

Stabilization of fractional hybrid systems with applications in biomedical dynamics



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Abstract

This work introduces a novel analytical framework for analyzing the stability of Caputo fractional dynamic equations on time scales (CFDET) using a two-measure approach combined with comparison principle. By applying paired measures (m_0, m) and vector Lyapunov functions, we derive sufficient conditions for both (m_0, m) -stability and asymptotic stability. The method simplifies analysis by relating the system to a well-understood comparison system, reducing the task to verifying quasimonotonicity and avoiding the need for explicit solutions. The framework's effectiveness is demonstrated through two biological models, an immune response system and a 3D hypothalamic-pituitary-adrenal (HPA) axis, highlighting its ability to handle nonlinearities, hybrid time scales, and varying system dimensions. This study bridges theoretical stability analysis with practical biomedical applications, advancing the understanding of fractional-order hybrid dynamics.

Keywords: Immune response model, neuro-endocrine regulation model, two measure stabilization, fractional calculus, time scales, comparison principle.

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1. Introduction

Time scale calculus offers a unified methodology for investigating dynamical systems, encompassing purely continuous, purely discrete, and mixed-type behaviors within a single theoretical framework. Introduced in [8], the theory of time scales provides a comprehensive mathematical structure to unify and extend the theories of differential equations [2] and difference equations [21]. This approach has found widespread applications in various fields, including biology, economics, engineering, and physics, where processes evolve over time in a manner that is neither purely continuous nor purely discrete.

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Stability analysis is a cornerstone of dynamical systems theory as it provides critical insights into the long term behavior of solutions to dynamic equations [15, 24, 25]. Understanding the stability properties of a system is essential for predicting its behavior under perturbations, designing control strategies, and ensuring the reliability of mathematical models. Although significant work has been done on the stability of integer-order dynamic equations on time scales, recent trends have shifted toward fractional-order dynamic equations, which offer a more generalized framework for modeling physical phenomena [14, 17]. Fractional calculus, particularly the Caputo fractional derivative, has gained prominence due to its ability to capture memory effects and hereditary properties, making it a powerful tool for describing complex systems [1, 4, 5]. Several stability approaches for fractional dynamic equations have been recently explored, including stability, uniform stability, eventual stability, practical stability, and asymptotic stability [18, 22, 23].

Among the various methods developed for stability analysis, the comparison principle is an elegant and effective tool [13]. This method leverages the properties of auxiliary comparison equations to determine the stability of the original system, often simplifying complex analyses and providing clear and interpretable results. By employing Lyapunov methods and vector Lyapunov functions, the comparison principle enables the reduction of stability analysis to checking the quasimonotonicity property of the comparison system [27]. This approach not only broadens the scope of stability analysis but also provides a unified framework applicable to a wide range of time scales. Consider the Caputo fractional dynamic system for $0 < \mathfrak{J} < 1$ as

$${}^C\Delta^{\mathfrak{J}}\wp = \mathcal{R}(t, \wp), \quad t \in \mathbb{T}, \quad \wp(t_0) = \wp_0, \quad t_0 \geq 0, \quad (1.1)$$

where $\mathcal{R} \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N]$ with $\mathcal{R}(t, 0) \equiv 0$, $\mathbb{T}$ is known as the time scale (any close subset of $\mathbb{R}$) and ${}^C\Delta^{\mathfrak{J}}\wp$ denotes the Caputo fractional delta derivative (Fr Δ D) of $\wp \in \mathbb{R}^N$ of order $\mathfrak{J}$. Let $\wp(t) = \wp(t, t_0, \wp_0) \in C_{rd}^{\mathfrak{J}}[\mathbb{T}, \mathbb{R}^N]$ be the unique solution of (1.1)-existence and uniqueness results can be found in [3, 7, 11].

In this work, we explore two-measure stability analysis of the CFrDET (1.1) using the comparison principle approach. By employing two measures (m_0, m) , we establish sufficient conditions for (m_0, m) -stability and (m_0, m) -asymptotic stability of solutions using vector Lyapunov functions. To achieve this using our comparison approach, we consider a comparison system of the form

$${}^C\Delta^{\mathfrak{J}}\mathfrak{s} = \mathcal{G}(t, \mathfrak{s}), \quad \mathfrak{s}(t_0) = \mathfrak{s}_0 \geq 0, \quad (1.2)$$

where $\mathcal{G} : \mathbb{T} \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ with $\mathcal{G}(t, 0) \equiv 0$, $n \leq N$. Assuming a unique solution $\mathfrak{s}(t) = \mathfrak{s}(t; t_0, \mathfrak{s}_0) \in C_{rd}^{\mathfrak{J}}[\mathbb{T}, \mathbb{R}_+^n]$ exists (see [9, 20, 26]).

The assumptions on (1.2) are that it represents a simpler system whose qualitative properties, including the existence of a unique solution and stability, are either already established or can be easily determined, as demonstrated in [19]. This allows us to draw parallels between the behavior of the main system (1.1) and that of the well-understood comparison system (1.2), enabling us to infer stability properties without directly solving (1.1). This approach not only simplifies the analysis but also unifies various stability concepts under a single framework through the use of our two measures (m_0, m) . The practical utility of this methodology is demonstrated through applications in a nonlinear immune response model, describing T-cell-pathogen interactions, and a 3D neuro-endocrine HPA axis model both of which highlights the framework's adaptability to varying dimensions, nonlinearities, and biological contexts. By bridging theoretical criteria with biomedical systems, this work advances stability analysis of Caputo fractional dynamics on time scales.

This work is arranged as follows. In Section 2, we provide the necessary preliminaries on time scales calculus, Fr Δ D, and two-measure stability and also introduce the comparison lemma. In Section 3, we then present our main results, including sufficient conditions for (m_0, m) -stability and (m_0, m) -asymptotic stability. In Section 4, we illustrate the theoretical findings with concrete examples which is then applied in Section 5 to two real world applications in bio-medicines.

2. Methods

Dynamic equations on time scales represent a class of differential equations that provide a unified and flexible framework for analyzing system behavior across both continuous and discrete time domains, as highlighted in [10]. For a given time scale (any closed subset of real numbers) $\mathbb{T}$, we present the following key definitions.

Definition 2.1 ([6]). For a time scale $\mathbb{T}$ and $t \in \mathbb{T}$, the *forward jump operator* is defined as $p(t) = \inf\{a \in \mathbb{T} \mid a > t\}$, while the *backward jump operator* is given by $\varpi(t) = \sup\{a \in \mathbb{T} \mid a < t\}$. A point $t \in \mathbb{T}$ is classified as:

- *right-dense* (rd) if $p(t) = t$;
- *right-scattered* (rs) if $p(t) > t$;
- *left-dense* (ld) if $\varpi(t) = t$;
- *left-scattered* (ls) if $\varpi(t) < t$.

Additionally, the *graininess function* $\omega(t)$ is defined as $\omega(t) = p(t) - t$.

