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A methodology of hybrid fixed point theorems for solving Fredholm-Volterra integral equations

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Abstract

This paper studies a new notion of Jaggi-type hybrid (θ-ϕ)-contraction and demonstrates its roles in proving fixed point theorems within the context of a generalized metric space. We prove, using comparative examples that under special instances, the ideas presented herein can be reduced to some known results in the existing literature. To show a possible application of our main contractive inequality, iterative methods are developed for addressing the existence of solutions to a class of mixed nonlinear fixed point problems involving Volterra-Fredholm integral equation.

Keywords: Fixed point, metric space, θ-contraction, nonlinear integral equation, Volterra-Fredholm integral.

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1. Introduction

The fixed point theorem (FPT) commonly referred to as the Banach contraction principle [7] is based on the solution of a simple (fixed point) equation: $\tilde{S}(u) = u$, for a self-mapping $\tilde{S}$ on a nonempty set $\mathcal{X}$. Inherently, a FP equation $\tilde{S}(u) - u = 0$ can be reformulated as $H(u) = \tilde{S}(u) - u$, where H is also a self-mapping on $\mathcal{X}$. Over the years, several generalizations of the idea of Banach contraction principle have been made. In some generalizations, the contractiveness of the map is lessened (e.g., see [27, 29] and the reference therein). In other generalizations, the structure is lessened (e.g., see [11, 28] and the reference therein). Not long ago, Tijjani et al. [2] introduced the concept of C^*-algebra-valued perturbed metric spaces as an extension of perturbed metric spaces. On a related development, Ahmad et al. [3] presented the idea of C^*-algebra-valued modular metric spaces and proved some new fixed point theorems in such spaces.

Branciari [9] introduced the idea of generalized metric spaces, where the triangle inequality is substituted by the inequality $\sigma(u,v) \leqslant \sigma(u,a) + \sigma(a,b) + \sigma(b,v)$ for all pairs of distinct points $u, v, a, b \in \mathcal{X}$. Since then, a number of FP conclusions on such spaces have been developed; Kirk and Shahzad [20] said a

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'generalized metric space' is a semimetric space which does not satisfy the triangle inequality, but satisfies a weaker assumption. Samet [24] also talked about exposing incorrect property of the generalized metric space as introduced by Branciari. Aydi et al. [6] studied the concept of generalized (α, ψ)-contractive mappings and proved some fixed point theorems, by taking note of the weakness of the topology of generalized metrics. On similar development, Shatanawi [26] examined new conditions for the existence of unique fixed points for mappings satisfying generalized weak contractions in the setting of a generalized metric space of Branciari-type. For other related discussions, see [5, 23] and some references therein.

In the last decades, a number of FP results have been published to extend the Banach FP theorem. Among them are following. Jaggi [15] proved a FP theorem in which the rational expression was considered for the first time. Karapinar [18] introduced the notion of Kannan-type interpolative contraction that maximizes the rate of convergence. Yelsilkaya [30] presented an example on interpolative Hardy-Rogers contractive of the Suzuki-type mapping. Not long ago, motivated by the outcome in [18], Mitrović et al. [21] introduced and investigated a hybrid contraction that combines a Reich-type contraction and interpolative-type contraction, while Karapinar and Fulga [19] provided a new hybrid contraction by combining Jaggi-type contraction and interpolative-type contraction.

In spite of the significant generality of the ideas of θ-contraction and the corresponding fixed point theorems, in certain situations (e.g., connected with problems from existence results of differential and integral equations), the concept of θ-contractions on generalized spaces or spaces with an additional metric structure has not been adequately examined. Therefore, inspired by the ideas of θ-contraction introduced by Jleli and Samet [17] and the work of Karapinar and Fulga [19] in metric space, the aim of this paper is to present a hybrid version of (θ-ϕ) contraction of Jaggi-type and establish various FP results for such map in the setting of complete generalized metric space. One of our obtained results is further applied to discuss some techniques of solving a mixed-type integral equations.

The organisation of this article is as follows. The introduction and a summary of relevant literary works are provided in Section 1. Section 2 compiles the fundamental ideas, such as definitions and lemmas, that are required for this work. The main findings and some implications of the FP theorem are presented in Section 3. The existence and uniqueness of a solution to a nonlinear Volterra-Fredholm integral equation are investigated as an application in Section 4, with the aid of one of the results found here. In Section 5, a numerical example is developed to substantiate the conclusions made therein. Deductions, suggestions, and conclusions are provided in Section 6.

2. Preliminaries

In this section, some fundamental definitions, terminology and notations that will be deployed subsequently are recalled. In this work, each set of $\mathcal{X}$ is considered nonempty and $\mathbb{N}$ is the set of natural numbers.

Bianciari [9] introduced the concept of generalized metric space, where the triangular inequality is replaced with rectangular inequality. The definition is as follows.

Definition 2.1 ([9])**.** Let $\mathcal{X}$ be a nonempty set and $\sigma : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ be a mapping such that for all $u, v \in \mathcal{X}$ and for all distinct points $a, b \in \mathcal{X}$, each of them different from u and v, we have

i. $\sigma(u, v) = 0 \Leftrightarrow u = v$;
ii. $\sigma(u, v) = \sigma(v, u)$;
iii. $\sigma(u, v) \leqslant \sigma(u, a) + \sigma(a, b) + \sigma(b, v)$.

Then, $(\mathcal{X}, \sigma)$ is referred to as a generalized metric space.

Recently, Jleli and Samet [17] presented a novel kind of contraction known as θ-contraction and established some new FP theorems for such contraction in the context of generalized metric spaces as introduced by Bianciari [9]. Let Θ be the set of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfying the following conditions:

(Θ_1) θ is non-decreasing;
(Θ_2) for each sequence $\{t_\iota\} \subset (0,\infty)$, $\lim_{\iota\to\infty}\theta(t_\iota) = 1$ if and only if $\lim_{\iota\to\infty}(t_\iota) = 0^+$;
(Θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+}\frac{\theta(t)-1}{t^r} = \ell$.

Definition 2.2 ([17])**.** Let $(\mathcal{X},\sigma)$ be a generalized metric space. A mapping $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ is called θ-contraction if there exist $\theta \in \Theta$ and $r \in (0,1)$ so that for every $u,v \in \mathcal{X}$,

$$\sigma(\tilde{S}u,\tilde{S}v) \neq 0 \Rightarrow \theta(\sigma(\tilde{S}u,\tilde{S}v)) \leqslant [\theta(\sigma(u,v))]^r.$$

The main theorem of Jleli and Samet [17] is as follows.

