J. Math. Computer Sci., 41 (2026), 284–306

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

The study of the product t-norm in the compositional rule of inference with various implications

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Abstract

Fuzzy Inference Systems (FIS) are used to help people to take decisions in complex situations or when a human expert is needed. Their particularity is that they can manage the imprecision and vagueness of knowledge by applying approximate reasoning. The main approach of approximate reasoning is the Compositional Rule of Inference (CRI), whose definition contains two operators as parameters: a t-norm and a fuzzy implication. However, since its creation, the fuzzy community considers only one combination of (t-norm, implication) in fuzzy applications, which is (min, min). For that, we are interested in studying the behavior of other combinations (t-norm, implication) and in checking their efficiency. In this paper, we combine the product t-norm with fifteen implications in the CRI. Then, for every combination, we check the satisfaction of the axiomatics of approximate reasoning. This axiomatics is a set of criteria that model human intuitions. This study allows us to identify the best combinations that coincide with human reasoning in order to guarantee an inference result close to the expert's opinion.

Keywords: Compositional rule of inference, fuzzy logic, approximate reasoning, product t-norm, fuzzy implications.

2020 MSC: 03B52, 68T37.

1. Introduction

Fuzzy logic [52] was introduced by Zadeh, inspired by the human mind in the way he evaluates knowledge. Unlike classical logic, fuzzy logic is able to infer results from imprecise concepts. Since, many intelligent systems were extended to be based on fuzzy logic, as fuzzy inference systems [19, 23], the successors of expert systems. We can find in the literature an infinity of successful applications of fuzzy inference systems in various domains like medicine [1, 7, 20], geology [2, 34], ecology [25], supply chain network [4, 31], manufacturing [37], intelligent transport [41, 48], networks [3] etc. Nowadays, with the immersion of big data, the fuzzy community prefers to generate rule bases using neural networks instead of getting them from the expert [46], which results in a revival of fuzzy inference systems. We can cite, for example, Adaptive Neuro-Fuzzy Inference System (ANFIS) whose applications are countable [15, 30, 49].

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doi: 10.22436/jmcs.041.03.01

Received: 2024-06-24 *Revised:* 2025-07-20 *Accepted:* 2025-08-20

In fuzzy logic, approximate reasoning [53] is used to have a process similar to human reasoning when situations are complex and vague. It is included in the inference engine of fuzzy inference systems to solve problems that do not have a precise solution [6]. More precisely, approximate reasoning is the process that, from an imprecise rule and a specific observation, allows deducing a possible imprecise conclusion [11, 32, 36, 38]. The first and principal approach of approximate reasoning is the compositional rule of inference (CRI) [13], whose general definition contains two operators: a t-norm and a fuzzy implication. However, since the proposition of the CRI, all the works relating to fuzzy inference systems adopt the Mamdani method or the Sugeno method, and both of these methods use the operator *Min* as t-norm and implication at the same time. This is due to the fact that this combination gives satisfactory results and is fast in terms of algorithmic complexity. Nevertheless, it is demonstrated in [29] that using *Min* operator as a t-norm and implication does not fit the axiomatics of approximate reasoning and especially the classical Modus Ponens. We are convinced, therefore, that studying other combinations of (t-norm, implication) could improve the inference mechanism in fuzzy inference systems, and could affect their performance, especially with the occurrence of big data.

In the literature, we can find some works performed by Mizumoto et al. [27–29] that studied some combinations of (t-norm, implication). Indeed, they tested three t-norms (min t-norm [29], drastic t-norm [28] and Lukasiewicz t-norm [27]), and combined them with fifteen fuzzy implications. After, they verified for each combination if it satisfies the axiomatics of approximate reasoning. This axiomatics was proposed by Fukami et al. [13] as a set of criteria that model human intuitions. According to the realized study, some combinations do not satisfy all the requested criteria.

According to our literature search, we noticed that the combinations treated before in the compositional rule of inference do not cover the product t-norm. For that, we proposed in [54] a study of the product t-norm with four implications in the compositional rule of inference. In this work, we expand our study and we combine it with fifteen implications that were considered before in the works of Mizumoto et al. [27–29]. By using the product t-norm, our aim is to demonstrate when we can get reasonable consequences in the compositional rule of inference that coincides with human intuitions. We show the results obtained from different combinations using the product t-norm, and we verify the satisfaction of the axiomatics of approximate reasoning. After that, we give the combinations that we judge the best from the ones treated. This work, associated with the work of Mizumoto et al. [27–29], will form a complete guide to help the developers of fuzzy inference systems in the choice of the suitable parameters (t-norm and implication). Indeed, it is primordial to use compatible couples to get the best results and actions that are close to the decision that could be given by the expert.

The remainder of this paper is organized as follows. In Section 2, we present the compositional rule of inference, and we illustrate the combinations studied before in the literature. In Section 3, we present our study about the product t-norm in the compositional rule of inference in combination with various implications, and we check the verification of approximate reasoning axiomatics. Section 4 is for comparing and discussing the obtained results. Finally, the paper ends with Section 5 devoted to a conclusion and some perspectives.

2. Preliminary considerations

In this section, we start by defining the approximate reasoning, then the compositional rule of inference and its components. After that, we show a synthesis of the combinations studies (t-norm, implication) treated in the literature.

2.1. Compositional rule of inference

Approximate reasoning [18, 22, 43] is a process used in fuzzy inference systems to give a result from imprecise data. It aims to be similar to human reasoning in complex situations, where classical Modus Ponens does not solve them. For that, approximate reasoning is based on a pattern of reasoning called

The Generalized Modus Ponens (GMP) [24]. Its principle is to allow inferring using an observation and a rule's premise which are different. The GMP has the following form:

$$\begin{array}{ll} \textbf{Ant 1}: & \text{If } X \text{ is } A \text{ then } Y \text{ is } B \\ \textbf{Ant 2}: & X \text{ is } A' \\ \hline \textbf{Cons}: & \qquad\qquad Y \text{ is } B' \end{array}$$

where X and Y are linguistic variables, A and A' are fuzzy sets belonging to the universe of discourse U, B and B' are fuzzy sets belonging to the universe of discourse V. To obtain the result B', Zadeh proposed the Compositional Rule of Inference (CRI)[13], which is the principal method of approximate reasoning [32]. From that, the inference conclusion B' is determined as follows:

$$\forall v \in V \mu_{B'}(v) = \sup_{u \in U} T(\mu_{A'}(u), I(\mu_A(u), \mu_B(v))), \tag{2.1}$$

where $\mu_A(u)$, $\mu_{A'}(u)$, $\mu_B(v)$, and $\mu_{B'}(v)$ are membership functions of the fuzzy sets A, A', B, and B' respectively, T is a t-norm and I is a fuzzy implication.

