\zeta(\eta\nu),\mathcal{J}^{\sigma}\zeta(\eta\nu))\right)\right)d\nu$$
$$+\mathcal{J}_{w}^{\alpha}\left(\mathcal{J}_{\chi}^{\delta}\left(\mathcal{G}_{\lambda}(w,\tilde{\zeta}(w),\tilde{\zeta}(\eta w),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta w))-\mathcal{G}_{\lambda}(w,\zeta(w),\zeta(\eta w),\mathcal{J}^{\sigma}\zeta(\eta w))\right)\right)$$
$$\left.-\frac{w^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\chi}^{\delta}\left(\mathcal{G}_{\lambda}(w,\tilde{\zeta}(w),\tilde{\zeta}(\eta w),\mathcal{J}^{\sigma}\tilde{\zeta}(\eta w))-\mathcal{G}_{\lambda}(w,\zeta(w),\zeta(\eta w),\mathcal{J}^{\sigma}\zeta(\eta w))\right)\right)\right|.$$

Now, using (H3), (H5), and (3.10), we have

$$\left|\tilde{\zeta}(w)-\zeta(w)\right|\leqslant\mu\theta_{\psi}\psi(w)+(1+\|\varphi_{\mathcal{B}}\|_{L^1})\frac{2L_{\lambda}\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^{\sigma}}{\Gamma(\sigma+1)}+2\right)\|\tilde{\zeta}-\zeta\|,$$

which implies that

$$\left(1-(1+\|\varphi_{\mathcal{B}}\|_{L^1})\frac{2L_{\lambda}\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^{\sigma}}{\Gamma(\sigma+1)}+2\right)\right)\|\tilde{\zeta}-\zeta\|\leqslant\mu\theta_{\psi}\psi(w).$$

Consequently, we can write

$$\|\tilde{\zeta}-\zeta\|\leqslant\frac{\theta_{\psi}}{\left(1-(1+\|\varphi_{\mathcal{B}}\|_{L^1})\frac{2L_{\lambda}\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^{\sigma}}{\Gamma(\sigma+1)}+2\right)\right)}\mu\psi(w).$$

Since the condition (3.5), there exist

$$\delta_{\mathcal{G}_{\lambda}}=\frac{\theta_{\psi}}{\left(1-(1+\|\varphi_{\mathcal{B}}\|_{L^1})\frac{2L_{\omega}\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^{\sigma}}{\Gamma(\sigma+1)}+1\right)\right)}>0,$$

such that $\|\tilde{\zeta}-\zeta\|\leqslant\delta_{\mathcal{G}_{\lambda}}\mu\psi(w)$. Then, by the Definition 2.11, the abstract sequential pantograph TFDE (1.4) is UHR-S. □

4. The case $\mathcal{B}\equiv\kappa$

In this case we have the following problem:

$$\begin{cases}\mathcal{D}^{\delta}\left(\mathcal{D}^{\alpha}-\kappa\right)\zeta(w)=\mathcal{G}_{\lambda}(w,\zeta(w),\zeta(\eta w),\mathcal{J}^{\sigma}(\zeta(\eta w))), & w\in[0,\chi],\\ \zeta(0)=\zeta_0, \quad \zeta(\chi)=\kappa\mathcal{J}_{\chi}^{\alpha}(\zeta),\end{cases}\tag{4.1}$$

with $\mathcal{D}^{\alpha}$, $\mathcal{D}^{\beta}$ being two CFD and $\mathcal{J}^{\sigma}$ is integral of Sturm-Liouville, $0<\alpha<1$, $0<\delta,\sigma<1$, $\kappa\in\mathbb{R}$, $0<\eta<1$, and $\mathcal{G}_{\lambda}\in C([0,\chi],Z)$ depending on a parameter $\kappa\geqslant 0$ with $\mathcal{G}_{\lambda}:[0,\chi]\times Z^3\to Z$ is continuous function. The problem (4.1) is equivalent to

$$\zeta(w)=\kappa\mathcal{J}_{w}^{\alpha}(\zeta(w))+\mathcal{J}_{w}^{\alpha}\left(\mathcal{J}_{\chi}^{\delta}\left(\mathcal{G}_{\lambda}(w,\zeta(w),\zeta(\eta w),\mathcal{J}^{\sigma}\zeta(\eta w))\right)\right)$$
$$-\frac{w^{\alpha}}{\chi^{\alpha}}\mathcal{J}_{\chi}^{\alpha}\left(\mathcal{J}_{\chi}^{\delta}\left(\mathcal{G}_{\lambda}(w,\zeta(w),\zeta(\eta w),\mathcal{J}^{\sigma}\zeta(\eta w))\right)\right)+\zeta_0\left(1-\frac{w^{\alpha}}{\chi^{\alpha}}\right),\quad w\in[0,\chi].$$