Theorem 2.3 ([17])**.** *Let $(\mathcal{X},\sigma)$ be a complete generalized metric space and $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ be a θ-contraction. Then $\tilde{S}$ has a unique FP in $\mathcal{X}$.*

They showed that the Banach contraction is a particular case of θ-contraction while there are θ-contractions which are not Banach contraction.

Example 2.4 ([17])**.** Let $\Theta : (0,\infty) \longrightarrow (1,\infty)$ be defined by

(1) $\theta(t) := e^{\sqrt{t}}$;
(2) $\theta(t) := e^{\sqrt{te^t}}$;
(3) $\theta(t) := 2 - \frac{2}{\pi}\arctan(\frac{1}{t^\alpha})$, $0 < \alpha < 1$, $t > 0$.

Then (1)-(3) satisfy all the properties of Θ.

In 1977, Jaggi[15] defined a new concept of a generalized Banach contraction principle, now called Jaggi contraction, which is one of the first known rational contractive inequalities.

Definition 2.5 ([13])**.** Suppose we have a metric space $(\mathcal{X},\sigma)$. Jaggi contraction describes a continuous self-mapping $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ if, for any $u,v \in \mathcal{X}$, $u \neq v$, there exist $\alpha_1,\alpha_2 \in [0,1)$ with $\alpha_1+\alpha_2 < 1$ such that

$$\sigma(\tilde{S}u,\tilde{S}v) \leqslant \alpha_1\frac{\sigma(u,\tilde{S}u).\sigma(v,\tilde{S}v)}{\sigma(u,v)} + \alpha_2\sigma(u,v).$$

Definition 2.6 ([4])**.** A mapping $\phi : [0,\infty) \longrightarrow [0,\infty)$ is called a (c)-comparison function if the following requirements are met:

(a) ϕ is nondecreasing;
(b) the series $\sum_{\iota=1}^{\infty}\phi^\iota(z)$ is convergent for $z \geqslant 0$.

Lemma 2.7 ([8])**.** *Let $\phi \in \Phi$ and Φ be the family of (c)-comparison functions. Then, the following conditions apply:*

(i) $\phi^\iota(z) \longrightarrow 0$ *as* $\iota \longrightarrow \infty$ *for all* $z \geqslant 0$;
(ii) $\phi(z) < z$ *for all* $z > 0$;
(iii) ϕ *is continuous;*
(iv) $\phi(z) = 0 \iff z = 0$;
(v) *the series* $\sum_{\iota=1}^{\infty}\phi^\iota(z) \geqslant 0$.

Definition 2.8 ([19])**.** A self-mapping $\tilde{S}$ on a metric space $(\mathcal{X},\sigma)$ referred to as a Jaggi-type hybrid contraction if there is $\phi \in \Phi$ such that for all distinct $u,v \in \mathcal{X}$,

$$\sigma(\tilde{S}u,\tilde{S}v) \leqslant \phi(M_{\alpha_i}(u,v,s,\tilde{S})), \tag{2.1}$$

where $\alpha_i \geqslant 0, i = 1,2$, such that $\alpha_1+\alpha_2 = 1$ and

$$M_{\alpha_i}(u,v,s,\tilde{S}) = \begin{cases} \left[\alpha_1\left(\frac{\sigma(u,\tilde{S}u).\sigma(v,\tilde{S}v)}{\sigma(u,v)}\right)^s + \alpha_2(\sigma(u,v))^s\right]^{\frac{1}{s}}, & \text{for } s > 0,\\ (\sigma(u,\tilde{S}u))^{\alpha_1}.(\sigma(v,\tilde{S}v))^{\alpha_2}, & \text{for } s = 0,\ u,v \in \mathcal{X}\backslash \text{fix}(\tilde{S}). \end{cases}$$

Here, $\text{fix}(\tilde{S}) = \{u \in \mathcal{X} : u = \tilde{S}u\}$.

3. Main results

In this section, the idea of Jaggi-type hybrid (θ-ϕ)-contraction in the structure of a generalized metric space is introduced, and new constraints for the existence and uniqueness of fixed points are discussed. In this manner, several useful and classical results are generalized. Also, a concrete example is created to furnish our results.

Definition 3.1. Let $(\mathcal{X}, \sigma)$ be a generalized metric space. A mapping $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ is regarded as a Jaggi-type hybrid (θ-ϕ)-contraction if there exist $\theta \in \Theta$, $\phi \in \Phi$ and $r \in (0, 1)$ such that for any $u, v \in \mathcal{X} \backslash \text{fix}(\tilde{S})$,

$$\sigma(\tilde{S}u, \tilde{S}v) \neq 0 \Rightarrow \theta(\sigma(\tilde{S}u, \tilde{S}v)) \leqslant [\theta(\phi(M_{\alpha_i}(u, v, s, \tilde{S})))]^r, \tag{3.1}$$

where $M_{\alpha_i}(u, v, s, \tilde{S})$ is as defined in (2.1).

Theorem 3.2. *Let $(\mathcal{X}, \sigma)$ be a complete generalized metric space and the mapping $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ be a continuous Jaggi-type hybrid (θ-ϕ)-contraction, then $\tilde{S}$ possesses a unique FP in $\mathcal{X}$.*

Proof. Suppose that u_0 is an arbitrary fixed element in $\mathcal{X}$. Define a sequence $\{u_\iota\}_{\iota \in \mathbb{N}}$ in $\mathcal{X}$ by $u_\iota = \tilde{S}^\iota u_0$, $\iota \geqslant 1$. If we can find some $\iota_0 \in \mathbb{N}$ such that $u_{\iota_0+1} = u_{\iota_0}$, then, $\tilde{S}^{\iota+1} u_0 = \tilde{S}^\iota u_0$. This implies that $\tilde{S}^\iota u_0$ is a FP of $\tilde{S}$ and the proof is complete. Now, suppose on the contrary that $u_{\iota+1} \neq u_\iota$ for all $\iota \in \mathbb{N}$, then $\sigma(u_\iota, u_{\iota+1}) > 0$ for all $\iota \in \mathbb{N}$. By examining two distinct cases, the theorem shall be proved in the following manner.