Definition 2.1. Triangular norm (or t-norm for short) [5, 16] is an extension of the classical Boolean conjunction connective. It is a binary function $[0, 1] \times [0, 1] \to [0, 1]$, used to aggregate truth degrees of a conjunction of propositions, having the following properties:

- $T(u, v) = T(v, u)$ (commutativity);
- $T(T(u, v), z) = T(u, T(v, z))$ (associativity);
- $T(u, v) \leqslant T(u, z)$, whenever $v \leqslant z$ (monotonicity);
- $T(u, 1) = u$ (boundary condition).

Some of the most known t-norms are in Table 1 ([5]).

Table 1: The most known t-norms.

Notation	Name	Function
$\wedge$	Zadeh	$\min(u, v)$
$\odot$	Lukasiewicz	$\max(u + v - 1, 0)$
$\cdot$	Goguen	$u.v$
$\dot{\wedge}$	Drastic	$\begin{cases} u, & \text{if } v = 1, \\ v, & \text{if } u = 1, \\ 0, & \text{else,} \end{cases}$

Definition 2.2. A fuzzy implication operator [51] is defined as a binary operation on $[0, 1]$, which extends the boolean implication, and which can be expressed as function of connectives $\vee$, $\wedge$ and negation. It is a binary operation $[0, 1] \times [0, 1] \to [0, 1]$ having the following properties [47]:

- $I(0, 0) = 1$;
- $I(0, 1) = 1$;
- $I(1, 1) = 1$;
- $I(1, 0) = 0$.

The Table 2 illustrates fuzzy implications found in the literature [9, 29, 35, 50] [1]. All these implications verify the properties cited above except Mamdani operator, but it has been successfully used in fuzzy systems [26].

[1]We should mention that the implications I_{sg}, I_{gs}, I_{gg}, and I_{ss} have different definitions in [17]. But in this paper, we use those considered in Mizumoto's work [27–29].

Table 2: List of the used implications.

Name	Notation	Function
Zadeh	I_m	$\max(1-u, \min(u, v))$
Lukasiewicz	I_a	$\min(1-u+v, 1)$
Mamdani	I_c	$\min(u, v)$
Rescher-Gaines	I_s	$\begin{cases} 1, & \text{if } u \leqslant v, \\ 0, & \text{else,} \end{cases}$
Brouwer-Gödel	I_g	$\begin{cases} 1, & \text{if } u \leqslant v, \\ v, & \text{else,} \end{cases}$
Kleene-Dienes	I_b	$\max(1-u, v)$
Fukami	I_{sg}	$\min(I_s(u,v), I_g(1-u, 1-v))$
	I_{gg}	$\min(I_g(u,v), I_g(1-u, 1-v))$
	I_{gs}	$\min(I_g(u,v), I_s(1-u, 1-v))$
	I_{ss}	$\min(I_s(u,v), I_s(1-u, 1-v))$
Mizumoto	$I_{\triangle}$	$\begin{cases} 1, & \text{if } u \leqslant v, \\ v/u, & \text{else,} \end{cases}$
	$I_{\blacktriangle}$	$\begin{cases} \min(1, \frac{v}{u}, \frac{1-u}{1-v}), & \text{if } u > 0 \text{ and } 1-v > 0, \\ 1, & \text{if } u = 0 \text{ or } 1-v = 0, \end{cases}$
	$I_{\bigstar}$	$1-u+u.v$
	$I_{\#}$	$\max(\min(u,v), \min(1-u, 1-v), \min(v, 1-u))$
	$I_{\square}$	$\begin{cases} 1, & \text{if u< 1 or v=1,} \\ 0, & \text{if u=1 and v<1,} \end{cases}$

2.2. *Studies of* (T, I) *combinations with CRI*

Fuzzy inference systems that use the compositional rule of inference as reasoning method obtain different results according to the chosen parameters, which are the t-norm and the fuzzy implication of formula (2.1). For that, the compatibility of the combinations of t-norm and implication in the CRI has been studied in many works. Indeed, the authors in [8, 10, 17, 33] were interested in the CRI using a t-norm T and its associated implication I_T. They demonstrated that in this case, the Modus Ponens is always satisfied. In [12], the authors added that, when using a t-norm and its associated implication, the conclusion B' cannot be less restrictive than B, i.e., $B' \supseteq B$, even when $A' \subseteq A$. Jenei [14] proved, by two metrics of distance (the uniform metric and the Hausdorff metric), the continuity of the CRI using an Archimedean t-norm and its associated residual fuzzy implication. In papers [40] and [39], the authors study the CRI in a different manner, they considered it as a system of fuzzy relation equation. For that, they use two implications which are the product operator and the residual implication. They try to find sufficient and necessary conditions to solve the system. In the same direction, authors in [44, 45], and by using crisp inputs in the CRI, they employ the modifiers “at-least” and “at-most” in the rule base to guarantee the monotonicity of the resulting function.

Others works [9, 21, 42] focused on fuzzy control systems, and tried to evaluate the results when changing their parameters, which are the implication, the t-norm, the t-conorm and the defuzzification operator. They aim to check whether it is possible to get the same or approximately the same crisp results when defuzzification is applied. Li et al. [21] defined this problem by *system function*, where the objective is to obtain a function $v_0 = G(u_0)$, $u_0 \in U$ is the crisp input of the system and $v_0 \in V$ is the crisp output. In this context, the CRI has only one parameter, which is the implication operator. This is due to the fact that in fuzzy control systems, the input is crisp: $\mu_A(u_0) = 1$ and $\mu_A(u) = 0$, $\forall u \neq u_0$. Thus, the CRI is simplified to be resolved by the function $\forall v \in V, \mu_{B'}(v) = I(\mu_A(u_0), \mu_B(v))$, since 1 is the neutral element of t-norms. In these works, the parameter t-norm is used for connecting propositions in conjunctive rules.

Some criteria that CRI should verify were defined by Fukami et al. [13]. They aim to formalize human reasoning in order to assign to intelligent systems a natural reasoning, and thereby, to guarantee a result that will be similar to human induction. According to this axiomatics, the membership value of B' depends on the relation between A and B and on the membership value of A'. The criteria are the following:

$$\text{Criterion 1 (C1): } A' = A \Rightarrow B' = B \text{ (modus ponens)}, \tag{2.2}$$
$$\text{Criterion 2-1 (C2-1): } A' = \text{ very } A \Rightarrow B' = \text{ very } B, \tag{2.3}$$
$$\text{Criterion 2-2 (C2-2): } A' = \text{ very } A \Rightarrow B' = B, \tag{2.4}$$
$$\text{Criterion 3 (C3): } A' = \text{ more or less } A \Rightarrow B' = \text{ more or less } B, \tag{2.5}$$
$$\text{Criterion 4-1 (C4-1): } A' = \text{ not } A \Rightarrow B' = \text{ unknown}, \tag{2.6}$$
$$\text{Criterion 4-2 (C4-2): } A' = \text{ not } A \Rightarrow B' = \text{ not } B, \tag{2.7}$$

where "A' is very A" means that $A' = A^2$, "A' is more or less A" is for $A' = A^{0.5}$ and "A' is not A' expresses that $A' = 1 - A$. From these criteria, four types of approximate reasoning are defined depending on the criteria that they satisfy. The types are the following [13]:

- Type 1: criteria C1, C2-1, C3, and C4-1;
- Type 2: criteria C1, C2-2, C3, and C4-1;
- Type 3: criteria C1, C2-1, C3, and C4-2;
- Type 4: criteria C1, C2-2, C3, and C4-2.