Definition 2.2 ([6]). We define the following class of functions:

$$\begin{aligned}\mathcal{K} &= \{\psi \in [[0, r], [0, \infty)] : \psi(t) \text{ is strictly increasing on } [0, r] \text{ and } \psi(0) = 0, \\ \mathcal{CK} &= \{a \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+] : a(t, s) \in \mathcal{K} \text{ for each } t\}, \\ \Lambda &= \{m \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+] : \inf_{t, \varphi} m(t, \varphi) = 0\}.\end{aligned}$$

Definition 2.3. We define the Caputo Fr Δ DiD of the Lyapunov function, $\aleph(t, \varphi) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}_+^N]$ with respect to the solutions of system (1.1) given as:

$${}^C\Delta_+^{\mathfrak{J}} \aleph(t, \varphi) = \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left[\sum_{p=0}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} [\aleph(p(t) - p\omega, \varphi(p(t)) - \omega^{\mathfrak{J}}\mathcal{R}(t, \varphi(t)) - \aleph(t_0, \varphi_0))] \right],$$

and can be expanded as

$$\begin{aligned}{}^C\Delta_+^{\mathfrak{J}} \aleph(t, \varphi) &= \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ \aleph(p(t), \varphi(p(t))) - \aleph(t_0, \varphi_0) \right. \\ &\quad \left. - \sum_{p=1}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^{p+1} \binom{\mathfrak{J}}{p} [\aleph(p(t) - p\omega, \varphi(p(t)) - \omega^{\mathfrak{J}}\mathcal{R}(t, \varphi(t)) - \aleph(t_0, \varphi_0)] \right\},\end{aligned}\tag{2.1}$$

where $t \in \mathbb{T}$, $\varphi, \varphi_0 \in \mathbb{R}^N$, $\omega = p(t) - t$, and $\varphi(p(t)) - \omega^{\mathfrak{J}}\mathcal{R}(t, \varphi) \in \mathbb{R}^N$. Applying (2.3) to (2.1), we obtain

$$\begin{aligned}{}^C\Delta_+^{\mathfrak{J}} \aleph(t, \varphi) &= \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ \aleph(\eta(t), \varphi(p(t))) \right. \\ &\quad \left. + \sum_{p=1}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} [\aleph(p(t) - p\omega, \varphi(p(t)) - \omega^{\mathfrak{J}}\mathcal{R}(t, \varphi(t)))] \right\} - \frac{\aleph(t_0, \varphi_0)(t - t_0)^{-\mathfrak{J}}}{\Gamma(1 - \mathfrak{J})}.\end{aligned}$$

Remark 2.4. In [12, 16], it has been established that

$$\lim_{\omega \rightarrow 0^+} \sum_{p=1}^{\left[\frac{(t-t_0)}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} = -1,\tag{2.2}$$

and

$$\limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \sum_{p=0}^{\left[\frac{(t-t_0)}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} = {}^{\text{RL}}\Delta^{\mathfrak{J}}(1) = \frac{(t - t_0)^{-\mathfrak{J}}}{\Gamma(1 - \mathfrak{J})}, \quad t \geq t_0.\tag{2.3}$$

Definition 2.5. Let $\wp(t) = \wp(t; t_0, \wp_0)$ be any solution of (1.1), then the fractional dynamic system (1.1) is said to be

- (M₁) (m_0, m) -stable if, for each $\epsilon > 0$, $t_0 \in \mathbb{T}$, $\exists$ a positive function $\delta = \delta(t_0, \epsilon)$ that is rd-continuous in t_0 for each ϵ such that $m_0(t_0, \wp) < \delta \implies m(t, \wp(t)) < \epsilon$, $t \geq t_0$;
- (M₂) (m_0, m) -asymptotically stable if, it is stable and there exist positive functions $\delta_0 = \delta_0(t_0, \epsilon)$ and $T = T(t_0, \epsilon)$ that are rd-continuous in t_0 for each ϵ : for $t \geq t_0 + T$, the inequality $m_0(t_0, \wp) < \delta$ implies $m(t, \wp(t)) < \epsilon$.

Definition 2.6. Let $m_0, m \in \Lambda$. Then we say that

- (i) m_0 is finer than m if there exists a $\gamma > 0$ and a function $\tau \in \mathcal{CK}$ such that $m_0(t, \wp) < \gamma$ implies $m(t, \wp) \leq \tau(t, m_0(t, \wp))$;
- (ii) m_0 is uniformly finer than m if there exists a $\gamma > 0$ and a function $\tau \in \mathcal{CK}$ such that $m_0(t, \wp) < \gamma$ implies $m(t, \wp) \leq \tau(m_0(t, \wp))$.

Definition 2.7. The Lyapunov function $\aleph \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}_+^N]$ is said to be

- (i) m -positive definite if there exists a $\gamma > 0$ and a function $\phi \in \mathcal{K}$ such that $\phi(m(t, \wp)) \leq \aleph(t, \wp)$ whenever $m(t, \wp) < \gamma$;
- (ii) m -decescent if there exists a $\gamma > 0$ and a function $\chi \in \mathcal{K}$ such that $\aleph(t, \wp) \leq \chi(m(t, \wp))$ whenever $m(t, \wp) < \gamma$;
- (iii) m -weakly decescent if there exists a $\gamma > 0$ and a function $\chi \in \mathcal{CK}$ such that $\aleph(t, \wp) \leq \chi(m(t, \wp))$ whenever $m(t, \wp) < \gamma$.

Lemma 2.8 ([16]). Assume that

- (i) $\mathcal{G} \in C_{rd}[\mathbb{T} \times \mathbb{R}_+^n, \mathbb{R}_+^n]$ with $\mathcal{G}(t, \mathfrak{s})\omega$ being non-decreasing with respect to $\mathfrak{s}$;
- (ii) $\aleph \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}_+^N]$ is locally Lipschitz continuous in its second argument, satisfying

$${}^C\Delta_+^1 \aleph(t, \wp) \leq \mathcal{G}(t, \aleph(t, \wp)), \quad \forall (t, \wp) \in \mathbb{T} \times \mathbb{R}^N;$$

- (iii) the maximal solution $\omega(t) = \omega(t; t_0, \mathfrak{s}_0)$ of system (1.2) exists on $\mathbb{T}$.

Then, the inequality

$$\aleph(t, \wp(t)) \leq \omega(t), \quad \forall t \geq t_0, \quad (2.4)$$

holds whenever $\aleph(t_0, \wp_0) \leq \mathfrak{s}_0$, where $\wp(t) = \wp(t; t_0, \wp_0)$ denotes any solution of (1.1) defined for $t \in \mathbb{T}$ with $t \geq t_0$.

3. Results

Theorem 3.1. Suppose the following hypotheses hold.

- (1) For $\aleph(t, \wp(t)) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}_+^N]$:
- (i) $\aleph$ satisfies a local Lipschitz condition in $\wp$ and vanishes at zero ($\aleph(t, 0) \equiv 0$);
- (ii) there exists $\phi \in \mathcal{K}$ such that $\phi(m(t, \wp)) \leq \aleph_0(t, \wp)$, where $\aleph_0(t, \wp) = \sum_{j=1}^N \aleph_j(t, \wp(t))$;
- (2) for the measures $m_0, m \in \Lambda$:
- (i) m_0 is uniformly finer than m ;
- (ii) $\aleph(t, \wp)$ exhibits m_0 -decescent behavior;
- (3) the function $\mathcal{G} \in C_{rd}[\mathbb{T} \times \mathbb{R}_+^n, \mathbb{R}_+^n]$ is:
- (i) quasimonotone nondecreasing in its second argument $\forall t \in \mathbb{T}$;
- (ii) vanishes at zero ($\mathcal{G}(t, 0) \equiv 0$);
- (iii) satisfies the differential inequality: ${}^C\Delta_+^1 \aleph(t, \wp(t)) \leq \mathcal{G}(t, \aleph(t, \wp(t)))$;
- (4) stability holds for the trivial solution of comparison equation (1.2).

Then, system (1.1) achieves (m_0, m) -stability.