Case 1. For $s > 0$, let $u = u_{\iota-1}$ and $v = u_\iota$ for all $\iota \in \mathbb{N}$, then (3.1) becomes:

$$\theta(\sigma(\tilde{S}u_{\iota-1}, \tilde{S}u_\iota)) = \theta(\sigma(u_\iota, u_{\iota+1})) \leqslant [\theta(\phi(M_{\alpha_i}(u_{\iota-1}, u_\iota)))]^r, \tag{3.2}$$

where

$$\begin{aligned} M_{\alpha_i}(u_{\iota-1}, u_\iota) &= \left[\alpha_1\left(\frac{\sigma(u_{\iota-1}, \tilde{S}u_{\iota-1}).\sigma(u_\iota, \tilde{S}u_\iota)}{\sigma(u_{\iota-1}, u_\iota)}\right)^s + \alpha_2(\sigma(u_{\iota-1}, u_\iota))^s\right]^{\frac{1}{s}} \\ &= \left[\alpha_1\left(\frac{\sigma(u_{\iota-1}, u_\iota).\sigma(u_\iota, u_{\iota+1})}{\sigma(u_{\iota-1}, u_\iota)}\right)^s + \alpha_2(\sigma(u_{\iota-1}, u_\iota))^s\right]^{\frac{1}{s}} \\ &= \left[\alpha_1(\sigma(u_\iota, u_{\iota+1}))^s + \alpha_2(\sigma(u_{\iota-1}, u_\iota))^s\right]^{\frac{1}{s}}. \end{aligned} \tag{3.3}$$

Assume that $\sigma(u_\iota, u_{\iota+1}) \geqslant \sigma(u_{\iota-1}, u_\iota)$. Then, from (3.2) and (3.3), we have

$$\begin{aligned} \theta(\sigma(u_\iota, u_{\iota+1})) &\leqslant [\theta(\phi(M_{\alpha_i}(u_{\iota-1}, u_\iota)))]^r \\ &= \left[\theta(\phi((\alpha_1(\sigma(u_\iota, u_{\iota+1}))^s + \alpha_2(\sigma(u_{\iota-1}, u_\iota))^s)^{\frac{1}{s}}))\right]^r \\ &\leqslant \left[\theta(\phi((\alpha_1(\sigma(u_\iota, u_{\iota+1}))^s + \alpha_2(\sigma(u_\iota, u_{\iota+1}))^s)^{\frac{1}{s}}))\right]^r \\ &= [\theta(\phi((\alpha_1 + \alpha_2)(\sigma(u_\iota, u_{\iota+1}))^s)^{\frac{1}{s}}]^r = [\theta(\phi(\sigma(u_\iota, u_{\iota+1})))]^r < [\theta(\sigma(u_\iota, u_{\iota+1}))]^r. \end{aligned} \tag{3.4}$$

That is, $\theta(\sigma(u_\iota, u_{\iota+1})) < [\theta(\sigma(u_\iota, u_{\iota+1}))]^r$, which is a contradiction for all $r \in (0, 1)$. Therefore, $\max\{\sigma(u_\iota, u_{\iota+1}), \sigma(u_{\iota-1}, u_\iota)\} = \sigma(u_{\iota-1}, u_\iota)$. Hence, (3.2) becomes:

$$\begin{aligned} \theta(\sigma(u_\iota, u_{\iota+1})) &\leqslant [\theta(\phi(\sigma(u_{\iota-1}, u_\iota)))]^r \\ &\leqslant [\theta(\phi(\phi(\sigma(u_{\iota-2}, u_{\iota-1}))))]^{r^2} \\ &= [\theta(\phi^2(\sigma(u_{\iota-2}, u_{\iota-1})))]^{r^2} \leqslant \cdots \leqslant [\theta(\phi^\iota(\sigma(u_0, u_1)))]^{r^\iota} \text{ for all } \iota \in \mathbb{N}. \end{aligned}$$

Thus, we have

$$1 \leqslant \theta(\sigma(u_\iota, u_{\iota+1})) \leqslant [\theta(\phi^\iota(\sigma(u_0, u))]^{r^\iota}. \tag{3.5}$$

Letting $\iota \longrightarrow \infty$ in (3.5) and using Sandwich theorem, (3.5) becomes:

$$\theta(\sigma(u_\iota, u_{\iota+1})) \to 1 \text{ as } \iota \longrightarrow \infty,$$

which implies from (Θ_2) that $\lim_{\iota\to\infty} \sigma(u_\iota, u_{\iota+1}) = 0$. From condition (Θ_3), there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{\iota\to\infty} \frac{\theta(\sigma(u_\iota, u_{\iota+1})) - 1}{[\sigma(u_\iota, u_{\iota+1})]^r} = \ell.$$

Suppose that $\ell < \infty$. In this instance, let $B = \frac{\ell}{2} > 0$. As stated in the definition of the limit, there exists $\iota_0 1\iota$ such that

$$\left| \frac{\theta(\sigma(u_\iota, u_{\iota+1})) - 1}{[\sigma(u_\iota, u_{\iota+1})]^r} - \ell \right| \leqslant B, \quad \text{for all } \iota \geqslant \iota_0.$$

This implies that

$$\frac{\theta(\sigma(u_\iota, u_{\iota+1})) - 1}{[\sigma(u_\iota, u_{\iota+1})]^r} \geqslant \ell - B = B, \text{ for all } \iota \geqslant \iota_0.$$

Then,

$$\iota[\sigma(u_\iota, u_{\iota+1})]^r \leqslant An[\theta(\sigma(u_\iota, u_{\iota+1})) - 1], \text{ for all } \iota \geqslant \iota_0,$$

$A = \frac{1}{B}$ in this case. Let us now assume that $\ell = \infty$. Let an arbitrary positive value $B > 0$. According to the limit's definition, there is a $\iota_0 \geqslant 1$ such that

$$\frac{\theta(\sigma(u_\iota, u_{\iota+1})) - 1}{(\sigma(u_\iota, u_{\iota+1}))^r} \geqslant B, \text{ for all } \iota \geqslant \iota_0.$$

This implies that

$$\iota[\sigma(u_\iota, u_{\iota+1})]^r \leqslant An[\theta(\sigma(u_\iota, u_{\iota+1})) - 1], \text{ for all } \iota \geqslant \iota_0,$$

where $A = \frac{1}{B}$. Thus, in all cases, there exists $A > 0$ such that

$$\iota[\sigma(u_\iota, u_{\iota+1})]^r \leqslant An[\theta(\sigma(u_\iota, u_{\iota+1})) - 1], \text{ for all } \iota \geqslant \iota_0.$$

Using (3.5), we obtain

$$\iota[\sigma(u_\iota, u_{\iota+1})]^r \leqslant An([\theta(\sigma(u_\iota, u_{\iota+1}))]^{r^\iota} - 1), \quad \text{for all } \iota \geqslant \iota_0. \tag{3.6}$$

Letting $\iota \to \infty$ in (3.6), we obtain $\lim_{\iota\to\infty} \iota[\sigma(u_\iota, u_{\iota+1})]^r = 0$. Consequently, there is $\iota_1 \in \mathbb{N}$ such that

$$\sigma(u_\iota, u_{\iota+1}) \leqslant \frac{1}{\iota^{\frac{1}{r}}}, \text{ for all } \iota \geqslant \iota_1. \tag{3.7}$$