Mizumoto has tested the satisfaction of the criteria (2.2)-(2.7) using min t-norm [29], drastic t-norm [28], and Lukasiewicz t-norm [27] with various implications. In what follows, we will expose these studies.

Table 3: The result B' with $\wedge$ t-norm [29].

Relation	A	Very A	More or less A	not A
I_m	$0.5 \vee \mu_B(v)$	$\frac{3-\sqrt{5}}{2} \vee \mu_B(v)$	$\frac{\sqrt{5}-1}{2} \vee \mu_B(v)$	1
I_a	$\frac{1+\mu_B(v)}{2}$	$\frac{3+2\mu_B(v)-\sqrt{5+4\mu_B(v)}}{2}$	$\frac{\sqrt{5+4\mu_B(v)}-1}{2}$	1
I_c	$\mu_B(v)$	$\mu_B(v)$	$\mu_B(v)$	$0.5 \wedge \mu_B(v)$
I_s	$\mu_B(v)$	$\mu_B^2(v)$	$\sqrt{\mu_B(v)}$	1
I_g	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	1
I_{sg}	$\mu_B(v)$	$\mu_B^2(v)$	$\sqrt{\mu_B(v)}$	$1-\mu_B(v)$
I_{gg}	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	$1-\mu_B(v)$
I_{gs}	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	$1-\mu_B(v)$
I_{ss}	$\mu_B(v)$	$\mu_B^2(v)$	$\sqrt{\mu_B(v)}$	$1-\mu_B(v)$
I_b	$0.5 \vee \mu_B(v)$	$\frac{3-\sqrt{5}}{2} \vee \mu_B(v)$	$\frac{\sqrt{5}-1}{2} \vee \mu_B(v)$	1
$I_\triangle$	$\sqrt{\mu_B(v)}$	$\mu_B(v)$	$\mu_B^{1/3}(v)$	1
$I_\blacktriangle$	$\sqrt{\mu_B(v)} \wedge \frac{1}{2-\mu_B(v)}$	$\mu_B(v)^{2/3} \wedge \left[\frac{\sqrt{5-4\mu_B(v)}-1}{2(1-\mu_B(v))}\right]^2$	$\mu_B(v)^{1/3} \wedge \frac{\sqrt{\mu_B(v)^2-2\mu_B(v)+5}+\mu_B(v)-1}{2}$	1
$I_\star$	$\frac{1}{2-\mu_B(v)}$	$\left[\frac{\mu_B(v)-1+\sqrt{(1-\mu_B(v))^2+4}}{2}\right]^2$	$\frac{\sqrt{5-4\mu_B(v)}-1}{2(1-\mu_B(v))}$	1
$I_\#$	$0.5 \vee \mu_B(v)$	$\frac{3-\sqrt{5}}{2} \vee \mu_B(v)$	$\mu_B(v) \vee \left[(1-\mu_B(v)) \wedge \frac{\sqrt{5}-1}{2}\right]$	$\mu_B(v) \vee (1-\mu_B(v))$
$I_\square$	1	1	1	1

Mizumoto and Zimmermann [29] studied the use of min t-norm in the compositional rule of inference. They combined this t-norm with the fifteen implications of Table 2. From this study, they found that only some combinations satisfy the criteria (2.2)-(2.7). Table 3 shows the result of each combination in all the considered cases. The implications that give a reasonable consequence and satisfy the criteria when they are combined with min t-norm are: I_s, I_g, I_{sg}, I_{gg}, I_{gs}, and I_{ss}. Nevertheless, the combinations with the other implications do not verify the principal criteria and cannot be classified according to the approximate reasoning typology.

Mizumoto [28] had studied the drastic t-norm in the compositional rule of inference. Combined with the same set of implications, it gave the results shown in Table 4. The combinations satisfying the criteria are with the implications: I_a, I_s, I_g, I_{sg}, I_{gg}, I_{gs}, I_{ss}, $I_\triangle$, $I_\blacktriangle$. For the rest of the implications, they do not satisfy all the criteria when they are combined with the drastic t-norm.

Table 4: The result B' with $\wedge\!\!\!\!\wedge$ t-norm [28].

Relation	A	Very A	More or less A	not A
I_m	$\mu_B(v)$	$\mu_B(v)$	$\mu_B(v)$	1
I_a	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	1
I_c	$\mu_B(v)$	$\mu_B(v)$	$\mu_B(v)$	$\varnothing$
I_s	$\mu_B(v)$	$\mu_B^2(v)$	$\sqrt{\mu_B(v)}$	1
I_g	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	1
I_{sg}	$\mu_B(v)$	$\mu_B^2(v)$	$\sqrt{\mu_B(v)}$	1-$\mu_B(v)$
I_{gg}	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	1-$\mu_B(v)$
I_{gs}	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	1-$\mu_B(v)$
I_{ss}	$\mu_B(v)$	$\mu_B^2(v)$	$\sqrt{\mu_B(v)}$	1-$\mu_B(v)$
I_b	$\mu_B(v)$	$\mu_B(v)$	$\mu_B(v)$	1
$I_\triangle$	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	1
$I_\blacktriangle$	$\mu_B(v)$	$\mu_B^2(v)$	$\sqrt{\mu_B(v)}$	1
$I_\bigstar$	$\mu_B(v)$	$\mu_B(v)$	$\mu_B(v)$	1
$I_\#$	$\mu_B(v)$	$\mu_B(v)$	$\mu_B(v)$	$\mu_B(v) \vee (1-\mu_B(v))$
$I_\square$	1	1	1	1

In [27], Mizumoto had used the t-norm of Lukasiewicz in the compositional rule of inference combined with the same fifteen implications. We cite the results of the study in Table 5. From the obtained results, we see that the combinations satisfying the criteria are with the following implications: I_s, I_g, I_{sg}, I_{gg}, I_{gs}, I_{ss}, $I_\triangle$, $I_\blacktriangle$. However, the combinations with the other implications do not check approximate reasoning axiomatics.

Table 5: The result B' with $\odot$ t-norm [27].