Proof. Consider an arbitrary $\epsilon > 0$. The stability of the trivial solution $s = 0$ of system (1.2) guarantees that for any $\phi(\epsilon) > 0$ and initial time $t_0 \in \mathbb{T}$, there exists $\lambda = \lambda(t_0, \epsilon) > 0$ satisfying

$$\sum_{j=1}^n s_j(t; t_0, s_0) < \phi(\epsilon), \quad \forall t \geq t_0, \quad (3.1)$$

provided that $\sum_{j=1}^n s_{0j} < \lambda$, where $s(t) = s(t, t_0, s_0)$ represents an arbitrary solution of the comparison system (1.2). By the m_0 -decreasing property of $\aleph(t, \varphi(t))$ and since m_0 is uniformly finer than m , then we can find a positive number γ and functions $\chi \in \mathcal{K}$ and $\beta \in \mathcal{CK}$, such that

$$\aleph_0(t_0, \varphi_0) \leq \chi(m_0(t_0, \varphi_0)) \text{ if } m_0(t_0, \varphi_0) < \gamma, \text{ and } m(t_0, \varphi_0) \leq \beta(m_0(t_0, \varphi_0)). \quad (3.2)$$

Combining (3.2) and assumption 1 (ii) above for $(t_0, \varphi_0) \in (\mathbb{T}, \mathbb{R}^N)$, we obtain

$$\phi(m(t_0, \varphi_0)) \leq \aleph_0(t_0, \varphi_0) \leq \chi(m_0(t_0, \varphi_0)),$$

whenever $m_0(t_0, \varphi_0) < \gamma$. Now, we claim that for any solution $\varphi(t) = \varphi(t; t_0, \varphi_0)$, and functions $\delta = \delta(t_0, \epsilon) \in (0, \gamma]$, $\chi(\delta) < \lambda$, such that

$$m(t, \varphi(t)) < \epsilon, \text{ whenever } m_0(t_0, \varphi_0) < \delta. \quad (3.3)$$

If this assertion is invalid, we could find a time instant $t_1 > t_0$ satisfying:

$$\begin{cases} m(t_1, \varphi(t_1)) \geq \epsilon, \\ m(t, \varphi(t)) < \epsilon, \end{cases} \quad \text{for all } t \in [t_0, t_1]. \quad (3.4)$$

However, from Lemma 2.8, we have that

$$\aleph(t, \varphi(t)) \leq \omega(t), \quad (3.5)$$

for $t \in [t_0, t_1]$, where $\omega(t) = \omega(t; t_0, s_0)$ is the maximal solution of (1.2). Combining assumption 1 (ii), (3.5), (3.4), and (3.1), at time t_1 , we get

$$\phi(\epsilon) \leq \aleph_0(t_1, \varphi(t_1)) \leq \omega_0(t_1) < \phi(\epsilon), \text{ where, } \omega_0(t_1) = \sum_{j=1}^n \omega_j(t_1),$$

which is a contradiction so the claim (3.3) is true, and therefore, (1.1) is (m_0, m) -stable. $\square$

Theorem 3.2. Let the following assumptions hold.

- (1) For the function $\aleph(t, \varphi(t))$ is C_{rd} in $[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}_+^N]$:
 - (i) $\aleph$ satisfies local Lipschitz condition in φ and vanishes identically at zero;
 - (ii) there exists $\phi \in \mathcal{K}$ such that $\phi(m(t, \varphi)) \leq \aleph_0(t, \varphi)$, with $\aleph_0(t, \varphi) = \sum_{j=1}^N \aleph_j(t, \varphi(t))$.
- (2) Concerning the measures $m_0, m \in \Lambda$:
 - (a) m_0 is uniformly finer than m ;
 - (b) $\aleph(t, \varphi)$ exhibits m_0 -decreasing behavior.
- (3) The mapping $\mathcal{G} \in C_{rd}[\mathbb{T} \times \mathbb{R}_+^n, \mathbb{R}_+^n]$ is:
 - (a) quasimonotone nondecreasing in s for all $t \in \mathbb{T}$;
 - (b) identically zero at the origin;
 - (c) satisfies the differential inequality: ${}^C\Delta_+^\tau \aleph(t, \varphi(t)) \leq \mathbf{0}$, where $\mathbf{0} = (0, 0, \dots, 0)^T$.
- (4) The trivial solution of (1.2) is asymptotically stable.

Then, the system (1.1) achieves (m_0, m) -asymptotic stability.

Proof. By the assumption of the asymptotic stability of the zero solution of system (1.2), we deduce that for $\phi(\epsilon) > 0$, we can find positive numbers $\lambda_1 = \lambda_1(t_0)$ and $T = T(t_0, \epsilon) > 0$ such that

$$\sum_{j=1}^n s_{0j} < \lambda_1 \implies \sum_{j=1}^n s_j(t) < \phi(\epsilon) \text{ for } t \geq t_0 + T, \quad (3.6)$$

where $s(t) = s(t; t_0, s_0)$ is any solution of (1.2). To show that (1.1) is (m_0, m) -asymptotically stable, it suffices to show that it is (m_0, m) -stable and (m_0, m) -attractive, that is given $\epsilon > 0$, and for some positive functions $\delta_0^* = \delta_0^* \in (0, \gamma]$ and $T = T(\epsilon)$, we can find a sequence $\{t_i\}$, $t_i \geq t_0 + T$, $t_i \rightarrow \infty$ as $i \rightarrow \infty$ such that

$$m(t_i, \varphi(t_i)) < \epsilon, \quad (3.7)$$

whenever $m_0(t_0, \varphi_0) < \delta_0$, where $\varphi(t) = \varphi(t; t_0, \varphi_0)$ is any solution of (1.1). If this is false, then the sequence of time $\{t_i\}$, would be such that

$$m(t_i, \varphi_i(t_i)) \geq \epsilon, \quad (3.8)$$

whenever $m_0(t_0, \varphi_0) < \delta_0$. However, by the m -positive definite property of $\aleph(t, \varphi)$, (2.4) of Lemma 2.8, and (3.6) we obtain

$$\phi(m(t, \varphi(t))) \leq \aleph_0(t, \varphi) \leq \omega_0(t) < \phi(\epsilon), \quad (3.9)$$

where $\omega_0(t) = \sum_{i=1}^n \omega_i(t)$ is the maximal solution of (1.2), which is a solution of (1.2) and so will satisfy (3.6). Now, from (3.8), (3.9) becomes $\phi(\epsilon) \leq \aleph_0(t, \varphi) \leq \omega_0(t) < \phi(\epsilon)$, which is a contradiction and concertizing that (3.7) is indeed correct and so system (1.1) is (m_0, m) -asymptotically stable. $\square$

4. Illustrations

4.1. Illustration 1

Consider the Caputo fractional dynamic system

$${}^C \Delta^\alpha \varphi_1(t) = -6\varphi_1 - \varphi_2 \cos 2\varphi_1 - 2\varphi_2 \sin^2 \varphi_1, \quad {}^C \Delta^\alpha \varphi_2(t) = -\varphi_2 \cos 2\varphi_1 - 2\varphi_2 \cos^2 \varphi_1, \quad (4.1)$$

for $t \geq t_0$, with initial conditions $\varphi_1(t_0) = \varphi_{10}$ and $\varphi_2(t_0) = \varphi_{20}$, where $\varphi = (\varphi_1, \varphi_2)$ and $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$. We can also choose a vector Lyapunov candidate function $\aleph = (\aleph_1, \aleph_2)^T$, where $\aleph_1 = |\varphi_1|$ and $\aleph_2 = |\varphi_2|$, for $t \in \mathbb{T}$ and $(\varphi_1, \varphi_2) \in \mathbb{R}^2$. So that

$$\aleph_0(\varphi_1, \varphi_2) = \sum_{i=1}^2 \aleph_i(\varphi_1, \varphi_2) = |\varphi_1| + |\varphi_2|.$$

The Caputo Fr Δ DiD of $\aleph_1 = |\varphi_1|$ is obtained as follows:

$$\begin{aligned} {}^C \Delta_+^\mathfrak{J} \aleph_1(t, \varphi_1) &= \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^\mathfrak{J}} \left\{ \aleph_1(p(t), \varphi_1(p(t))) \right. \\ &\quad \left. + \sum_{p=1}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^p \binom{\mathfrak{J}}{p} [\aleph_1(p(t) - p\omega, \varphi_1(p(t)) - \omega^\mathfrak{J} \mathcal{R}_1(t, \varphi_1(t)))] \right\} - \frac{\aleph_1(t_0, \varphi_{10})(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} \\ &\leq \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^\mathfrak{J}} \left\{ |\varphi_1(p(t))| + \sum_{p=1}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^p \binom{\mathfrak{J}}{p} [|\varphi_1(p(t))| + |\omega^\mathfrak{J} \mathcal{R}_1(t, \varphi_1(t))|] \right\} - \frac{|\varphi_{10}|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} \end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ |\wp_1(\mathfrak{p}(t))| + \sum_{\mathfrak{p}=1}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^{\mathfrak{p}} \binom{\mathfrak{J}}{\mathfrak{p}} |\wp_1(\mathfrak{p}(t))| \right. \\
&\quad \left. + \sum_{\mathfrak{p}=1}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^{\mathfrak{p}} (\mathfrak{J} C_{\mathfrak{p}}) |\omega^{\mathfrak{J}} \mathcal{R}_1(t, \wp_1)| \right\} - \frac{|\wp_{10}|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} \\
&\leq |\wp_1(\mathfrak{p}(t))| \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \sum_{\mathfrak{p}=0}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^{\mathfrak{p}} \binom{\mathfrak{J}}{\mathfrak{p}} + |\mathcal{R}_1(t, \wp_1)| \limsup_{\omega \rightarrow 0^+} \sum_{\mathfrak{p}=1}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^{\mathfrak{p}} \binom{\mathfrak{J}}{\mathfrak{p}} - \frac{|\wp_{10}|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})}.
\end{aligned}$$