Now, we'll show that there is a periodic point for $\tilde{S}$. If this isn't the case, then for each $\iota, m \in \mathbb{N}$ such that $\iota \neq m$, $u_\iota \neq u_m$. Applying (3.4), we arrive at

$$1 \leqslant \theta(\sigma(u_\iota, u_{\iota+2})) \leqslant [\theta(\phi(M_{\alpha_i}(u_{\iota-1}, u_{\iota+1})))]^r \leqslant [\theta(\phi^2(M_{\alpha_i}(u_{\iota-2}, u_\iota)))]^{r^2} \leqslant \cdots \leqslant [\theta(\phi^\iota(\sigma(u_0, u_2)))]^{r^n}.$$

Letting $\iota \longrightarrow \infty$ in the above inequality and using (Θ_2), we have $\lim_{\iota\to\infty} \sigma(u_\iota, u_{\iota+2}) = 0$. In a similar vein, $\iota_2 \in \mathbb{N}$ exists from condition (Θ_3) such that

$$\sigma(u_\iota, u_{\iota+2}) \leqslant \frac{1}{\iota^{\frac{1}{r}}}, \text{ for all } \iota \geqslant n_2. \tag{3.8}$$

For $H = \max\{\iota_0, \iota_1\}$, set it. We examine two cases.
(A) Writing $m = 2l + 1$, $l \geqslant 1$, then applying (3.7) for all $\iota \geqslant H$, if $m > 2$ is odd, yields

$$\begin{aligned}\sigma(u_\iota, u_{\iota+m}) &\leqslant \sigma(u_\iota, u_{\iota+1}) + \sigma(u_{\iota+1}x, u_{\iota+2}) + \cdots + \sigma(u_{\iota+2l}, u_{\iota+2l+1}) \\ &\leqslant \frac{1}{\iota^{\frac{1}{r}}} + \frac{1}{(\iota+1)^{\frac{1}{r}}} + \cdots + \frac{1}{(\iota+2l)^{\frac{1}{r}}} = \sum_{i=1}^{\iota+2l} \frac{1}{i^{\frac{1}{r}}} \leqslant \sum_{i=\iota}^{\infty} \frac{1}{i^{\frac{1}{r}}}.\end{aligned}$$

(B) If $m > 2$ is even, then writing $m = 2l$, $l \geqslant 2$ and using (3.7) and (3.8), yields

$$\begin{aligned}\sigma(u_\iota, u_{\iota+m}) &\leqslant \sigma(u_\iota, u_{\iota+2}) + \sigma(u_{\iota+2}, u_{\iota+3}) + \cdots + \sigma(u_{\iota+2l-1}, u_{\iota+2l}) \\ &\leqslant \frac{1}{\iota^{\frac{1}{r}}} + \frac{1}{(\iota+2)^{\frac{1}{r}}} + \cdots + \frac{1}{(\iota+2l-1)^{\frac{1}{r}}} \leqslant \sum_{i=\iota}^{\infty} \frac{1}{i^{\frac{1}{r}}}.\end{aligned}$$

Thus, combining all the instances, leads to

$$\sigma(u_\iota, u_{\iota+m}) \leqslant \sum_{i=\iota}^{\infty} \frac{1}{i^{\frac{1}{r}}} \text{ for all } \iota \geqslant H,\ m \in \mathbb{N}.$$

The series $\sum_{i=\iota}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ converges since $\frac{1}{r} > 1$, which shows that $\{u_\iota\}$ is a Cauchy sequence. From the completeness of the space, there exists $z \in \mathcal{X}$ such that $u_\iota \longrightarrow z$ as $\iota \longrightarrow \infty$. Since $\tilde{S}$ is a continuous mapping, it follows that

$$\sigma(z, \tilde{S}z) = \lim_{\iota\to\infty} \sigma(u_\iota, \tilde{S}u_\iota) = \lim_{\iota\to\infty} \sigma(u_\iota, u_{\iota+1}) = \sigma(z, z) = 0.$$

This implies that $z = \tilde{S}z$, which is a contradiction with the assumption $\tilde{S}$ does not have a periodic point. Thus, $\tilde{S}$ has at least a periodic point. That is, the FP exists.
Case 2. For $s = 0$, we let $u = u_{\iota-1}$ and $v = u_\iota$ in (3.1), for all $\iota \in \mathbb{N}$. Then,

$$\theta(\sigma(\tilde{S}u_{\iota-1}, \tilde{S}u_\iota)) = \theta(\sigma(u_\iota, u_{\iota+1})) \leqslant [\theta(\phi(M_{\alpha_i}(u_{\iota-1}, u_\iota)))]^r,$$

where

$$M_{\alpha_i}(u_{\iota-1}, u_\iota) = (\sigma(u_{\iota-1}, \tilde{S}u_{\iota-1}))^{\alpha_1}.(\sigma(u_\iota, \tilde{S}u_\iota))^{\alpha_2} = (\sigma(u_{\iota-1}, u_\iota))^{\alpha_1}.(\sigma(u_\iota, u_{\iota+1}))^{\alpha_2}.$$

It follows from (3.2) that

$$\theta(\sigma(u_\iota, u_{\iota+1})) \leqslant [\theta(\phi(\sigma(u_{\iota-1}, u_\iota))^{\alpha_1}.(\sigma(u_\iota, u_{\iota+1}))^{\alpha_2}]^r. \tag{3.9}$$

Assume that $\sigma(u_\iota, u_{\iota+1}) \geqslant \sigma(u_{\iota-1}, u_\iota)$. Then, from (3.9), we have

$$\begin{aligned}\theta(\sigma(u_\iota, u_{\iota+1})) &\leqslant [\theta(\phi(\sigma(u_\iota, u_{\iota+1}))^{\alpha_1}(\sigma(u_\iota, u_{\iota+1}))^{\alpha_2})]^r \\ &= [\theta(\phi(\sigma(u_\iota, u_{\iota+1}))^{\alpha_1+\alpha_2})]^r = [\theta(\phi(\sigma(u_\iota, u_{\iota+1})))]^r < [\theta(\sigma(u_\iota, u_{\iota+1}))]^r,\end{aligned}$$

which is a contradiction. This implies from (3.9) that

$$\begin{aligned}\theta(\sigma(u_\iota, u_{\iota+1})) &\leqslant [\theta(\phi(\sigma(u_{\iota-1}, u_\iota)))]^r \\ &\leqslant [\theta(\phi(\phi(\sigma(u_{\iota-2}, u_{\iota-1}))))]^{r^2} \\ &= \theta(\phi^2(\sigma(u_{\iota-2}, u_{\iota-1})))]^{r^2} \leqslant \cdots \leqslant [\theta(\phi^\iota(\sigma(u_0, u_1)))]^{r^\iota} \text{ for all } \iota \in \mathbb{N}.\end{aligned}$$

the identical tools as used in the instance of $s > 0$, we can establish the fact that $\{u_\iota\}$ forms a Cauchy sequence. Since the metric is complete, $\lim_{\iota\to\infty} u_\iota = u$, which is a point in $\mathcal{X}$. Also, from the continuity of $\tilde{S}$, it's evident that this point is a FP of $\tilde{S}$.