Corollary 4.1. *Assume* (H2)-(H4) *hold true. Then, the system* (4.1) *owns a mild solution.*

Corollary 4.2. *Assuming conditions* (H1)-(H4) *hold, for* $0<\delta,\alpha,\sigma<1$, *the abstract sequential TFDE* (4.1) *is UH-S and it follows that it is also generalized UH-S.*

5. Applications

We investigate the existence result of solution for the pantograph TFDE with CFD, which is formulated as follows.

Example 5.1. Let the pantograph sequential fractional problem with CFD:

$$\begin{cases} \mathcal{D}^{\frac{3}{5}}\left(\mathcal{D}^{\frac{1}{2}}+\frac{1}{2}\right)\zeta(\mathtt{w})=\dfrac{1}{\lambda}\left(\cos(\zeta(\frac{1}{2}\mathtt{w}))+\mathcal{J}_{\mathtt{w}}^{\frac{1}{2}}\left(\zeta(\frac{1}{2}\mathtt{w})\right)\right), & \mathtt{w}\in[0,1], \\ \zeta(0)=\frac{3}{5}, \quad \zeta(1)=-\frac{1}{2}\mathcal{J}_{\chi}^{\frac{1}{2}}(\zeta(\mathtt{w})), \end{cases} \tag{5.1}$$

where $\delta=\frac{3}{5}, \alpha=\frac{1}{2}, \sigma=\frac{1}{2}, \eta=\frac{1}{2}$, $\mathcal{B}\zeta=-\frac{1}{2}\zeta$, and $\lambda>1$ parameter large enough. The problem (5.1) associated differentiable R-operators $(\mathcal{R}(\mathtt{w}))_{\mathtt{w}\geqslant 0}$ and $\exists\mathcal{K}>0$ such that $\|\mathcal{R}'(\mathtt{w})z\|\leqslant\mathcal{K}\|z\|$, $\forall\mathtt{w}>0, z\in D(\mathcal{B})$. Hence the assumptions (H1) and (H2) are satisfied. Now for λ large enough, we have

$$\mathcal{G}_\lambda(\mathtt{w},\zeta(\mathtt{w}),\zeta(\eta\mathtt{w}),\mathcal{J}^\sigma\zeta(\eta\mathtt{w}))=\frac{1}{\lambda}\left(\cos(\zeta(\frac{1}{2}\mathtt{w}))+\mathcal{J}_{\mathtt{w}}^{\frac{1}{2}}\left(\zeta(\frac{1}{2}\mathtt{w})\right)\right)$$

and

$$\|\mathcal{G}_\lambda(\mathtt{w},z_1,z_2,z_3)-\mathcal{G}_\lambda(\mathtt{w},x_1,x_2,x_3)\|\leqslant\frac{1}{\lambda}\left(\|z_1-x_1\|+\|z_2-x_2\|+\|z_3-x_3\|\right) \text{ with } L_\lambda=\frac{1}{\lambda}.$$

So, (H3) is satisfied and for (H4), we have

$$\begin{aligned}(1+\|\varphi_{\mathcal{B}}\|_{L^1})\frac{2L_\lambda\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^\sigma}{\Gamma(\sigma+1)}+2\right)&\leqslant(1+\mathcal{K})\frac{1}{\lambda}\frac{2}{\Gamma\left(\frac{11}{10}+1\right)}\left(\frac{1}{\Gamma\left(\frac{1}{2}+1\right)}+2\right)\\ &\leqslant\frac{(1+\mathcal{K})}{\lambda}\frac{2}{1.04}\left(\frac{1}{0.88}+2\right)\leqslant 6.03\times\frac{(1+\mathcal{K})}{\lambda}.\end{aligned}$$

Then $\exists\,\lambda^*=6,03\,(1+\mathcal{K})>0$, such that, $\forall\lambda>\lambda^*$,

$$(1+\|\varphi_{\mathcal{B}}\|_{L^1})\frac{2L_\lambda\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^\sigma}{\Gamma(\sigma+1)}+2\right)<1.$$

Hence, based on Theorem 3.4, the problem (5.1) possesses a solution on $[0,1]$. On the other hand, in view of Theorem 3.5 this problem is UH-S.