Using (2.2) and (2.3), we get

$$\begin{aligned}
{}^C \Delta_+^{\mathfrak{J}} \mathfrak{K}_1(t, \wp_1) &= \frac{|\wp_1(\mathfrak{p}(t))|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} - |\mathcal{R}_1(t; \wp_1)| - \frac{|\wp_{10}|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})}, \\
{}^C \Delta_+^{\mathfrak{J}} \mathfrak{K}_1 &\leq \frac{|\wp_1(\mathfrak{p}(t))|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} - |\mathcal{R}_1(t; \wp_1)|.
\end{aligned}$$

when $t \rightarrow \infty$, $\frac{|\wp_1(\mathfrak{p}(t))|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} \rightarrow 0$, so that

$$\begin{aligned}
{}^C \Delta_+^{\mathfrak{J}} \mathfrak{K}_1 &\leq -|\mathcal{R}_1(t; \wp_1)| = -|-6\wp_1 - \wp_2 \cos 2\wp_1 - 2\wp_2 \sin^2 \wp_1| \\
&\leq -[6\wp_1] + \wp_2 [1 - 2\sin^2 \wp_1 + 2\sin^2 \wp_1] \leq -6|\wp_1| - |\wp_2|, \\
{}^C \Delta_+^{\mathfrak{J}} \mathfrak{K}_1 &\leq -6\mathfrak{K}_1 + \mathfrak{K}_2.
\end{aligned} \tag{4.2}$$

We also compute the Caputo Fr Δ DiD of $\mathfrak{K}_2 = |\wp|$ as follows:

$$\begin{aligned}
{}^C \Delta_+^{\mathfrak{J}} \mathfrak{K}_2(t, \wp_2) &= \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ \mathfrak{K}_2(\mathfrak{p}(t), \wp_2(\mathfrak{p}(t))) \right. \\
&\quad \left. + \sum_{\mathfrak{p}=1}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^{\mathfrak{p}} \binom{\mathfrak{J}}{\mathfrak{p}} [\mathfrak{K}_2(\mathfrak{p}(t) - \mathfrak{p}\omega, \wp_2(\mathfrak{p}(t)) - \omega^{\mathfrak{J}} \mathcal{R}_2(t, \wp_2(t)))] \right\} - \frac{\mathfrak{K}_2(t_0, \wp_{2,0})(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} \\
&= \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ |\wp_2(\mathfrak{p}(t))| + \sum_{\mathfrak{p}=1}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^{\mathfrak{p}} \binom{\mathfrak{J}}{\mathfrak{p}} [|\wp_2(\mathfrak{p}(t)) - \omega^{\mathfrak{J}} \mathcal{R}_2(t, \wp_2)|] \right\} - \frac{|\wp_{2,0}|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} \\
&\leq \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ |\wp_2(\mathfrak{p}(t))| + \sum_{\mathfrak{p}=1}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^{\mathfrak{p}} \binom{\mathfrak{J}}{\mathfrak{p}} [|\wp_2(\mathfrak{p}(t))| + |\omega^{\mathfrak{J}} \mathcal{R}_2(t, \wp_2)|] \right\} - \frac{|\wp_{2,0}|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} \\
&\leq \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ |\wp_2(\mathfrak{p}(t))| + \sum_{\mathfrak{p}=1}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^{\mathfrak{p}} \binom{\mathfrak{J}}{\mathfrak{p}} |\wp_2(\mathfrak{p}(t))| \right. \\
&\quad \left. + \sum_{\mathfrak{p}=1}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^{\mathfrak{p}} \binom{\mathfrak{J}}{\mathfrak{p}} |\omega^{\mathfrak{J}} \mathcal{R}_2(t, \wp_2)| \right\} - \frac{|\wp_{2,0}|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} \\
&\leq |\wp_2(\mathfrak{p}(t))| \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \sum_{\mathfrak{p}=0}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^{\mathfrak{p}} \binom{\mathfrak{J}}{\mathfrak{p}} + |\mathcal{R}_2(t, \wp_2)| \limsup_{\omega \rightarrow 0^+} \sum_{\mathfrak{p}=1}^{\lceil \frac{t-t_0}{\omega} \rceil} (-1)^{\mathfrak{p}} \binom{\mathfrak{J}}{\mathfrak{p}} - \frac{|\wp_{2,0}|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})}.
\end{aligned}$$

Using (2.2) and (2.3) we obtain

$${}^C \Delta_+^{\mathfrak{J}} \mathfrak{K}_2 = \frac{|\wp_2(\mathfrak{p}(t))|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} - |\mathcal{R}_2(t; \wp_2)| - \frac{|\wp_{2,0}|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})},$$

$${}^C\Delta_+^{\mathfrak{J}} \mathfrak{K}_2 \leq \frac{|\wp_2(\mathfrak{p}(t))|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} - |\mathcal{R}_2(t; \wp_2)|,$$

when $t \rightarrow \infty$, $\frac{|\wp_2(\mathfrak{p}(t))|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} \rightarrow 0$, obtaining

$${}^C\Delta_+^{\mathfrak{J}} \mathfrak{K}_2 \leq -|\mathcal{R}_2(t; \wp_2)| = -[|-\wp_2 \cos 2\wp_1 - 2\wp_2 \cos^2 \wp_1|] \leq -[|-\wp_2(2 \cos^2 \wp_1 - 1 - 2 \cos^2 \wp_1)|] \leq -|\wp_2|.$$

Therefore,

$${}^C\Delta_+^{\mathfrak{J}} \mathfrak{K}_2 \leq 0\mathfrak{K}_1 - \mathfrak{K}_2. \quad (4.3)$$

From (4.2) and (4.3), we obtain

$${}^C\Delta_+^{\mathfrak{J}} \mathfrak{K} \leq \begin{pmatrix} -6 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathfrak{K}_1 \\ \mathfrak{K}_2 \end{pmatrix} = \mathcal{G}(t, \mathfrak{K}).$$

Choosing comparison system, such that

$${}^C\Delta_+^{\mathfrak{J}} \mathfrak{s} = \mathcal{G}(t, \mathfrak{s}) = \mathcal{A}\mathfrak{s}, \quad (4.4)$$

where $\mathcal{A} = \begin{pmatrix} -6 & 1 \\ 0 & -1 \end{pmatrix}$, since the eigen values of $\mathcal{A}$ are -6 and -2 , then the comparison system (4.4) is stable and by Theorem 3.1, we can immediately infer the (m_0, m) -stability of (4.1).

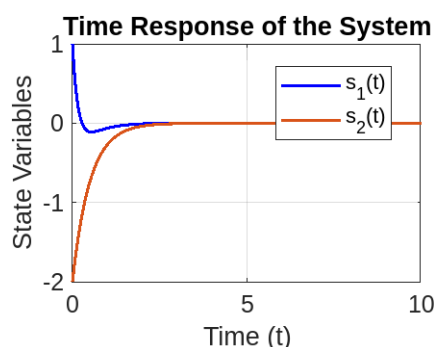


Figure 1: Time response of the comparison system: decay of state variables $s_1(t)$ and $s_2(t)$ demonstrating (m_0, m) -stability.