Suppose $\tilde{S}$ has more than one FP. Indeed, if $u, v \in \mathcal{X}$ are two FP of $\tilde{S}$ such that $\tilde{S}u = u \neq v = \tilde{S}v$, then $\sigma(u, v) = \sigma(\tilde{S}u, \tilde{S}v) > 0$. Using (3.1),

$$\theta(\sigma(u, v)) = \theta(\sigma(\tilde{S}u, \tilde{S}v)) \leqslant [\theta(\phi(M_{\alpha_i}(u, v)))]^r. \tag{3.10}$$

Case 1. For $s > 0$,

$$\begin{aligned} M_{\alpha_i}(u, v) &= \left[\alpha_1 \left(\frac{\sigma(u, \tilde{S}v).\sigma(u, \tilde{S}v)}{\sigma(u, v)}\right)^s + \alpha_2(\sigma(u, v))^s\right]^{\frac{1}{s}} \\ &= \left[\alpha_1 \left(\frac{\sigma(u, v).\sigma(u, v)}{\sigma(u, v)}\right)^s + \alpha_2(\sigma(u, v))^s\right]^{\frac{1}{s}} = [\alpha_2(\sigma(u, v))^s]^{\frac{1}{s}} = [\alpha_2^{\frac{1}{s}}(\sigma(u, v))] \leqslant \sigma(u, v). \end{aligned}$$

Hence (3.10) becomes $\theta(\sigma(u, v)) \leqslant [\theta(\phi(\sigma(u, v)))]^r < \theta(\sigma(u, v))$, which is a contradiction.
Case 2. For $s = 0$,

$$M_{\alpha_i}(u, v) = (\sigma(u, \tilde{S}v))^{\alpha_1}.(\sigma(u, \tilde{S}v))^{\alpha_2} = (\sigma(u, v))^{\alpha_1}.(\sigma(u, v))^{\alpha_2} = 0.$$

Therefore, from (3.10), we have $\theta(\sigma(u, v)) \leqslant [\theta(\phi(0))]^r \leqslant [\theta(0)]^r < \theta(0)$, which is a contradiction. Hence, the FP of $\tilde{S}$ is unique. □

Listed below are some immediate consequences of Theorem 3.2.

Corollary 3.3. *Let $(\mathcal{X}, \sigma)$ be a complete metric space and $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ be a given map. Assume that there exist $\theta \in \Theta$, $\phi \in \Phi$, and $r \in (0, 1)$ such that for any $u, v \in \mathcal{X} \backslash \text{fix}(\tilde{S})$,*

$$\theta(\sigma(\tilde{S}u, \tilde{S}v)) \leqslant [\theta(\phi(M_{\alpha_i}(u, v, s, \tilde{S})))]^r,$$

where $M_{\alpha_i}(u, v, s, \tilde{S})$ is defined as in (2.1). Then $\tilde{S}$ has a unique FP in $\mathcal{X}$.

Corollary 3.4. *Let $(\mathcal{X}, \sigma)$ be a complete generalized metric space and $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ be a continuous mapping satisfying the condition:*

$$\theta(\sigma(\tilde{S}u, \tilde{S}v)) \leqslant [\theta(\mu M_{\alpha_i}(u, v))]^r \text{ for all } u, v \in \mathcal{X} \text{ and } r \in (0, 1).$$

Then $\tilde{S}$ has a unique FP in $\mathcal{X}$.

Proof. By taking $\phi(t) = \mu t$, for all $t \geqslant 0$ and for some $\mu \in (0, 1)$, the proof follows from Theorem 3.2. □

Corollary 3.5. *Given $(\mathcal{X}, \sigma)$ to be a complete generalized metric space and $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ a continuous mapping satisfying the condition:*

$$\theta(\sigma(\tilde{S}u, \tilde{S}v)) \leqslant [\theta(\mu\sqrt{\sigma(u, \tilde{S}u)(\sigma(v, \tilde{S}v))})]^r \text{ for all } u, v \in \mathcal{X},$$

then $\tilde{S}$ has a unique FP in $\mathcal{X}$.

Proof. Consider Case 2 of Theorem 3.2 and by taking $\alpha_1 = \alpha_2 = \frac{1}{2}$, then the conclusion is immediate from Corollary (3.4). □

Corollary 3.6 ([17])**.** *Let $(\mathcal{X}, \sigma)$ be a complete generalized metric space and $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ be a given mapping. Assume that there exit $\theta \in \Theta$ and $r \in (0, 1)$ such that*

$$u, v \in \mathcal{X}, \sigma(\tilde{S}u, \tilde{S}v) \neq 0 \Rightarrow \theta(\sigma(\tilde{S}u, \tilde{S}v)) \leqslant [\theta(\mu\sigma(u, v))]^r.$$

Then, $\tilde{S}$ has a unique FP in $\mathcal{X}$.

Proof. Consider the case where $s > 0$ in Theorem 3.2. Then by taking $\alpha_1 = 0$, $\alpha_2 = 1$ and $\phi(t) = \mu t$, for all $t > 0$ and some $\mu \in (0, 1)$, the conclusion follows. □

The example that follows is created to assist the hypotheses of Corollary 3.3.