Example 5.2. Consider the below sequential pantograph TFDE with CFD in the space $Z=L^2([0,\pi])$:

$$\begin{cases} \dfrac{\partial^{\frac{1}{2}}}{\partial\mathtt{w}^{\frac{1}{2}}}\left(\dfrac{\partial^{\frac{3}{4}}}{\partial\mathtt{w}^{\frac{3}{4}}}-\dfrac{\partial^2}{\partial x^2}\right)\zeta(x,\mathtt{w})=\dfrac{1}{\lambda}\left(\sin(\zeta(x,\frac{1}{3}\mathtt{w}))+\mathcal{J}_{\mathtt{w}}^{\frac{1}{2}}\left(\zeta(x,\frac{1}{3}\mathtt{w})\right)\right), \\ \zeta(0,\mathtt{w})=\zeta(\pi,\mathtt{w})=0, \quad \mathtt{w}\in[0,1], \\ \zeta(x,0)=e^x, \zeta(x,1)=\mathcal{J}_1^{\frac{3}{4}}\left(\dfrac{\partial^2\zeta}{\partial x^2}(x,1)\right), \quad x\in[0,\pi]. \end{cases} \tag{5.2}$$

where $\delta=\frac{1}{2}, \alpha=\frac{3}{4}, \sigma=\frac{1}{2}, \eta=\frac{1}{3}$, and $\lambda>1$ parameter large enough. Let $\mathcal{B}\zeta=\zeta''$ with

$$D(\mathcal{B})=\left\{\zeta\in Z, \zeta''\in Z, \quad \zeta(0)=\zeta(\pi)=0\right\}.$$

Due to [11], problem (5.2) is associated analytic and differentiable R-operators $(\mathcal{R}(\mathtt{w}))_{\mathtt{w}\geqslant 0}$ and $\exists\mathcal{K}>0$ such that

$$\left\|\mathcal{R}'(\mathtt{w})z\right\|\leqslant\mathcal{K}\|z\|, \forall\mathtt{w}>0, z\in D(\mathcal{B}).$$

Hence, the assumptions (H1) and (H2) are satified. Now for λ large enough, we have

$$\mathcal{G}_\lambda(\mathtt{w}, \zeta(\mathtt{w}), \zeta(\eta\mathtt{w}), \mathcal{J}^\sigma\zeta(\eta\mathtt{w})) = \frac{1}{\lambda}\left(\sin(\zeta(x, \frac{1}{3}\mathtt{w})) + \mathcal{J}^{\frac{1}{2}}\zeta(x, \frac{1}{3}\mathtt{w})\right)$$

and

$$\|\mathcal{G}_\lambda(\mathtt{w}, z_1, z_2, z_3) - \mathcal{G}_\lambda(\mathtt{w}, x_1, x_2, x_3)\| \leqslant \frac{1}{\lambda}\left(\|z_1 - x_1\| + \|z_2 - x_2\| + \|z_3 - x_3\|\right), \text{ with } L_\lambda = \frac{1}{\lambda}.$$

On the other hand, we have

$$\begin{aligned}(1 + \|\varphi_{\mathcal{B}}\|_{L^1})\frac{2L_\lambda\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^\sigma}{\Gamma(\sigma+1)} + 2\right) &\leqslant (1 + \mathcal{K})\frac{1}{\lambda}\frac{2}{\Gamma\left(\frac{5}{4}+1\right)}\left(\frac{1}{\Gamma\left(\frac{1}{2}+1\right)} + 2\right)\\ &\leqslant \frac{(1+\mathcal{K})}{\lambda}\frac{2}{1.13}\left(\frac{1}{0.88} + 2\right) \leqslant 5.5 \times \frac{(1+\mathcal{K})}{\lambda}.\end{aligned}$$