Figure 1 depicts the time response of the comparison system (4.4) defined by the matrix $\mathcal{A}$, with state variables $s_1(t)$ and $s_2(t)$. Both curves decay to zero over time, confirming the (m_0, m) -stability of the system. This behavior arises because the eigenvalues of $\mathcal{A}$ are -6 and -2 , both of which have negative real parts. The plot visually demonstrates that the system asymptotically returns to equilibrium after any initial disturbance, highlighting its stability properties in a clean and intuitive manner.

4.2. Illustration 2

Consider the Caputo fractional dynamic system

$${}^C\Delta^{\mathfrak{J}} \mathfrak{J}_1(t) = 6\mathfrak{J}_1 - \frac{2\mathfrak{J}_2^2}{\mathfrak{J}_1}, \quad {}^C\Delta^{\mathfrak{J}} \mathfrak{J}_2(t) = -\frac{2\mathfrak{J}_1^2}{\mathfrak{J}_2} + 4\mathfrak{J}_2 - \frac{2\mathfrak{J}_3^2}{\mathfrak{J}_2}, \quad {}^C\Delta^{\mathfrak{J}} \mathfrak{J}_3(t) = -\frac{2\mathfrak{J}_2^2}{\mathfrak{J}_3} + 2\mathfrak{J}_3, \quad (4.5)$$

for $t \geq t_0$, with initial conditions $\mathfrak{J}_1(t_0) = \mathfrak{J}_{10}$, $\mathfrak{J}_2(t_0) = \mathfrak{J}_{20}$, and $\mathfrak{J}_3(t_0) = \mathfrak{J}_{30}$, where $\mathfrak{J} = (\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3)$, and the right hand side is given as $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$. We also consider a vector Lyapunov candidate function $\mathfrak{K} = (\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3)^T$, where $\mathfrak{K}_1 = \mathfrak{J}_1^2$, $\mathfrak{K}_2 = \mathfrak{J}_2^2$, and $\mathfrak{K}_3 = \mathfrak{J}_3^2$, for $(\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3) \in \mathbb{R}^3$. Then we compute the Caputo FrΔDiD for $\mathfrak{K}_1 = \mathfrak{J}_1^2$ as follows:

$${}^C\Delta_+^{\mathfrak{J}} \mathfrak{K}_1 = \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ [(\mathfrak{J}_1(\mathfrak{p}(t)))^2] - [(\mathfrak{J}_{10})^2] + \sum_{p=1}^{\left\lceil \frac{t-t_0}{\omega} \right\rceil} (-1)^p \binom{\mathfrak{J}}{p} [(\mathfrak{J}_1(\mathfrak{p}(t)) - \omega^{\mathfrak{J}} \mathcal{R}_1(t, \mathfrak{J}))^2] - [(\mathfrak{J}_{10})^2] \right\}$$

$$\begin{aligned}
&= \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ [(\mathfrak{J}_1(\mathfrak{p}(t)))^2] - [(\mathfrak{J}_{10})^2] + \sum_{p=1}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} [(\mathfrak{J}_1(\mathfrak{p}(t)))^2 - 2\mathfrak{J}_1(\mathfrak{p}(t))\omega^{\mathfrak{J}}\mathcal{R}_1(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3)] \right. \\
&\quad \left. + \omega^{2\mathfrak{J}}(\mathcal{R}_1(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3))^2] - [(\mathfrak{J}_{10})^2] \right\} \\
&= -\limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ \sum_{p=0}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} [(\mathfrak{J}_{10})^2] \right\} + \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ \sum_{p=0}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} [(\mathfrak{J}_1(\mathfrak{p}(t)))^2] \right\} \\
&\quad - \limsup_{\omega \rightarrow 0^+} \left\{ \sum_{p=1}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} [2\mathfrak{J}_1(\mathfrak{p}(t))\omega^{\mathfrak{J}}\mathcal{R}_1(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3)] \right\}.
\end{aligned}$$

Using (2.2) and (2.3), we obtain

$$\leq \frac{(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} [(\mathfrak{J}_1(\mathfrak{p}(t)))^2] - [2\mathfrak{J}_1(\mathfrak{p}(t))\mathcal{R}_1(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3)].$$

When $t \rightarrow \infty$, $\frac{(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} [(\mathfrak{J}_1(\mathfrak{p}(t)))^2] \rightarrow 0$, so that

$${}^C\Delta_+^{\mathfrak{J}} \mathfrak{K}_1 \leq -2[\mathfrak{J}_1(\mathfrak{p}(t))\mathcal{R}_1(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3)].$$

Using the fact that $\mathfrak{J}(\mathfrak{p}(t)) \leq \omega {}^C\Delta^{\mathfrak{J}}\mathfrak{J}(t) + \mathfrak{J}(t)$,

$$\begin{aligned}
{}^C\Delta_+^{\mathfrak{J}} \mathfrak{K}_1 &= -2 \left[\omega(t)\mathcal{R}_1^2(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3) + \mathfrak{J}_1(t)\mathcal{R}_1(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3) \right] \\
&= -2 \left[\omega(t) \left(6\mathfrak{J}_1 - \frac{2\mathfrak{J}_2^2}{\mathfrak{J}_1} \right)^2 + \mathfrak{J}_1 \left(5\mathfrak{J}_1 - \frac{\mathfrak{J}_2^2}{\mathfrak{J}_1} \right) \right] = -2\omega(t) \left[\left(6\mathfrak{J}_1 - \frac{2\mathfrak{J}_2^2}{\mathfrak{J}_1} \right)^2 - 2\mathfrak{J}_1 \left[6\mathfrak{J}_1 - \frac{2\mathfrak{J}_2^2}{\mathfrak{J}_1} \right] \right]. \quad (4.6)
\end{aligned}$$

Setting $\mathbb{T} = \mathbb{R}$, then $\omega = 0$, so that (4.6) becomes:

$${}^C\Delta_+^{\mathfrak{J}} \mathfrak{K}_1 = -2\mathfrak{J}_1 \left[6\mathfrak{J}_1 - \frac{2\mathfrak{J}_2^2}{\mathfrak{J}_1} \right] = -12\mathfrak{J}_1^2 + 4\mathfrak{J}_2^2 + 0\mathfrak{J}_3^2 = (-12 \ 4 \ 0) \cdot (\mathfrak{K}_1 \ \mathfrak{K}_2 \ \mathfrak{K}_3)^T. \quad (4.7)$$

Setting $\mathbb{T} = \mathbb{N}_0$, then $\omega = 1$, so that (4.6) becomes:

$${}^C\Delta_+^{\mathfrak{J}} \mathfrak{K}_1 = -2 \left[\left(5\mathfrak{J}_1 - \frac{\mathfrak{J}_2^2}{\mathfrak{J}_1} \right)^2 \right] - 2\mathfrak{J}_1 \left[5\mathfrak{J}_1 - \frac{\mathfrak{J}_2^2}{\mathfrak{J}_1} \right] \leq -2\mathfrak{J}_1 \left[5\mathfrak{J}_1 - \frac{\mathfrak{J}_2^2}{\mathfrak{J}_1} \right].$$