Relation	A	Very A	More or less A	not A
I_m	$\mu_B(v)$	$\mu_B(v)$	$\frac{1}{4} \vee \mu_B(v)$	1
I_a	$\mu_B(v)$	$\mu_B(v)$	$\begin{cases} \mu_B(v)+\frac{1}{4}, & \text{if } \mu_B(v) \leqslant \frac{1}{4}, \\ \sqrt{\mu_B(v)}, & \text{else,} \end{cases}$	1
I_c	$\mu_B(v)$	$\mu_B(v)$	$\mu_B(v)$	$\varnothing$
I_s	$\mu_B(v)$	$\mu_B^2(v)$	$\sqrt{\mu_B(v)}$	1
I_g	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	1
I_{sg}	$\mu_B(v)$	$\mu_B^2(v)$	$\sqrt{\mu_B(v)}$	1-$\mu_B(v)$
I_{gg}	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	1-$\mu_B(v)$
I_{gs}	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	1-$\mu_B(v)$
I_{ss}	$\mu_B(v)$	$\mu_B^2(v)$	$\sqrt{\mu_B(v)}$	1-$\mu_B(v)$
I_b	$\mu_B(v)$	$\mu_B(v)$	$\frac{1}{4} \vee \mu_B(v)$	1
$I_\triangle$	$\mu_B(v)$	$\mu_B(v)$	$\sqrt{\mu_B(v)}$	1
$I_\blacktriangle$	$\mu_B(v)$	$\mu_B^2(v)$	$\sqrt{\mu_B(v)}$	1
$I_\bigstar$	$\mu_B(v)$	$\mu_B(v)$	$\begin{cases} \frac{1}{4(1-\mu_B(v))}, & \text{if } \mu_B(v) \leqslant \frac{1}{2}, \\ \mu_B(v), & \text{else,} \end{cases}$	1
$I_\#$	$\mu_B(v)$	$\mu_B(v)$	$\frac{1}{4} \vee \mu_B(v)$	$\mu_B(v) \vee (1-\mu_B(v))$
$I_\square$	1	1	1	1

Table 6 represents the satisfaction or not of the cited combinations above, where $(\surd)$ means that the criterion is satisfied by the combination, and $(\times)$ means that the criterion is not satisfied. As we can see in Table 6, whatever the t-norm, some implications always verify the axiomatics, which are I_s, I_g, I_{gg}, I_{gs}, and I_{ss}. Also, we remark that each implication of this set always keeps the same type whatever the t-norm. Other implications verify the axiomatics but not with all the t-norms, like $I_\triangle$ and $I_\blacktriangle$ which are compatible with the Lukasiewicz t-norm and the drastic t-norm, and I_a which is compatible with the drastic t-norm. For the remaining implications, they check some criteria, but cannot be classified.

Table 6: The satisfaction of the criteria by the combinations (t-norm,implication).

t-norm	Criteria	I_m	I_a	I_c	I_s	I_g	I_{sg}	I_{gg}	I_{gs}	I_{ss}	I_b	$I_\triangle$	$I_\blacktriangle$	$I_\bigstar$	$I_\#$	$I_\square$
Min $(\wedge)$	C1	×	×	√	√	√	√	√	√	√	×	×	×	×	×	×
	C2-1	×	×	×	√	×	√	×	×	√	×	×	×	×	×	×
	C2-2	×	×	√	×	√	×	√	√	×	×	×	×	×	×	×
	C3	×	×	×	√	√	√	√	√	√	×	×	×	×	×	×
	C4-1	√	√	×	√	√	×	×	×	×	√	√	√	√	×	√
	C4-2	×	×	×	×	×	√	√	√	√	×	×	×	×	×	×
Type		-	-	-	1	2	3	4	4	3	-	-	-	-	-	-
Lukasiewicz $(\odot)$	C1	√	√	√	√	√	√	√	√	√	√	√	√	√	√	×
	C2-1	×	×	×	√	×	√	×	×	√	×	×	√	×	×	×
	C2-2	√	√	√	×	√	×	√	√	×	√	√	×	√	√	×
	C3	×	×	×	√	√	√	√	√	√	×	√	√	×	×	×
	C4-1	√	√	×	√	√	×	×	×	×	√	√	√	√	×	√
	C4-2	×	×	×	×	×	√	√	√	√	×	×	×	×	×	×
Type		-	-	-	1	2	3	4	4	3	-	2	1	-	-	-
Drastic $(\dot\wedge)$	C1	√	√	√	√	√	√	√	√	√	√	√	√	√	√	×
	C2-1	×	×	×	√	×	√	×	×	√	×	×	√	×	×	×
	C2-2	√	√	√	×	√	×	√	√	×	√	√	×	√	√	×
	C3	×	√	×	√	√	√	√	√	√	×	√	√	×	×	×
	C4-1	√	√	×	√	√	×	×	×	×	√	√	√	√	×	√
	C4-2	×	×	×	×	×	√	√	√	√	×	×	×	×	×	×
Type		-	2	-	1	2	3	4	4	3	-	2	1	-	-	-

3. Proposal method: CRI with product t-norm

To get a convenient inference result from fuzzy inference systems and to find the best couples (T,I) for the CRI, we need to test all the possible combinations. This work is a continuity of Mizumoto et al.'s works [27–29], where we focus on the combinations that were not treated by him. In this work, we shall use the product t-norm in the compositional rule of inference. We combine it with various implications to see the satisfaction of the criteria (2.2)-(2.7). For that, we consider the fifteen implications treated by Mizumoto et al. [27–29] (see Table 2).

To get the membership function of the inference result B', the function of the CRI when the product t-norm is used is the following:

$$\forall v \in V \mu_{B'}(v) = \sup_{u \in U} \left(\mu_{A'}(u) \cdot I(\mu_A(u), \mu_B(v))\right), \tag{3.1}$$

where $\mu_A(u)$, $\mu_{A'}(u)$, $\mu_B(v)$, and $\mu_{B'}(v)$ are membership functions of the fuzzy sets A, A', B, and B' respectively. To take into consideration the modifications made by linguistic modifiers, we use a variable α with a value $\alpha = 1$ when A' is A ($A' = A^1$), $\alpha = 2$ when A' is very A ($A' = A^2$) and $\alpha = 0.5$ in the case where A' is more or less A ($A' = A^{0.5}$). The equation (3.1) becomes the following:

$$\forall v \in V \mu_{B'}(v) = \sup_{u \in U}(\mu_A^{\alpha}(u) \cdot I(\mu_A(u), \mu_B(v))).$$

In the CRI, we shall evaluate the inferred conclusions obtained by the couple $(\cdot, I)$. Our aim is to see the results obtained by each couple and to test the satisfaction of the criteria (2.2)-(2.7), using all the implications of Table 2. For this reason, we suppose two subsets U_1 and U_2 for U which satisfy the following conditions [13]:

$$U_1 \cup U_2 = U \text{ and } U_1 \cap U_2 = \varnothing,$$

$$\forall u \in U_1 \ \mu_A(u) \leqslant \mu_B(v), \tag{3.2}$$

$$\forall u \in U_2 \ \mu_A(u) > \mu_B(v). \tag{3.3}$$

3.1. I_s implication

We begin by studying the implication I_s with the product t-norm. Let us note that the following demonstrations are applicable also for the other t-norms. The membership function of the result B', using the implication I_s in the CRI, is obtained as follows:

$$B'_s = \int_V \sup_{u \in U}(\mu_{A'}(u) \cdot I_s(\mu_A(u), \mu_B(v)))/v = \int_V \bigvee_{u \in U} \mu_{A'}(u) \cdot \begin{cases} 1, & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ 0, & \text{else.} \end{cases} /v.$$

With the defined subsets U_1 and U_2 of (3.2) and (3.3), B'_s becomes:

$$B'_s = \int_V \bigvee_{u \in U_1} (\mu_{A'}(u) \cdot 1) \vee \bigvee_{u \in U_2} (\mu_{A'} \cdot 0)/v.$$

This gives

$$B'_s = \int_V \bigvee_{u \in U_1} \mu_{A'}(u)/v. \tag{3.4}$$

Using this equation, we verify the satisfaction of the criteria C1, C2-1, C2-2, C3, C4-1, and C4-2 by the combination $(\cdot, I_s)$.