Example 3.7. Let $\mathcal{X} = \left\{u_\iota = \frac{\iota(\iota+1)(2\iota+1)}{2} \text{ for all } \iota \in \mathbb{N}\right\}$. If $\mathcal{X}$ is endowed with the standard metric $\sigma(u, \nu) = |u - \nu|$ for all $u, \nu \in \mathcal{X}$, consider the mapping $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ defined by $\tilde{S}u_1 = u_1$ and $\tilde{S}u_\iota = u_{\iota-1}$ if $\iota \geqslant 2$. Obviously, the Banach contraction is not satisfied, since

$$\lim_{\iota\to\infty} \frac{\sigma(\tilde{S}u_\iota, \tilde{S}u_1)}{\sigma(u_\iota, u_1)} = \lim_{\iota\to\infty} \frac{2\iota^3 - 3\iota^2 + \iota - 6}{2\iota^3 + 3\iota^2 + \iota - 6} = 1.$$

To show that $\tilde{S}$ is a Jaggi-type hybrid (θ-ϕ)-contraction, it is sufficient to show that

$$\theta(\sigma(\tilde{S}u_\iota, \tilde{S}u_m)) \leqslant [\theta(\phi(M_{\alpha_i}(u_\iota, u_m)))]^r, \tag{3.11}$$

holds for some $r \in (0, 1)$ and for all $\iota, m \in \mathbb{N}$. So, let $\theta(t) = \sqrt{te^t}$, $t > 0$, then (3.11) yields:

$$\sigma(\tilde{S}u_\iota, \tilde{S}u_m)e^{\sigma(\tilde{S}u_\iota,\tilde{S}u_m)} \leqslant r^2[\phi(M_{\alpha_i}(u_\iota, u_m))e^{\phi(M_{\alpha_i}(u_\iota,u_m))}] < r^2(M_{\alpha_i}(u_\iota, u_m)e^{M_{\alpha_i}(u_\iota,u_m)}. \tag{3.12}$$

Let $\alpha_1 = 0$ and $\alpha_2 = 1$ for the case when $s = 0$ in Corollary (3.4). Then, $M_{\alpha_i}(u_\iota, u_m) = \sigma(u_\iota, u_m)$. This implies that (3.12) can be written as

$$\sigma(\tilde{S}u_\iota, \tilde{S}u_m)e^{\sigma(\tilde{S}u_\iota,\tilde{S}u_m)} \leqslant r^2\sigma(u_\iota, u_m)e^{\sigma(u_\iota,u_m)}. \tag{3.13}$$

Now, two cases are considered.

(A) $\iota = 1$ and $m \geqslant 2$. From (3.13), we have

$$\frac{\sigma(\tilde{S}u_1, \tilde{S}u_m)e^{\sigma(\tilde{S}u_1,\tilde{S}u_m)-\sigma(u_1,u_m)}}{\sigma(u_1, u_m)} = \frac{2m^3 - 3m^2 + m - 6}{2m^3 + 3m^2 + m - 6}e^{-3m^2} \leqslant e^{-1}.$$

(B) $m > \iota > 1$. From (3.13), we have

$$\frac{\sigma(\tilde{S}u_\iota, \tilde{S}u_m)e^{\sigma(\tilde{S}u_\iota,\tilde{S}u_m)-\sigma(u_\iota,u_m)}}{\sigma(u_\iota, u_m)} = \frac{(2m^3 - 3m^2 + m) - (2\iota^3 - 3\iota^2 + \iota)}{(2m^3 + 3m^2 + m) - (2\iota^3 + 3\iota^2 + \iota)}e^{-3(m^2-\iota^2)} \leqslant e^{-1},$$

considering $r = e^{-\frac{1}{2}}$. As such, $\tilde{S}$ has a unique FP in $\mathcal{X}$ and is a Jaggi-type hybrid (θ-ϕ)-contraction. Therefore, every condition stated in Corollary 3.3 is met. Here, u_1 represents the distinct FP of $\tilde{S}$. In contrast, $\tilde{S}$ is not a hybrid contraction of the Jaggi type according to Karapinar and Fulga [15]. Define a mapping $\phi \in \Phi$ such that, for all $t \geqslant 0$, $\phi(t) = \frac{t}{4}$. Next, based on 3.11, we possess that

$$\begin{aligned}\sigma(\tilde{S}u_\iota, \tilde{S}u_m)e^{\sigma(\tilde{S}u_\iota,\tilde{S}u_m)} &\leqslant r^2\phi(M_{\alpha_i}(u_\iota, u_m))e^{\phi(M_{\alpha_i}(u_\iota,u_m))} \\ &= r^2\frac{1}{4}(M_{\alpha_i}(u_\iota, u_m))e^{\frac{1}{4}(M_{\alpha_i}(u_\iota,u_m)} < r^2M_{\alpha_i}(u_\iota, u_m)e^{M_{\alpha_i}(u_\iota,u_m)}.\end{aligned} \tag{3.14}$$

Obviously, (3.13) and (3.14) coincide. Here, under the value of $\alpha_1 = 0$ and $\alpha_2 = 1$, it follows that the mapping T is a Jaggi-type hybrid (θ-ϕ)-contraction. On the other hand, for all $u, \nu \in \mathcal{X} \setminus \text{fix}(T)$, where (2.1) yields

$$\sigma(\tilde{S}u, \tilde{S}\nu) \leqslant \phi(\sigma(u, \nu)) = \frac{1}{4}\sigma(u, \nu). \tag{3.15}$$

Now, pick two points $u = u_1$ and $\nu = u_3$, then evidently, (3.15) becomes

$$\frac{4\sigma(\tilde{S}u_1, \tilde{S}u_3)}{\sigma(u_1, u_3)} = \frac{4\sigma(3, 15)}{\sigma(3, 42)} = \frac{4|3 - 15|}{|3 - 42|} = \frac{4(12)}{39} = \frac{48}{39} > 1,$$

which shows that (2.1) fails.

4. Applications to a nonlinear Volterra-Fredholm integral equation

Fixed point results of Lipschitze-type inequalities have been very useful in the existence theory of both linear and nonlinear equations. Consequently, more than a handful of such applications have been presented. Not long ago, Hammad and De le Sen [12] established a coupled fixed point theorem for F-contraction mappings and applied the obtained result to analyzing new conditions for the existence of a unique solution to a nonlinear integral equation. In [14], a graphic-type fixed point results was employed to examine solvability criteria of a system of ordinary differential equations with infinite delay. For some related applications of Lipschitz-type inequalities, we refer to [16, 22, 25]. In this section, one of our obtained findings is used to investigate conditions for the existence and uniqueness of a solution to nonlinear Volterra-Fredholm equation. To this effect, consider the nonlinear Volterra-Fredholm integral equation of the first kind, given as

$$u(t) = \lambda_1 \int_0^t k_1(t,s,u(s))ds + \lambda_2 \int_0^h k_2(t,s,u(s))ds, \quad t \in [0,h], \tag{4.1}$$

where $u(t)$ is the unknown solution, $k_i(t,s,u(s))$ are smooth functions, λ_i, h are constants. Assume that the set of all continuous real valued functions defined on $\mathcal{X}$ with the supremum norm is $\mathcal{X} = C([0,h])$. The metric $\sigma : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}_+$, which is defined as $\sigma(u,v) = \sup\{|u(t) - v(t)|, t \in [0,h]\}$, is equipped with $\mathcal{X}$, the completion of $(\mathcal{X}, \sigma)$ occurs. A mapping $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ should be defined by

$$\tilde{S}u(t) = \lambda_1 \int_0^t k_1(t,s,u(s))ds + \lambda_2 \int_0^h k_2(t,s,u(s))ds. \tag{4.2}$$

Then, a point z s regarded as a FP of $\tilde{S}$ if and only if z is a solution to (4.1). Now, we study solvability criteria of the nonlinear Volterra-Fredholm integral equation (4.1) under the following assumptions.