This likewise results in the same conclusion as (4.11). Evidently, this approach remains valid for any other discrete time instance. Also, we compute the Caputo FrADiD for $\mathfrak{K}_2(\mathfrak{J}) = \mathfrak{J}_2^2$ as follows:

$$\begin{aligned}
{}^C\Delta_+^{\mathfrak{J}} \mathfrak{K}_2(\mathfrak{J}) &= \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ [(\mathfrak{J}_2(\mathfrak{p}(t)))^2] - [(\mathfrak{J}_{20})^2] + \sum_{p=1}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} [(\mathfrak{J}_2(\mathfrak{p}(t)) - \omega^{\mathfrak{J}}\mathcal{R}_2(t, \mathfrak{J}))^2] - [(\mathfrak{J}_{20})^2] \right\} \\
&= \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ [(\mathfrak{J}_2(\mathfrak{p}(t)))^2] - [(\mathfrak{J}_{20})^2] + \sum_{p=1}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} [(\mathfrak{J}_2(\mathfrak{p}(t)))^2 \right. \\
&\quad \left. - 2\mathfrak{J}_2(\mathfrak{p}(t))\omega^{\mathfrak{J}}\mathcal{R}_2(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3) + \omega^{2\mathfrak{J}}(\mathcal{R}_2(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3))^2] - [(\mathfrak{J}_{20})^2] \right\}
\end{aligned}$$

$$\begin{aligned}
&= -\limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ \sum_{p=0}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} [(\mathfrak{J}_{20})^2] \right\} + \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ \sum_{p=0}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} [(\mathfrak{J}_2(p(t)))^2] \right\} \\
&\quad - \limsup_{\omega \rightarrow 0^+} \left\{ \sum_{p=1}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^p \binom{\mathfrak{J}}{p} [2\mathfrak{J}_2(p(t))\omega^{\mathfrak{J}}\mathcal{R}_2(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3)] \right\}.
\end{aligned}$$

From (2.2) and (2.3) we obtain

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{K}_2 \leq \frac{(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} [(\mathfrak{J}_2(p(t)))^2] - [2\mathfrak{J}_2(p(t))\mathcal{R}_2(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3)].$$

When $t \rightarrow \infty$, $\frac{(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} [(\mathfrak{J}_2(p(t)))^2] \rightarrow 0$, and

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{K}_2 \leq -2[\mathfrak{J}_2(p(t))\mathcal{R}_2(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3)].$$

Using $\mathfrak{J}(p(t)) \leq \omega^C \Delta^{\mathfrak{J}}\mathfrak{J}(t) + \mathfrak{J}(t)$,

$$\begin{aligned}
{}^C\Delta_+^{\mathfrak{J}}\mathfrak{K}_2 &= -2 \left[\omega(t)\mathcal{R}_2^2(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3) + \mathfrak{J}_2(t)\mathcal{R}_2(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3) \right] \\
&= -2 \left[\omega(t) \left(-\frac{2\mathfrak{J}_1^2+}{\mathfrak{J}_2} + 4\mathfrak{J}_2 - \frac{2\mathfrak{J}_3^2}{\mathfrak{J}_2} \right)^2 + \mathfrak{J}_2 \left(-\frac{2\mathfrak{J}_1^2+}{\mathfrak{J}_2} + 4\mathfrak{J}_2 - \frac{2\mathfrak{J}_3^2}{\mathfrak{J}_2} \right) \right] \\
&= -2\omega(t) \left[\left(-\frac{2\mathfrak{J}_1^2+}{\mathfrak{J}_2} + 4\mathfrak{J}_2 - \frac{2\mathfrak{J}_3^2}{\mathfrak{J}_2} \right)^2 \right] - 2\mathfrak{J}_2 \left[-\frac{2\mathfrak{J}_1^2+}{\mathfrak{J}_2} + 4\mathfrak{J}_2 - \frac{2\mathfrak{J}_3^2}{\mathfrak{J}_2} \right].
\end{aligned} \tag{4.8}$$

For $\mathbb{T} = \mathbb{R}$, then $\omega = 0$, then from (4.8) we obtain

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{K}_2 = -2\mathfrak{J}_2 \left[-\frac{2\mathfrak{J}_1^2+}{\mathfrak{J}_2} + 4\mathfrak{J}_2 - \frac{2\mathfrak{J}_3^2}{\mathfrak{J}_2} \right] = 4\mathfrak{J}_1^2 - 8\mathfrak{J}_2^2 + 4\mathfrak{J}_3^2 = (4 \quad -8 \quad 4) \cdot (\mathfrak{K}_1 \quad \mathfrak{K}_2 \quad \mathfrak{K}_3)^T. \tag{4.9}$$

For $\mathbb{T} = \mathbb{N}_0$, then $\omega = 1$, then from (4.6) we obtain:

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{K}_2 = -2 \left[\left(-\frac{2\mathfrak{J}_1^2+}{\mathfrak{J}_2} + 4\mathfrak{J}_2 - \frac{2\mathfrak{J}_3^2}{\mathfrak{J}_2} \right)^2 \right] - 2\mathfrak{J}_2 \left[-\frac{2\mathfrak{J}_1^2+}{\mathfrak{J}_2} + 4\mathfrak{J}_2 - \frac{2\mathfrak{J}_3^2}{\mathfrak{J}_2} \right] \leq -2\mathfrak{J}_2 \left[-\frac{2\mathfrak{J}_1^2+}{\mathfrak{J}_2} + 4\mathfrak{J}_2 - \frac{2\mathfrak{J}_3^2}{\mathfrak{J}_2} \right].$$

This likewise results in the same conclusion as (4.11). Evidently, this approach remains valid for any other discrete time instance. Finally, we determine the Caputo Fr Δ DiD for $\mathfrak{K}_3 = \mathfrak{J}_3^2$ using the following computation:

$$\begin{aligned}
{}^C\Delta_+^{\mathfrak{J}}\mathfrak{K}_3 &= \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ [(\mathfrak{J}_3(p(t)))^2] - [(\mathfrak{J}_{30})^2] + \sum_{r=1}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^r \binom{\mathfrak{J}}{r} [(\mathfrak{J}_3(p(t)) - \omega^{\mathfrak{J}}\mathcal{R}_3(t, \mathfrak{J}))^2] - [(\mathfrak{J}_{30})^2] \right\} \\
&= \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ [(\mathfrak{J}_3(p(t)))^2] - [(\mathfrak{J}_{30})^2] + \sum_{r=1}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^r \binom{\mathfrak{J}}{r} [(\mathfrak{J}_3(p(t)))^2 - 2\mathfrak{J}_3(p(t))\omega^{\mathfrak{J}}\mathcal{R}_3(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3) \right. \\
&\quad \left. + \omega^{2\mathfrak{J}}(\mathcal{R}_3(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3))^2] - [(\mathfrak{J}_{30})^2] \right\} \\
&= -\limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ \sum_{r=0}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^r \binom{\mathfrak{J}}{r} [(\mathfrak{J}_{30})^2] \right\} + \limsup_{\omega \rightarrow 0^+} \frac{1}{\omega^{\mathfrak{J}}} \left\{ \sum_{r=0}^{\left[\frac{t-t_0}{\omega}\right]} (-1)^r \binom{\mathfrak{J}}{r} [(\mathfrak{J}_3(p(t)))^2] \right\}
\end{aligned}$$

$$-\limsup_{\omega \rightarrow 0^+} \left\{ \sum_{r=1}^{\left\lceil \frac{t-t_0}{\omega} \right\rceil} (-1)^r \binom{\mathfrak{J}}{r} [2\mathfrak{J}_3(\mathfrak{p}(t))\omega^{\mathfrak{J}}\mathcal{R}_3(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3)] \right\}.$$

From (2.2) and (2.3), we obtain

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{K}_3 \leq \frac{(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} [(\mathfrak{J}_3(\mathfrak{p}(t)))^2] - [2\mathfrak{J}_1(\mathfrak{p}(t))\mathcal{R}_3(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3)].$$

When $t \rightarrow \infty$, $\frac{(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} [(\mathfrak{J}_3(\mathfrak{p}(t)))^2] \rightarrow 0$, then

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{K}_3 \leq -2[\mathfrak{J}_3(\mathfrak{p}(t))\mathcal{R}_3(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3)].$$