Theorem 3.1. *For the implication I_s, when combining it with product t-norm, the criteria C1, C2-1, and C3 are satisfied.*

Proof. The criteria C1, C2-1 (and C2-2), and C3 consider that $A' = A$, $A' = A^2$ or $A' = A^{0.5}$, respectively. Consider α as the power of the observation $A' = A^{\alpha}$, so $\alpha = 1, 2$ or 0.5. We have

$$B'_s = \int_V \bigvee_{u \in U_1} \mu_A^{\alpha}(u)/v.$$

The fuzzy set A is supposed to be normal, so its membership function takes any value in $[0, 1]$. For that and from condition (3.2), the maximum value of $\mu_A(u)$ when $u \in U_1$ will not exceed $\mu_B(v)$. Thus, $\bigvee_{u \in U_1} \mu_A(u) = \mu_B(v)$, which gives $\bigvee_{u \in U_1} \mu_A^{\alpha}(u) = \mu_B^{\alpha}(v)$. We deduce from equation (3.4) that

$$B'_s = \int_V \mu_B^{\alpha}(v)/v = B^{\alpha}.$$

So for the couple $(\cdot, I_s)$, the consequence B' is equal to B, very B and more or less B, when A' is A, very A and more or less A, respectively. $\square$

Theorem 3.2. *When A' is equal to not A, the criterion C4-1 is satisfied by the combination $(\cdot, I_s)$.*

Proof. For the case of $A' = 1 - A$, we have from equation (3.4):

$$B'_s = \int_V \bigvee_{u \in U_1} (1 - \mu_A(u))/v.$$

Using condition (3.2), we can see that the maximum value of the expression $(1 - \mu_A(u))$ when $u \in U_1$ is 1, and is reached when $\mu_A(u) = 0$. So we get $B'_s = \int_V 1/v$. We deduce that the result B'_s of the CRI, when A' is not A, is unknown. □

3.2. I_g implication

To study the satisfaction of the criteria (2.2)-(2.7) with the implication I_g, the membership function of the result B' of the CRI is defined as

$$\begin{aligned} B'_g &= \int_V \sup_{u \in U} (\mu_{A'}(u) \cdot I_g(\mu_A(u), \mu_B(v)))/v \\ &= \int_V \bigvee_{u \in U} \mu_{A'}(u) \cdot \left\{ \begin{array}{ll} 1 & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ \mu_B(v) & \text{else.} \end{array} \right. /v = \int_V \bigvee_{u \in U_1} \mu_{A'}(u).1 \vee \bigvee_{u \in U_2} (\mu_{A'}(u) \cdot \mu_B(v))/v. \end{aligned}$$

So, the obtained B'_g is

$$B'_g = \int_V \bigvee_{u \in U_1} \mu_{A'}(u) \vee \bigvee_{u \in U_2} (\mu_{A'}(u) \cdot \mu_B(v))/v. \tag{3.5}$$

Theorem 3.3. *Using the implication I_g with the product t-norm, the criteria C1, C2-2, and C3 are satisfied.*

Proof. To test the satisfaction of the criteria C1, C2-1, C2-2, and C3, we use α as 1, 2, and 0.5 for the power of the observation $A' = A^\alpha$. For that, from equation (3.5), we have

$$B'_g = \int_V \bigvee_{u \in U_1} \mu_A^\alpha(u) \vee \bigvee_{u \in U_2} (\mu_A^\alpha(u) \cdot \mu_B(v))/v.$$

With condition (3.2), the maximum value of $\mu_A(u)$ is $\mu_B(v)$ when $u \in U_1$. So, $\bigvee_{u \in U_1} \mu_A(u) = \mu_B(v)$, which gives $\bigvee_{u \in U_1} \mu_A^\alpha(u) = \mu_B^\alpha(v)$. Consequently:

$$B'_g = \int_V \mu_B^\alpha(v) \vee \bigvee_{u \in U_2} (\mu_A^\alpha(u) \cdot \mu_B(v))/v.$$

Also, from condition (3.3), when $u \in U_2$ the maximum value of $\mu_A(u)$ is 1, since A is a normal fuzzy set. This implies that $\bigvee_{u \in U_2} \mu_A(u) = 1$, and consequently $\bigvee_{u \in U_2} \mu_A^\alpha(u) = 1$. So, we can deduce that $\bigvee_{u \in U_2} \mu_A^\alpha(u) \cdot \mu_B(v)$ is equal to $\mu_B(v)$. So we get

$$B'_g = \int_V \mu_B^\alpha(v) \vee \mu_B(v)/v.$$

The result in this case depends on α, so we have two cases. The first case is when $\alpha \geqslant 1$, and since $\mu_B(v) \in [0, 1]$, we get $\mu_B^\alpha(v) \leqslant \mu_B(v)$, and consequently $\mu_B^\alpha(v) \vee \mu_B(v) = \mu_B(v)$. The second case is when $\alpha < 1$, so $\mu_B^\alpha(v) > \mu_B(v)$, which gives $\mu_B^\alpha(v) \vee \mu_B(v) = \mu_B^\alpha(v)$. We deduce that

$$B'_g = \left\{ \begin{array}{ll} \int_V \mu_B(v)/v, & \text{if } \alpha \geqslant 1, \\ \int_V \mu_B^\alpha(v)/v, & \text{else.} \end{array} \right.$$

From this result, we can determine that when α is 1 or 2, the conclusion is $B'_g = B$. Then, when α is 0.5, we obtain $B'_g = \sqrt{B}$. We conclude that the combination $(\cdot, I_g)$ satisfies the criteria C1, C2-2, and C3 when A' is equal to A, A^2 and $A^{0.5}$, respectively. □

Theorem 3.4. *The couple* $(\cdot, I_g)$ *satisfy the criteria C4-1 when* $A' =$ *not* A.