Theorem 4.1. *Assume that the following conditions are satisfied:*

(i) $\tilde{S}$ *is a continuous mapping and* $k_i : [0,h] \times [0,h] \times \mathbb{R} \longrightarrow \mathbb{R}$*;*
(ii) *for some constant* A_i*, there exist* $r, \mu \in (0,1)$ *such that* $|k_i(t,s,u(t)) - k_i(t,s,v(u))| \leqslant A_i[r\mu|u(s) - v(s)|]$*, where* $i = 1,2$*;*
(iii) *suppose further that* $\delta t + \eta h < 1$*, where* $\tau_1 A_1 = \delta$*, and* $\tau_2 A_2 = \eta$*.*

Then, the integral equation (4.1) *has a unique solution in* $\mathcal{X}$*.*

Proof. Note that that every $u, v \in \mathcal{X}$, using (4.2) and the above hypotheses,

$$\begin{aligned}
&|\tilde{S}u(t) - \tilde{S}v(t)| \\
&= \left|\tau_1 \int_0^t k_1(t,s,u(s))ds + \tau_2 \int_0^h k_2(t,s,u(s))ds - \left[\tau_1 \int_0^t k_1(t,s,v(s))ds + \tau_2 \int_0^h k_2(t,s,v(s))ds\right]\right| \\
&= \left|\tau_1 \left[\int_0^t k_1(t,s,u(s))ds - k_1(t,s,v(s))ds\right] + \tau_2 \left[\int_0^h k_2(t,s,u(s))ds - k_2(t,s,v(s))ds\right]\right| \\
&\leqslant \tau_1 \int_0^t A_1 \left[r\mu|u(s) - v(s)|\right] ds + \tau_2 \int_0^h A_2 \left[r\mu|u(s) - v(s)|\right] ds \\
&\leqslant \tau_1 A_1 [r\mu\|u - v\|]t + \tau_2 A_2 [r\mu\|u - v\|]h \\
&= (\tau_1 A_1 t + \tau_2 A_2 h)[r\mu\|u - v\|] = (\delta t + \eta h)[r\mu\|u - v\|] < [r\mu\|u - v\|].
\end{aligned}$$

This yields that $\sup_{t\in[0,h]} |\tilde{S}u(t) - \tilde{S}v(t)| \leqslant [r\mu\sigma(u(t), v(t))]$. Hence,

$$\sigma(\tilde{S}u, \tilde{S}v) \leqslant [r\mu\sigma(u,v)]. \tag{4.3}$$

Now, from Corollary 3.4, for the case when $s = 0, \alpha_1 = 0, \alpha_2 = 1$, and $s = 1$, $M_{\alpha_i}(u,v) = \sigma(u,v)$. This implies that equation (4.3) becomes:

$$\sigma(\tilde{S}u, \tilde{S}v) \leqslant [r\mu(M_{\alpha_i}(u,v))]. \tag{4.4}$$

Taking exponential of both sides of (4.4), produces

$$e^{\sigma(\tilde{S}u,\tilde{S}v)} \leqslant [e^{\mu(M_{\alpha_i}(u,v))}]^r. \tag{4.5}$$

Define a mapping $\theta : [0,\infty) \longrightarrow [1,\infty)$ by $\theta(t) = e^t$, then (4.5) becomes

$$\theta(\sigma(\tilde{S}u, \tilde{S}v)) \leqslant [\theta(\mu(M_{\alpha_i}(u,v)))]^r.$$

Thus, all of the theories of Corollary 3.4 are satisfied and so $\tilde{S}$ has a unique FP in $\mathcal{X}$. This implies that there is a unique solution to the nonlinear Volterra-Fredholm integral equation. □

5. Numerical examples

This section establishes a numerical example to support the reliability of the given results, using a class of integral equation. The ideas used in this section are motivated by [1]. Let $\mathcal{X} = C([0,1],\mathbb{R})$ be the set of all continuous real-valued functions, and $\tilde{S} : \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping defined by

$$\tilde{S}u(t) = \psi(t) + \int_0^1 k(t,s)u(s)ds, \quad t,s \in [0,1]^2. \tag{5.1}$$

Let $\psi(t) = t^2 \sin(t)$ and $k(t,s)u(s) = t\cos(u(s))$. Then, by substitution, equation (5.1) becomes

$$\tilde{S}u(t) = t^2 \sin(t) + \int_0^1 t\cos(u(s))ds.$$

Let $\phi : [0,\infty) \longrightarrow [0,\infty)$ be a mapping satisfying the condition.

(A) $|\cos(u(s)) - \cos(v(s))| \leqslant r(\phi|u(s) - v(s)|)$ for some $r \in (0,1)$. Note that for any $u,v \in \mathcal{X}$ and using condition (A), we have

$$\begin{aligned}
\left|\tilde{S}u(t) - \tilde{S}v(t)\right| &= \left|t^2\sin(t) + \int_0^1 t\cos(u(t))ds - t^2\sin(t) - \int_0^1 t\cos(v(t))ds\right| \\
&= \left|\int_0^1 t\cos(u(t))ds - \int_0^1 t\cos(v(t))ds\right| \\
&= \left|\int_0^1 t[\cos(u(t)) - \cos(v(t))]ds\right| \\
&\leqslant \int_0^1 |t|\Big[|\cos(u(t)) - \cos(v(t))|\Big]ds \\
&\leqslant \int_0^1 r(\phi|u(t) - v(t)|)ds = r(\phi|u(s) - v(s)|)\int_0^1 ds = r(\phi|u(s) - v(s)|).
\end{aligned}$$

This implies that

$$\sigma(\tilde{S}u, \tilde{S}v) \leqslant r\phi(\sigma(u,v)). \tag{5.2}$$

Taking exponential of both sides of (5.2), yields

$$e^{\sigma(\tilde{S}u,\tilde{S}v)} \leqslant e^{r\phi(\sigma(u,v))} = e^{\phi(\sigma(u,v))^r}. \tag{5.3}$$

Define a mapping $\theta : [0,\infty) \longrightarrow [1,\infty)$ by $\theta(t) = e^t$. Then (5.3) becomes

$$\theta(\sigma(\tilde{S}u, \tilde{S}v)) \leqslant [\theta(\phi(\sigma(u,v)))]^r. \tag{5.4}$$

Now from Theorem 3.2, considering the case when $s > 0, \alpha_1 = 0, \alpha_2 = 1$, and $s = 1$, $M_{\alpha_i}(u,v) = \sigma(u,v)$.