Then $\exists\, \lambda^* = 5,5\,(1 + \mathcal{K}) > 0$, such that, $\forall\lambda > \lambda^*$,

$$(1 + \|\varphi_{\mathcal{B}}\|_{L^1})\frac{2L_\lambda\chi^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)}\left(\frac{\chi^\sigma}{\Gamma(\sigma+1)} + 2\right) < 1.$$

So, (H3) and (H4) are satisfied. For (H5), Let $\psi(\omega) = \omega \in C([0,\chi], \mathbb{R}_+)$, $\exists\theta_\psi = \frac{2\chi}{\alpha} > 0$ such that, $\forall \mathtt{w} \in [0, \chi]$,

$$\frac{1}{\Gamma(\alpha)}\int_0^{\mathtt{w}} (\mathtt{w} - \nu)^{\alpha-1}\nu d\nu \leqslant 2\frac{\chi}{\alpha}\psi(\mathtt{w}).$$

Then, due to Theorem 3.4, the problem (5.2) has at least one solution, also based on Theorems 3.5 and 3.6 this problem is UH-S and UHR-S.

6. Conclusions

In this paper, our results are expected to motivate the generation of the concept of TFDE via the CFD. In this case, we introduced an operator in one coefficient of a sequential pantograph equation. We have discussed the existence of mild solutions and the UH-S and UHR-S for the abstract sequential pantograph TFDE (1.4) with nonhomogeneous boundary conditions. The R-operators play a key role in this study. We have established the existence results by applying the Schaefer fixed point theorem, along with UH-S and UHR-S investigated by utilizing nonlinear topics. To justify our findings, we provide some applications to which our results can be applied. For future research, we will focus on establishing the uniqueness, existence, and stability of this equation, but with two operator coefficients by using the semi-group approach.

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References

[1] M. I. Abbas, M. Ghaderi, Sh. Rezapour, S. T. M. Thabet, *On a coupled system of fractional differential equations via the generalized proportional fractional derivatives*, J. Funct. Spaces, **2022** (2022), 10 pages. 1

[2] S. Aljawi, S. Aljohani, Kamran, A. Ahmed, N. Mlaiki,*Numerical solution of the linear fractional delay differential equation using Gauss–Hermite Quadrature*, Symmetry, **16** (2024), 1–17.

[3] R. Agarwal, S. Hristova, D. O'Regan,*Existence and Ulam type stability for nonlinear Riemann-Liouville fractional differential equations with constant delay*, Electron. J. Qual. Theory Differ. Equ., **2020** (2020), 18 pages. 1

[4] B. Ahmed, S. K. Ntouyas, *Existence and uniqueness of solutions for Caputo-Hadamard sequential fractional order neutral functional differential equations*, Electron. J. Differ. Equ., **2017** (2017), 11 pages. 1

[5] B. Ahmed, S. K. Ntouyas, A. Alsaedi,*Sequential fractional differential equationa and inclusions with semi-periodic and nonlocal integro-multipoint boundary conditions*, J. King Saud Univ.-Sci., **31** (2019), 184–193. 1

[6] A. Ali, I. Mahariq, K. Shah, T. Abdeljawad, B. Ali-Sheikh, *Stability analysis of initial value problem of pantograph-type implicit fractional differential equations with impulsive conditions*, Adv. Differ. Equ., **2021** (2021), 17 pages. 1

[7] A. Al-Khateeb, H. Zureigat, O. Ala'yed, S. Bawaneh, *Ulam-Hyers stability and uniqueness for nonlinear sequential fractional differential equations involving integral boundary conditions*, Fractal Fract., **5** (2021), 1–16. 1

[8] A. Anapali, Y. Ozturk, M. Gulsu, *Numerical approach for solving fractional pantograph equation*, Int. J. Comput. Appl., **113** (2015), 45–52. 1, 1

[9] M. H. Aqlan, A. Alsaedi, B. Ahmed, J. J. Nieto, *Existence theory for sequential fractional differential equations with anti-periodic type boundary conditions*, Open Math., **14** (2016), 723–735. 1

[10] M. Bahar Ali Khan, T. Abdeljawad, K. Shah, G. Ali, H. Khan, A. Khan, *Study of a nonlinear multi-terms boundary value problem of fractional pantograph differential equations*, Adv. Differ. Equ., **2021** (2021), 15 pages. 1

[11] K. Balachandran, S. Kiruthika, *Existence results for fractional integrodifferential equations with nonlocal condition via resolvent operators*, Comput. Math. Appl., **62** (2011), 1350–1358. 1, 5.2