Using $\mathfrak{J}(\mathfrak{p}(t)) \leq \omega {}^C\Delta^{\mathfrak{J}}\mathfrak{J}(t) + \mathfrak{J}(t)$,

$$\begin{aligned} {}^C\Delta_+^{\mathfrak{J}}\mathfrak{K}_3 &= -2 \left[\omega(t)\mathcal{R}_3^2(t, \mathfrak{J}_3, \mathfrak{J}_2, \mathfrak{J}_3) + \mathfrak{J}_3(t)\mathcal{R}_3(t, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3) \right] \\ &= -2 \left[\omega(t) \left(-\frac{\mathfrak{J}_1^2 + \mathfrak{J}_2^2}{\mathfrak{J}_3} + 3\mathfrak{J}_3 \right)^2 + \mathfrak{J}_3 \left(-\frac{\mathfrak{J}_1^2 + \mathfrak{J}_2^2}{\mathfrak{J}_3} + 3\mathfrak{J}_3 \right) \right] \\ &= -2\omega(t) \left[\left(-\frac{\mathfrak{J}_1^2 + \mathfrak{J}_2^2}{\mathfrak{J}_3} + 3\mathfrak{J}_3 \right)^2 \right] - 2\mathfrak{J}_3 \left[-\frac{\mathfrak{J}_1^2 + \mathfrak{J}_2^2}{\mathfrak{J}_3} + 3\mathfrak{J}_3 \right]. \end{aligned} \quad (4.10)$$

Set $\mathbb{T} = \mathbb{R}$, then $\omega = 0$, and (4.10) becomes

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{K}_3 = -2\mathfrak{J}_3 \left[-\frac{2\mathfrak{J}_2^2}{\mathfrak{J}_3} + 2\mathfrak{J}_3 \right] = 4\mathfrak{J}_2^2 - 4\mathfrak{J}_3^2 = (0 \ 4 \ -4) \cdot (\mathfrak{K}_1 \ \mathfrak{K}_2 \ \mathfrak{K}_3)^T. \quad (4.11)$$

Set $\mathbb{T} = \mathbb{N}_0$, then $\omega = 1$, and (4.6) becomes:

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{K}_3 = -2 \left[\left(-\frac{2\mathfrak{J}_2^2}{\mathfrak{J}_3} + 2\mathfrak{J}_3 \right)^2 \right] - 2\mathfrak{J}_3 \left[-\frac{2\mathfrak{J}_2^2}{\mathfrak{J}_3} + 2\mathfrak{J}_3 \right] \leq -2\mathfrak{J}_3 \left[-\frac{2\mathfrak{J}_2^2}{\mathfrak{J}_3} + 2\mathfrak{J}_3 \right].$$

This likewise results in the same conclusion as (4.11). Evidently, this approach remains valid for any other discrete time instance. From (4.7), (4.9), and (4.11), we obtain

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{K} \leq \begin{pmatrix} -10 & 2 & 0 \\ 4 & -6 & 4 \\ 0 & 4 & -4 \end{pmatrix} \begin{pmatrix} \mathfrak{K}_1 \\ \mathfrak{K}_2 \\ \mathfrak{K}_3 \end{pmatrix} = \mathcal{G}(t, \mathfrak{K}). \quad (4.12)$$

Set the comparison system for (4.12) to be

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{s} = \mathcal{G}(t, \mathfrak{s}) = \mathfrak{A}\mathfrak{s}, \quad (4.13)$$

where $\mathfrak{A} = \begin{pmatrix} -10 & 2 & 0 \\ 4 & -6 & 4 \\ 0 & 4 & -4 \end{pmatrix}$. Since $\mathfrak{A}$ has negative real eigen-values; $\zeta_1 = -4(2 + \sqrt{3})$, $\zeta_2 = -8$, $\zeta_3 = 4(\sqrt{3} - 2)$, then the comparison system (4.4) is asymptotically stable, inferring the (m_0, m) -asymptotic stability of the main system (4.5).

The Figure 2 above shows the time response of system (4.13) defined by the matrix $\mathfrak{A}$, with state variables $s_1(t)$, $s_2(t)$, and $s_3(t)$ plotted over time. All three curves decay to zero as time progresses, demonstrating the asymptotic stability of the system. This behavior confirms that the system, governed by eigenvalues with negative real parts, returns to equilibrium after any initial disturbance.

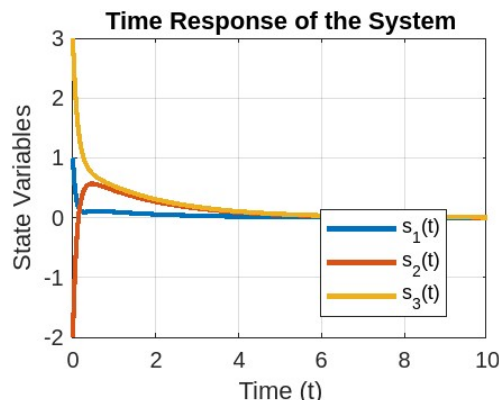


Figure 2: Time response of the system: decay of state variables $s_1(t)$, $s_2(t)$, and $s_3(t)$ demonstrating asymptotic stability.

5. Applications in Bio-medicines

Application 5.1. Consider a Caputo fractional-order Delta model of order $\mathfrak{J} \in (0, 1)$ describing the interaction between cytotoxic T-cells ($\mathcal{T}(t)$) and a viral pathogen ($\mathcal{P}(t)$) during infection clearance

$$\begin{aligned} {}^C\Delta^{\mathfrak{J}}\mathcal{T}(t) &= \underbrace{a\mathcal{P}}_{\text{T-cell activation}} - \underbrace{b\mathcal{T}}_{\text{T-cell death}}, \quad \mathcal{T}(t_0) = \mathcal{T}_0, \\ {}^C\Delta^{\mathfrak{J}}\mathcal{P}(t) &= \underbrace{-c\mathcal{P}}_{\text{Pathogen decay}} - \underbrace{d\mathcal{T}\mathcal{P}}_{\text{Pathogen elimination}}, \quad \mathcal{P}(t_0) = \mathcal{P}_0, \end{aligned} \quad (5.1)$$

where $a, b, c, d > 0$. Here, $\mathcal{T}_0 \geq 0$ and $\mathcal{P}_0 \geq 0$ are initial T-cell and pathogen concentrations. Choose a vector Lyapunov function to be $\mathfrak{N}(t, \mathcal{T}, \mathcal{P}) = (\mathfrak{N}_1, \mathfrak{N}_2)^T$, where $\mathfrak{N}_1(t, \mathcal{T}) = |\mathcal{T}|$, $\mathfrak{N}_2(t, \mathcal{P}) = |\mathcal{P}|$. Following through with the computation as in (4.1) and applying the definition of the Caputo Fr Δ DiD as given in Definition 2.3 we immediately obtain following. For $\mathfrak{N}_1 = |\mathcal{T}|$:

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{N}_1 \leq \frac{|\mathcal{T}(p(t))|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} - |a\mathcal{P} - b\mathcal{T}| - \frac{|\mathcal{T}_0|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})}.$$

As $t \rightarrow \infty$, the fractional terms vanish, leaving:

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{N}_1 \leq -b|\mathcal{T}| + a|\mathcal{P}| = -b\mathfrak{N}_1 + a\mathfrak{N}_2. \quad (5.2)$$

Also, for $\mathfrak{N}_2 = |\mathcal{P}|$:

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{N}_2 \leq \frac{|\mathcal{P}(p(t))|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})} - |-c\mathcal{P} - d\mathcal{T}\mathcal{P}| - \frac{|\mathcal{P}_0|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})}.$$

Since $-d|\mathcal{T}||\mathcal{P}| \leq 0$, asymptotically:

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{N}_2 \leq -c|\mathcal{P}| = -c\mathfrak{N}_2. \quad (5.3)$$

Combining (5.2) and (5.3) we obtain:

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{N} \leq \begin{pmatrix} -b & a \\ 0 & -c \end{pmatrix} \begin{pmatrix} \mathfrak{N}_1 \\ \mathfrak{N}_2 \end{pmatrix} = \mathcal{G}\mathfrak{N}. \quad (5.4)$$

Set the comparison system for (5.4) to be

$${}^C\Delta_+^{\mathfrak{J}}\mathfrak{N} = \mathcal{G}(t, \mathfrak{N}) = \mathcal{A}\mathfrak{N}, \quad (5.5)$$

where $\mathcal{A} = \begin{pmatrix} -b & a \\ 0 & -c \end{pmatrix}$. Clearly the eigenvalues λ of $\mathcal{A}$ are $\lambda_1 = -b$, $\lambda_2 = -c$. Both eigenvalues are strictly negative for $b, c > 0$, confirming asymptotic stability of the comparison system (5.5) and by Theorems 3.1

and 3.2, we infer the (m_0, m) -stability and (m_0, m) -asymptotic stability of the zero solution $(\mathcal{T}, \mathcal{P}) = (0, 0)$ of the immune system (5.1). This means that despite initial pathogen load $\mathcal{P}_0$, the system dynamics drive both T-cell activity and pathogen concentration to zero, reflecting successful immune clearance. This result aligns with biological expectations where resolved infections exhibit damped immune responses and pathogen eradication. Also, $\frac{|\mathcal{T}_0|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})}$ and $\frac{|\mathcal{P}_0|(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})}$ decay to zero as $t \rightarrow \infty$, ensuring stability is independent of initial concentrations.