Consequently, (5.4) becomes

$$\theta(\sigma(\tilde{S}u, \tilde{S}v)) \leqslant [\theta(\phi(M_{\alpha_i}(u, v)))]^r.$$

Thus, every requirement of Theorem 3.2 are satisfied. As a result, the integral equation (5.1) has a unique solution in $\mathcal{X}$. It can be easily checked that $u(t) = t$ is the exact solution of equation (5.1).

Next, the iterative method will be employed to examine the viability of our technique. Tables 1, 2, 3, 4, and 5 show the sequence of iteration of $u_{\iota+1}(t) = \tilde{S}u_\iota(t) = t^2\sin(t) + \int_0^1 t\cos(u_\iota(s))ds$. Let $u_0(t) = 1$ be an initial fixed solution.

Table 1: For $t = 0.2$.

n	$u_{\iota+1}(0.2)$	Approximation	Absolute Error
0	$u_1(0.2)$	0.200109	1.09×10^{-4}
1	$u_2(0.2)$	0.200138	1.38×10^{-4}
2	$u_3(0.2)$	0.200138	1.38×10^{-4}
3	$u_4(0.2)$	0.200138	1.38×10^{-4}

Table 2: For $t = 0.4$.

n	$u_{\iota+1}(0.4)$	Approximation	Absolute Error
0	$u_1(0.4)$	0.401056	1.056×10^{-3}
1	$u_2(0.4)$	0.401107	1.107×10^{-3}
2	$u_3(0.4)$	0.401107	1.107×10^{-3}
3	$u_4(0.4)$	0.401107	1.107×10^{-3}

Table 3: For $t = 0.6$.

n	$u_{\iota+1}(0.6)$	Approximation	Absolute Error
0	$u_1(0.6)$	0.603678	3.678×10^{-3}
1	$u_2(0.6)$	0.603736	3.736×10^{-3}
2	$u_3(0.6)$	0.603736	3.736×10^{-3}
3	$u_4(0.6)$	0.603736	3.736×10^{-3}

Table 4: For $t = 0.8$.

ι	$u_{\iota+1}(0.8)$	Approximation	Absolute Error
0	$u_1(0.8)$	0.808813	8.08813×10^{-3}
1	$u_2(0.8)$	0.808856	8.08856×10^{-3}
2	$u_3(0.8)$	0.808856	8.08856×10^{-3}
3	$u_4(0.8)$	0.808856	8.08856×10^{-3}

Table 5: For $t = 1$.

ι	$u_{\iota+1}(1)$	Approximation	Absolute Error
0	$u_1(1)$	1.017300	1.173×10^{-2}
1	$u_2(1)$	1.017294	1.7294×10^{-2}
2	$u_3(1)$	1.017294	1.7294×10^{-2}
3	$u_4(1)$	1.017294	1.7294×10^{-2}

Figures 1 and 2 illustrate the convergence behaviour of the sequence $u_{\iota+1}(t) = \tilde{S}u_\iota(t) = t^2\sin(t) + \int_0^1 t\cos(u_\iota(s))ds$.

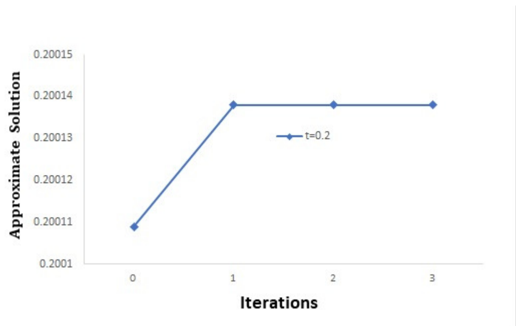

Figure 1

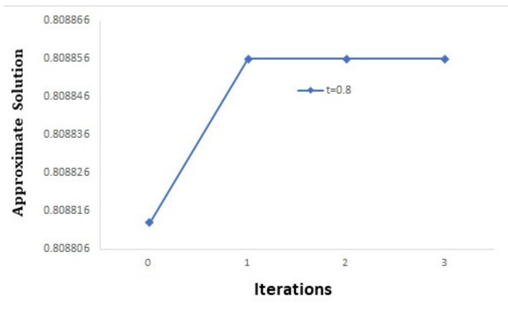

Figure 2

The graph shows that the sequence $u_{\iota+1}(t) = \tilde{S}u_\iota(t) = t^2\sin(t) + \int_0^1 t\cos(u_\iota(s))ds$ converges to the exact solution of $t = 0.2$.

The graph shows that the sequence $u_{\iota+1}(t) = \tilde{S}u_\iota(t) = t^2\sin(t) + \int_0^1 t\cos(u_\iota(s))ds$ converges to the exact solution of $t = 0.8$.

6. Conclusion

Jlelli and Samet [17] presented the idea of θ-contraction in an attempt to illustrate how the Banach contraction principle was improved in terms of the existence of fixed points in generalized metric space. In the context of complete metric space, Karapinar and Fulga [19] merged the concepts of interpolative type contraction and Jaggi type contraction to create a novel hybrid contraction. A novel concept known as Jaggi-type hybrid (θ-ϕ)-contraction has been established, in accordance with [17, 19]. Some fixed point results for such mappings in the context of generalized metric space were reported in this work. In terms of application, one of the inferred corollaries was used to prove the existence and uniqueness of solutions to nonlinear Volterra-Fredholm integral equations.

The main idea of this paper being presented in the context of Branciari-type metric space is, indeed, fundamental. Therefore, the proposed concepts herein can be extended to some other framework such as multivalued mappings, fuzzy mappings, intuitionist fuzzy set-valued mappings, soft sets, and so on. The underlying set can also be considered in some non-classical spaces such as fuzzy metric spaces with related applications.

Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Authors' contributions

Conceptualization: Mohammed Shehu Shagari, Maha Noorwali, and Irshad Ayoob; formal analysis: Nabil Mlaiki and Paul Oloche; investigation: Maha Noorwali, Irshad Ayoob, and Mohammed Shehu Shagari; Methodology: Nabil Mlaiki and Maha Noorwali; Writing-review and editing: Mohammed Shehu Shagari, Paul Oloche, and Nabil Mlaiki.

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