Proof. In the case of $A' = 1 - A$, we obtain from equation (3.5):

$$B'_g = \int_V \bigvee_{u \in U_1} (1 - \mu_A(u)) \vee \bigvee_{u \in U_2} ((1 - \mu_A(u)) \cdot \mu_B(v))/v.$$

The maximum value of the expression $(1 - \mu_A(u))$ when $u \in U_1$ (see condition (3.2)) is 1 and is reached when $\mu_A(u) = 0$. From that, we obtain:

$$B'_g = \int_V 1 \vee \bigvee_{u \in U_2} ((1 - \mu_A(u)) \cdot \mu_B(v))/v.$$

On the other hand, when $u \in U_2$ (see condition (3.3)), the maximum value of $(1 - \mu_A(u))$ is 0 and is reached when $\mu_A(u) = 1$. We deduce that $B'_g = \int_V 1/v$. Thus, in the CRI, using the product t-norm combined with I_g implication, the conclusion B' is unknown when A' is not A. □

3.3. I_{sg} *implication*

In order to study the satisfaction of the criteria (2.2)-(2.7), we combine the product t-norm with I_{sg} implication (see Table 2) in the CRI as follows:

$$\begin{aligned} B'_{sg} &= \int_V \sup_{u \in U} (\mu_{A'}(u) \cdot I_{sg}(\mu_A(u), \mu_B(v)))/v \\ &= \int_V \sup_{u \in U} [\mu_{A'}(u) \cdot \min(I_s(\mu_A(u), \mu_B(v)), I_g(1 - \mu_A(u), 1 - \mu_B(v)))]/v. \end{aligned}$$

On one hand, we have:

$$I_s(\mu_A(u), \mu_B(v)) = \begin{cases} 1, & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ 0, & \text{else.} \end{cases}$$

On the other hand:

$$I_g(1 - \mu_A(u), 1 - \mu_B(v)) = \begin{cases} 1, & \text{if } 1 - \mu_A(u) \leqslant 1 - \mu_B(v), \\ 1 - \mu_B(v), & \text{else.} \end{cases}$$

Since $1 - \mu_A(u) \leqslant 1 - \mu_B(v)$ is equivalent to $\mu_A(u) \geqslant \mu_B(v)$, we obtain

$$I_g(1 - \mu_A(u), 1 - \mu_B(v)) = \begin{cases} 1 - \mu_B(v), & \text{if } \mu_A(u) < \mu_B(v), \\ 1, & \text{else.} \end{cases}$$

So, the expression of B'_{sg} becomes:

$$\begin{aligned} B'_{sg} &= \int_V \bigvee_{u \in U} \mu_{A'}(u) \cdot \begin{cases} \min(1, 1 - \mu_B(v)), & \text{if } \mu_A(u) < \mu_B(v), \\ \min(1, 1), & \text{if } \mu_A(u) = \mu_B(v), \\ \min(0, 1), & \text{else.} \end{cases} /v \\ &= \int_V \bigvee_{u \in U} \mu_{A'}(u) \cdot \begin{cases} 1 - \mu_B(v), & \text{if } \mu_A(u) < \mu_B(v), \\ 1, & \text{if } \mu_A(u) = \mu_B(v), \\ 0, & \text{else.} \end{cases} /v \\ &= \int_V \bigvee_{\mu_A(u) < \mu_B(v)} [\mu_{A'}(u) \cdot (1 - \mu_B(v))] \vee \bigvee_{\mu_A(u) = \mu_B(v)} \mu_{A'}(u) \vee \bigvee_{\mu_A(u) > \mu_B(v)} [\mu_{A'}(u) \cdot 0]/v \\ &= \int_V \bigvee_{\mu_A(u) < \mu_B(v)} [\mu_{A'}(u) \cdot (1 - \mu_B(v))] \vee \bigvee_{\mu_A(u) = \mu_B(v)} \mu_{A'}(u)/v. \end{aligned}$$

Theorem 3.5. *The criteria C1, C2-1, and C3 are satisfied by the couple* $(\cdot, I_{sg})$ *when* $A' = A$, A^2, *and* $A^{0.5}$, *respectively.*

Proof. When α is equal to 1, 2, and 0.5, we get

$$\begin{aligned} B'_{sg} &= \int_V \bigvee_{\mu_A(u)<\mu_B(v)} [\mu_A^\alpha(u) \cdot (1-\mu_B(v))] \vee \bigvee_{\mu_A(u)=\mu_B(v)} \mu_A^\alpha(u)/v \\ &= \int_V \bigvee_{\mu_A(u)<\mu_B(v)} [\mu_A^\alpha(u) \cdot (1-\mu_B(v))] \vee \mu_B^\alpha(v). \end{aligned}$$

When $\mu_A(u) < \mu_B(v)$, we will find that $\bigvee \mu_A^\alpha(u) < \mu_B^\alpha(v)$, and since $1-\mu_B(v) < 1$, we obtain $\bigvee \mu_A^\alpha(u) \cdot (1-\mu_B(v)) < \mu_B^\alpha(v)$. We deduce that:

$$B'_{sg} = \int_V \mu_B^\alpha(v)/v = B^\alpha.$$

□

Theorem 3.6. *The criteria C4-2 is satisfied by the combination* $(\cdot, I_{sg})$ *when* $A' = 1-A$.

Proof. For the two criteria C4-1 and C4-2, we consider that A' is not A. So, we have:

$$\begin{aligned} B'_{sg} &= \int_V \bigvee_{\mu_A(u)<\mu_B(v)} [(1-\mu_A(u)) \cdot (1-\mu_B(v))] \vee \bigvee_{\mu_A(u)=\mu_B(v)} (1-\mu_A(u))/v \\ &= \int_V \bigvee_{\mu_A(u)<\mu_B(v)} [(1-\mu_A(u)) \cdot (1-\mu_B(v))] \vee (1-\mu_B(v))/v. \end{aligned}$$

As $1-\mu_A(u) \leqslant 1$ and $1-\mu_B(v) \leqslant 1$, we always have $(1-\mu_A(u)) \cdot (1-\mu_B(v)) \leqslant (1-\mu_B(v))$. We obtain:

$$B'_{sg} = \int_V 1-\mu_B(v)/v = 1-B.$$

Combining the product t-norm with I_{sg} implication in the CRI, the conclusion B'_{sg} is equal to not B when A' is equal not A. □

Following the same demonstration, we get that the couple $(\cdot, I_{ss})$ also satisfy the criteria C1, C2-1, C3, and C4-2.