[12] K. Balachandran, S. Kiruthika, J. J. Trujillo, *Existence of solutions of nonlinear fractional pantograph equations*, Acta Math. Sci. Ser. B (Engl. Ed.), **33** (2013), 712–720. 1

[13] Q. Dai, R. Gao, Z. Li, C. Wang, *Stability of Ulam-Hyers and Ulam-Hyers-Rassias for a class of fractional differential equations*, Adv. Differ. Equ., **2020** (2020), 1–15. 2

[14] G. Derfel, A. Iserles, *The pantograph equation in the complex plane*, J. Math. Anal. Appl., **213** (1997), 117–132. 1

[15] A. J. Gnanaprakasam, B. Ramalingam, G. Mani, O. Ege, R. George, *A numerical scheme and application to fractional integro-differential equation using fixed-point techniques*, Fractal Fract., **8** (2024), 1–17. 1

[16] E. Hernández, D. O'Regan, *On a New class of abstract impulsive differential equations*, Proc. Amer. Math. Soc., **141** (2013), 1641–1649. 1

[17] E. Hernández, D. O'Regan, K. Balachandran, *On recent developments in the theory of abstract differential equations with fractional derivatives*, Nonlinear Anal., **73** (2010), 3462–3471.

[18] E. Hernández, D. O'Regan, K. Balachandran, *Existence results for abstract fractional differential equations with nonlocal conditions via resolvent operators*, Indag. Math. (N.S.), **24** (2013), 68–82. 1

[19] M. Houas, *Sequential fractional pantograph differential equations with nonlocal boundary conditions: uniqueness and Ulam-Hyers-Rassias stability*, Results Nonlinear Anal., **5** (2022), 29–41. 1, 1, 2.3, 2.4, 2

[20] M. Houas, K. Ould Melha, *Existence and uniqueness results for a coupled system of Hadamard fractional differential equations with multi-point boundary conditions*, Facta Univ. Ser. Math. Inform., **35** (2020), 843–856. 1

[21] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publishing Co., Inc., River Edge, NJ, (2000). 1

[22] A. Iserles, *Exact and discretized stability of the pantograph equation*, Appl. Numer. Math., **24** (1997), 295–308. 1

[23] A. Iserles, Y. K. Liu, *On pantograph integro-differential equations*, J. Integral Equ. Appl., **6** (1994), 213–237. 1

[24] R. Khalil, S. Alshari, S. Khamis, *Second-order abstract Cauchy problem of conformable fractional type*, Int J. Nonlinear Anal. Appl., **13** (2022), 1143–1150. 1

[25] A. Khan, H. Khan, J. F. Gómez-Aguilar, T. Abdeljawad, *Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel*, Chaos Solitons Fractals, **127** (2019), 422–427. 2

[26] A. A. Kilbas, S. A. Marzan, *Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions*, Differ. Uravn., **41** (2005), 84–89. 2.1, 2.2, 2

[27] P. Kumar, D. N. Pandey, D. Bahuguna, *On a new class of abstract impulsive function differential equation of fractional order*, J. Nonlinear Sci. Appl., **7** (2014), 102–114. 1

[28] C. Lizama, F. Poblete, *Regularity of mild solution for a class of fractional order differential equations*, Appl. Math. Comput., **224** (2013), 803–816. 1

[29] G. Mani, P. Ganesh, K. Chidhambaram, S. Aljohani, N. Mlaiki, *Existence and uniqueness solutions of multi-term delay Caputo fractional differential equations*, Int. J. Anal. Appl., **22** (2024), 1–23.

[30] G. Mani, P. Ganesh, P. Ramasamy, S. Aljohani, N. Mlaiki, *Existence and stability results for a coupled multi-term Caputo fractional differential equations*, Fixed Point Theory Algorithms Sci. Eng., **2025** (2025), 33 pages.