Application 5.2. Consider a 3D Caputo fractional-order Delta model of order $\mathfrak{J} \in (0, 1)$ describing the interaction between corticotropin-releasing hormone (x_1), adrenocorticotrophic hormone (x_2), and cortisol (x_3) in the hypothalamic-pituitary-adrenal (HPA) axis. The Caputo fractional dynamic system is:

$$\begin{aligned} {}^C\Delta_{+}^{\mathfrak{J}}x_1(t) &= a_1 - b_1x_1 - c_1x_3, & x_1(t_0) &= x_{10}, \\ {}^C\Delta_{+}^{\mathfrak{J}}x_2(t) &= a_2x_1 - b_2x_2 - c_2x_3, & x_2(t_0) &= x_{20}, \\ {}^C\Delta_{+}^{\mathfrak{J}}x_3(t) &= a_3x_2 - b_3x_3, & x_3(t_0) &= x_{30}, \end{aligned}$$

with Initial conditions $x_{10}, x_{20}, x_{30} \geq 0$ which represent fasting hormone levels, where $a_i, b_i, c_i > 0$, a_1 are the Basal CRH secretion rates, b_1, b_2, b_3 are the hormone degradation rates, c_1, c_2 are the cortisol-mediated inhibition coefficients, and a_2, a_3 are the stimulatory feedback gains. Set the vector the Lyapunov function to be $\mathfrak{N}(t, x_1, x_2, x_3) = (\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3)^T$, where $\mathfrak{N}_1 = x_1^2$, $\mathfrak{N}_2 = x_2^2$, $\mathfrak{N}_3 = x_3^2$. Applying Definition 2.3 and following through the simplifications done for (4.5), then we obtain: for $\mathfrak{N}_1 = x_1^2$,

$${}^C\Delta_{+}^{\mathfrak{J}}\mathfrak{N}_1 \leq 2x_1(a_1 - b_1x_1 - c_1x_3) - \frac{x_{10}^2(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})}.$$

As $t \rightarrow \infty$, the transient term vanishes. Assuming $a_1 \leq \epsilon$ (small basal rate):

$${}^C\Delta_{+}^{\mathfrak{J}}\mathfrak{N}_1 \leq -2b_1x_1^2 - 2c_1x_1x_3 = -2b_1\mathfrak{N}_1 - 2c_1\sqrt{\mathfrak{N}_1\mathfrak{N}_3}. \quad (5.6)$$

Also, for $\mathfrak{N}_2 = x_2^2$:

$${}^C\Delta_{+}^{\mathfrak{J}}\mathfrak{N}_2 \leq 2x_2(a_2x_1 - b_2x_2 - c_2x_3) - \frac{x_{20}^2(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})}.$$

As $t \rightarrow \infty$:

$${}^C\Delta_{+}^{\mathfrak{J}}\mathfrak{N}_2 \leq 2a_2x_1x_2 - 2b_2x_2^2 - 2c_2x_2x_3 = a_2(\mathfrak{N}_1 + \mathfrak{N}_2) - 2b_2\mathfrak{N}_2 - c_2(\mathfrak{N}_2 + \mathfrak{N}_3). \quad (5.7)$$

Finally, for $\mathfrak{N}_3 = x_3^2$:

$${}^C\Delta_{+}^{\mathfrak{J}}\mathfrak{N}_3 \leq 2x_3(a_3x_2 - b_3x_3) - \frac{x_{30}^2(t-t_0)^{-\mathfrak{J}}}{\Gamma(1-\mathfrak{J})}.$$

As $t \rightarrow \infty$:

$${}^C\Delta_{+}^{\mathfrak{J}}\mathfrak{N}_3 \leq 2a_3x_2x_3 - 2b_3x_3^2 = a_3(\mathfrak{N}_2 + \mathfrak{N}_3) - 2b_3\mathfrak{N}_3. \quad (5.8)$$

So that combining (5.6), (5.7), and (5.8), we obtain

$${}^C\Delta_{+}^{\mathfrak{J}}V \leq \begin{pmatrix} -2b_1 & -c_1 & -c_1 \\ a_2 - c_2 & -2b_2 - c_2 & 0 \\ 0 & a_3 & -2b_3 \end{pmatrix} \begin{pmatrix} \mathfrak{N}_1 \\ \mathfrak{N}_2 \\ \mathfrak{N}_3 \end{pmatrix} = \mathcal{G}s.$$

Setting the comparison system to be

$${}^C\Delta_{+}^{\mathfrak{J}}\mathfrak{N} = \mathcal{G}(t, \mathfrak{N}) = \mathcal{A}s, \quad (5.9)$$

where $\mathcal{A} = \begin{pmatrix} -2b_1 & -c_1 & -c_1 \\ a_2 - c_2 & -2b_2 - c_2 & 0 \\ 0 & a_3 & -2b_3 \end{pmatrix}$. Now, the eigenvalues λ of $\mathcal{A}$ satisfy:

$$\det(\mathcal{A} - \lambda I) = (-\lambda - 2b_1)(-\lambda - 2b_2 - c_2)(-\lambda - 2b_3).$$

For $\alpha_2, \alpha_3 \ll b_i, c_i$ (physiological damping), all eigenvalues have negative real parts, leading to the conclusion of the asymptotic stability of the comparison system (5.9) and by Theorems 3.1 and 3.2, we can conclude the (m_0, m) -stability and (m_0, m) -asymptotic stability of the equilibrium point $(x_1, x_2, x_3) = (0, 0, 0)$ under physiological damping conditions. This reflects the HPA axis's homeostatic regulation, where cortisol feedback suppresses CRH/ACTH activity, stabilizing hormone levels. The result aligns with endocrine system dynamics during stress recovery.

6. Conclusion

In this work, we investigated the stability of FrDET using a comparison principle approach within a two-measure framework. By employing two measures (m_0, m) and vector Lyapunov functions, we established new sufficient conditions for (m_0, m) -stability and (m_0, m) -asymptotic stability, thereby unifying stability concepts across continuous and discrete domains. This framework simplifies the analysis by reducing the stability assessment to verifying the quasimonotonicity of a related comparison system, enhancing both computational efficiency and practical accessibility. Theoretical results were validated through two biologically relevant examples: an immune response model, illustrating stability in a nonlinear T-cell–pathogen interaction system, and a Neuro-Endocrine Regulation Model, demonstrating the scalability of the approach through a three-dimensional HPA axis system. These examples highlight the practical applicability of the proposed method in biomedical contexts and emphasize its ability to bridge theoretical stability analysis with complex, real-world biological systems. Moreover, by incorporating the Caputo fractional delta Dini derivative, the framework naturally accommodates hybrid time scales and discontinuous interventions. Overall, this study advances the understanding of fractional dynamic systems on time scales and offers a unified, robust tool for stability analysis across diverse fields, including pharmacokinetics and neuroendocrinology.

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