3.4. I_{gs} *implication*

The CRI equation, using the product t-norm and the I_{gs} implication, is given as follows:

$$\begin{aligned} B'_{gs} &= \int_V \sup_{u\in U} (\mu_{A'}(u) \cdot I_{gs}(\mu_A(u), \mu_B(v)))/v \\ &= \int_V \sup_{u\in U} [\mu_{A'}(u) \cdot \min(I_g(\mu_A(u), \mu_B(v)), I_s(1-\mu_A(u), 1-\mu_B(v)))]/v. \end{aligned}$$

We have

$$I_g(\mu_A(u), \mu_B(v)) = \begin{cases} 1, & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ \mu_B(v), & \text{else.} \end{cases}$$

We have also:

$$I_s(1-\mu_A(u), 1-\mu_B(v)) = \begin{cases} 1, & \text{if } 1-\mu_A(u) \leqslant 1-\mu_B(v), \\ 0, & \text{else,} \end{cases}$$

which gives

$$I_s(1-\mu_A(u),1-\mu_B(v)) = \begin{cases} 0, & \text{if } \mu_A(u) < \mu_B(v), \\ 1, & \text{else.} \end{cases}$$

Thus, B'_{gs} becomes as follows:

$$\begin{aligned} B'_{gs} &= \int_V \bigvee_{u\in U} \mu'_A(u) \cdot \begin{cases} \min(1,0), & \text{if } \mu_A(u) < \mu_B(v), \\ \min(1,1), & \text{if } \mu_A(u) = \mu_B(v), \\ \min(\mu_B(v),1), & \text{else.} \end{cases} /v \\ &= \int_V \bigvee_{u\in U} \mu'_A(u) \cdot \begin{cases} 0, & \text{if } \mu_A(u) < \mu_B(v), \\ 1, & \text{if } \mu_A(u) = \mu_B(v), \\ \mu_B(v), & \text{else.} \end{cases} /v \\ &= \int_V \bigvee_{\mu_A(u)=\mu_B(v)} \mu_{A'}(u) \vee \bigvee_{\mu_A(u)>\mu_B(v)} (\mu_{A'}(u)\cdot\mu_B(v))/v. \end{aligned}$$

Theorem 3.7. *The combination $(\cdot, I_{gs})$ satisfies the criteria C1, C2-2, and C3.*

Proof. With I_{gs} implication, and when α is equal to 1, 2, and 0.5, we get:

$$B'_{gs} = \int_V \bigvee_{\mu_A(u)=\mu_B(v)} \mu_A^\alpha(u) \vee \bigvee_{\mu_A(u)>\mu_B(v)} (\mu_A^\alpha(u)\cdot\mu_B(v))/v.$$

The maximum value of $\mu_A^\alpha(u)\cdot\mu_B(v)$ is reached when $\mu_A(u)=1$, which means that $\bigvee_{\mu_A(u)>\mu_B(v)} \mu_A^\alpha(u)\cdot\mu_B(v) = \mu_B(v)$. We get

$$B'_{gs} = \int_V \mu_B^\alpha(v) \vee \mu_B(v)/v = B^\alpha \cup B.$$

So, as explained in the proof of Theorem 3.3, the conclusion B'_{gs} is equal to B when A' is A or very A, and equal to more or less B when A' is more or less A. □

Theorem 3.8. *The couple $(\cdot, I_{gs})$ satisfies the criterion C4-2.*

Proof. To verify the satisfaction of the criteria C4-1 and C4-2, we put $A' = 1 - A$. For that we obtain:

$$\begin{aligned} B'_{gs} &= \int_V \bigvee_{\mu_A(u)=\mu_B(v)} (1-\mu_A(u)) \vee \bigvee_{\mu_A(u)>\mu_B(v)} [(1-\mu_A(u))\cdot\mu_B(v)]/v \\ &= \int_V (1-\mu_B(v)) \vee \bigvee_{\mu_A(u)>\mu_B(v)} [(1-\mu_A(u))\cdot\mu_B(v)]/v. \end{aligned}$$

When $\mu_A(u) > \mu_B(v)$, the maximum value of $(1-\mu_A(u))\cdot\mu_B(v)$ is reached when $\mu_A(u) \simeq \mu_B(v)$. So

$$B'_{gs} \simeq \int_V (1-\mu_B(v)) \vee [(1-\mu_B(v))\cdot\mu_B(v)]/v.$$

Since $\mu_B(v) < 1$, we will have $(1-\mu_B(v)) > [(1-\mu_B(v))\cdot\mu_B(v)]$. We conclude that

$$B'_{gs} = \int_V 1-\mu_B(v)/v = 1 - B.$$

We deduce that the conclusion B'_{gs} is equal to not B when A' is not A. □

In the same way we can demonstrate the satisfaction of the criteria C1, C2-2, C3, and C4-2 by the combination $(\cdot, I_{gg})$.

For the implications: I_s, I_g, I_{gs}, I_{sg}, I_{gg}, and I_{ss} the same demonstration is feasible with other t-norms.

3.5. The remaining implications

For the remaining implications of Table 2, there is no general demonstration for all the forms of the observation like the implications treated before. Indeed, we will see further that the results are irregular from a case to another. So we will study whether the combinations satisfy the criteria (2.2)-(2.7) case by case. We take $I_\triangle$ implication as an example and the same procedure is followed in the demonstrations of the other implications. To get the membership function of the result B', the CRI with the implication $I_\triangle$ becomes the following:

$$B'_\triangle = \int_V \bigvee_{u \in U} \mu_A^\alpha(u) \cdot I_\triangle(\mu_A(u), \mu_B(v))/v.$$

In order to check the satisfaction of the criteria (2.2)-(2.7), let's suppose the function $S_\triangle$ defined as follows:

$$S_\triangle(\mu_A(u), \alpha) = \mu_A^\alpha(u) \cdot I_\triangle(\mu_A(u), \mu_B(v)).$$

Theorem 3.9. *The criterion C1 of* (2.2) *is satisfied by the combination* $(\cdot, I_\triangle)$.

Proof. The criterion C1 considers that the fuzzy set of the observation A' is equal to the fuzzy set of the rule's premise A, it means that $A' = A$. We find with $\alpha = 1$ that the function $S_\triangle$ is:

$$S_\triangle(\mu_A(u), 1) = \mu_A^1(u) \cdot I_\triangle(\mu_A(u), \mu_B(v)). \tag{3.6}$$

Replacing the function of $I_\triangle$ from Table 2, we get

$$\begin{aligned} S_\triangle(\mu_A(u), 1) &= \mu_A(u) \cdot \begin{cases} 1, & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ \frac{\mu_B(v)}{\mu_A(u)}, & \text{else}, \end{cases} \\ &= \begin{cases} \mu_A(u), & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ \mu_A(u) \cdot \frac{\mu_B(v)}{\mu_A(u)}, & \text{else}, \end{cases} = \begin{cases} \mu_A(u), & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ \mu_B(v), & \text{else}. \end{cases} \end{aligned}$$

The inference result $B'_\triangle$ of the CRI is obtained by the maximum values of $S_\triangle(\mu_A(u), 1)$, so using (3.2) and (3.3):

$$\bigvee_{u \in U} S_\triangle(\mu_A(u), 1) = \bigvee_{u \in U_1} \mu_A(u) \vee \bigvee_{u \in U_2} \mu_B(v) = \Big(\bigvee_{u \in U_1} \mu_A(u)\Big) \vee \mu_B(v).$$

For $u \in U_1$, we have $\mu_A(u) \leqslant \mu_B(v)$, so $\left(\bigvee_{u \in U_1} \mu_A(u)\right) \leqslant \mu_B(v)$, which gives

$$\bigvee_{u \in U} S_\triangle(\mu_A(u), 1) = \mu_B(v).$$

We deduce that:

$$B'_\triangle = \int_V \bigvee_{u \in U} S_\triangle(\mu_A(u), 1)/v = \int_V \mu_B(v)/v = B.$$

Another way to proof this equality is to use a graphical based demonstration method used in [27–29]. The Figure 1 (a) is a representation of the function $S_\triangle(\mu_A(u), 1)$ according to the values of $\mu_A(u)$ and $\mu_B(v)$, where $\mu_A(u)$ takes all the values into the interval [0,1]. From Fig 1 (a) and the equation (3.6), we can see that when $\mu_B(v)$ is equal to 0.2, which is represented by the broken line '- - -', the maximum of this curve is 0.2. We deduce that the maximum value of $S_\triangle$ is equal to 0.2. Furthermore, when we take another value of $\mu_B(v)$ like 0.8, which is represented by '-.-.-', we find that the maximum of this curve is 0.8. Thus, the maximum value of the function $S_\triangle$ for this case is 0.8.