[31] G. Mani, A. J. Gnanaprakasam, B. Ramaligam, A. S. A. Omer, I. Khan, *Mathematical model of the lumpy skin disease using Caputo fractional-order derivative via invariant point technique*, Sci. Rep., **15** (2025), 1–16. 1

[32] A. Mohammed Djaouti, K. Ould Melha, M. A. Latif, *New results on the solvability of abstract sequential Caputo fractional differential equations with a resolvent-operator approach and applications*, Mathematics, **12** (2024), 18 pages. 1

[33] S. A. Murad, Z. A. Ameen, *Existence and Ulam stability for fractional differential equations of mixed Caputo-Riemann derivatives*, AIMS Math., **7** (2022), 6404–6419. 1

[34] G. M. N'Guérékata, *A Cauchy problem for some fractional abstract differential equation with non local conditions*, Nonlinear Anal., **70** (2009), 1873–1876. 1

[35] K. Ould Melha, A. Mohammed Djaouti, M. A. Latif, V. L. Chinchane, *Study of Uniqueness and Ulam-Type stability of abstract Hadamard fractional differential equations of Sobolev type via resolvent operators*, Axioms **13** (2024), 16 pages. 2.9, 2.10

[36] K. Ould Melha, M. Djilali, V. L. Chinchane, *Abstract fractional differential equations with Caputo-Fabrizio derivative*, Fract. Differ. Calc., **13** (2023), 149–162. 1

[37] K. Ould Melha, M. Djilali, V. L. Chinchane, A. B. Nale, S. T. M. Thabet, I. Kedim, *Uniqueness and stability results of the abstract fractional spatial heterogeneous viral infection model*, Bound. Value Probl., **2025** (2025), 17 pages. 2.11, 2.12

[38] M. I. Pasha, K. R. K. Rao, K. Nallasamy, G. Mani, S. Kumar, *Solution of boundary value problem via C^*-algebra valued bipolar metric spaces*, Appl. Gen. Topol., **26** (2025), 29–49. 1

[39] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, (1983). 2.7

[40] I. Podlubny, *Fractional Differential equations: An introduction to fractional derivatives, fractional differential equations, to method of their solutions and some of their applications*, Academic Press, San Diego, (1999). 1

[41] J. Prüss, *Evolutionary integral equations and applications*, Birkhäuser Verlag, Basel, (1993). 2.5, 2.6, 2.7, 2.7, 2.8

[42] S. Rezapour, S. T. M. Thabet, A. S. Rafeeq, I. Kedim, M. Vivas-Cortez, N. Aghazadeh, *Topology degree results on a $G-ABC$ implicit fractional differential equation under three-point boundary conditions*, PLoS ONE, **19** (2024), 19 pages 1

[43] M. Saeedian, M. Khalighi, N. Azimi-Tafreshi, G. R. Jafari, M. Ausloos, *Memory effects on epidemic evolution: the susceptible-infected-recovered epidemic model*, Phys. Rev. E, **95** (2017), 9 pages. 1

[44] A. Salem, L. Almaghamsi, *Solvability of sequential fractional differential equation at resonance*, Mathematics, **11** (2023), 1–18. 1

[45] F. A. Shah, Kamran, W. Boulila, A. Koubaa, N. Mlaiki, *Numerical solution of advection–diffusion equation of fractional order using chebyshev collocation method*, Fractal Fract., **7** (2023), 20 pages. 1

[46] D. R. Smart, *Fixed point theorems*, Cambridge University Press, London-New York, (1974). 1

[47] M. Tamsir, N. Dhiman, D. Nigam, A. Chauhan, *Approximation of Caputo time-fractional diffusion equation using redefined cubic exponential B-spline collocation technique*, AIMS Math., **6** (2021), 3805–3820. 1, 1

[48] S. T. M. Thabet, M. B. Dhakne, *On boundary value problems of higher order abstract fractional integro-differential equations*, Int. J. Nonlinear Anal. Appl., **7** (2016), 165–184. 1

[49] D. Vivek, E. M. Elsayed, K. Kanagarajan, *Existence and Ulam stability results for a class of boundary value problem of neutral pantograph equations with complex order*, SeMA J., **77** (2020), 243–256. 1

[50] Z. Wang, Z. Qiu, S. Qiu, Z. Ruan, *Multiple Terms Identification of Time-Fractional Diffusion Equation with Symmetric Potential from Nonlocal Observation*, Fractal Fract., **7** (2023), 15 pages. 1, 1

[51] A. Zada, M. Yar, T. Li, *Existence and stability analysis of nonlinear sequential coupled system of Caputo fractional differential equations with integral boundary conditions*, Ann. Univ. Paedagog. Crac. Stud. Math., **17** (2018), 103–125. 1