So in a general case, we notice that for all the values of $\mu_B(v)$ and when $\mu'_A(u) = \mu_A(u)$, the maximum value of the function $S_\triangle$ is equal to $\mu_B(v)$. We conclude that:

$$\bigvee_{u \in U} S_\triangle(\mu_A(u), 1) = \mu_B(v).$$

Consequently, the value of the result B' obtained from the CRI in the case of $A' = A$ is equal to B. □

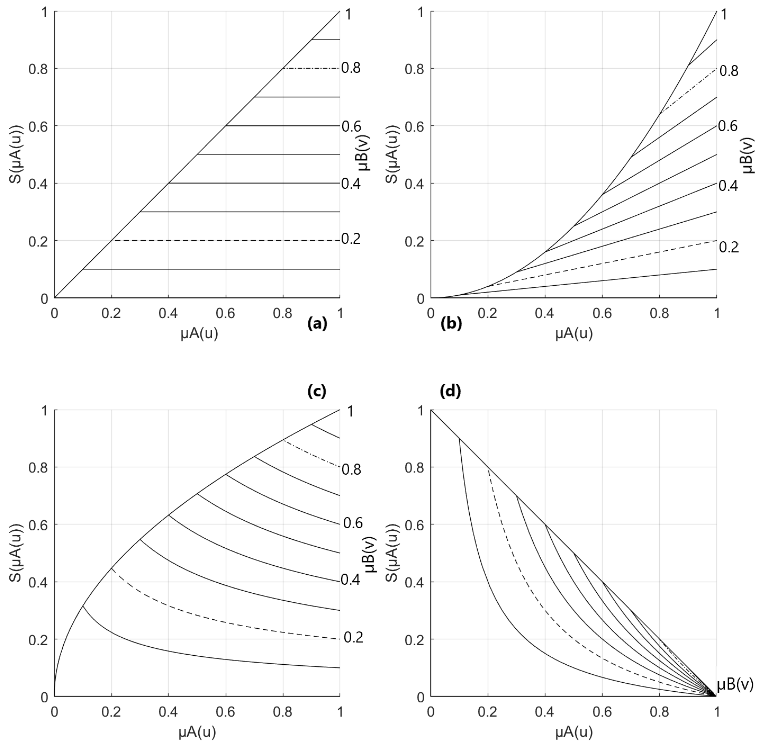

Figure 1: Representation of the result $S_{\triangle}(\mu_A(u), \alpha)$ when (a) $\mu_{A'} = \mu_A$; (b) $\mu_{A'} = \mu_A^2$; (c) $\mu_{A'} = \mu_A^{0.5}$; (d) $\mu_{A'} = 1 - \mu_A$.

Theorem 3.10. *The criterion C2-2 of* (2.4) *is satisfied by the combination* $(\cdot, I_{\triangle})$.

Proof. The criteria C2-1 and C2-2 concern the case where A' is very A, which means that $\alpha = 2$ and $A' = A^2$. We find that the function $S_{\triangle}$ when $\alpha = 2$ is:

$$S_{\triangle}(\mu_A(u), 2) = \mu_A^2(u) \cdot I_{\triangle}(\mu_A(u), \mu_B(v)).$$

Getting the definition of $I_{\triangle}$ from Table 2, we get:

$$S_{\triangle}(\mu_A(u), 2) = \mu_A(u)^2 \cdot \begin{cases} 1, & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ \frac{\mu_B(v)}{\mu_A(u)}, & \text{else}, \end{cases} = \begin{cases} \mu_A(u)^2, & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ \mu_A(u) \cdot \mu_B(v), & \text{else}. \end{cases}$$

$B'_{\triangle}$ is determined by the maximum values of $S_{\triangle}$ as follows:

$$\bigvee_{u \in U} S_{\triangle}(\mu_A(u), 2) = \bigvee_{u \in U_1} \mu_A(u)^2 \vee \bigvee_{u \in U_2} \mu_A(u).\mu_B(v).$$

We impose that the fuzzy set A is normalized, so $\mu_A(u)$ takes all values in $[0, 1]$. Furthermore with to the conditions (3.2) and (3.3), respectively, we find that $\bigvee_{u \in U_1} \mu_A(u)^2 = \mu_B(v)^2$ and $\bigvee_{u \in U_2} \mu_A(u) = 1$, which gives:

$$\bigvee_{u \in U} S_{\triangle}(\mu_A(u), 2) = \mu_B(v)^2 \vee \mu_B(v) = \mu_B(v).$$

We deduce that:

$$B'_{\triangle} = \int_V \bigvee_{u \in U} S_{\triangle}(\mu_A(u), 2)/v = \int_V \mu_B(v)/v = B.$$

In the same way, we give another demonstration based on the representation of $S_\triangle(\mu_A(u), 2)$. From Figure 1 (b) describing $S_\triangle(\mu_A(u), 2)$, we notice that for $\mu_B(v) = 0.2$ represented by the broken line '- - -', the maximum value of the function $S_\triangle$ is 0.2. The same when $\mu_B(v) = 0.8$, we find that $\bigvee_{u\in U} S_\triangle = 0.8$. So for all the values of $\mu_B(v)$, the maximum value of the function $S_\triangle$ when $\alpha = 2$ is $\mu_B(v)$. From that, we get:

$$\bigvee_{u\in U} S_\triangle(\mu_A(u), 2) = \mu_B(v).$$

So, the consequence B' for the combination $(.,I_\triangle)$ in the CRI when $A' = A^2$ is equal to B. □

Theorem 3.11. *The criterion C3 of (2.5) is satisfied by the combination $(\cdot,I_\triangle)$.*

Proof. The criterion C3 represents the case where A' is more or less A, thus $A' = A^{0.5}$ and $\alpha = 0.5$. We obtain that the function $S_\triangle$ is:

$$\begin{aligned} S_\triangle(\mu_A(u), 0.5) &= \mu_A^{0.5}(u) \cdot I_\triangle(\mu_A(u), \mu_B(v)) \\ &= \mu_A^{0.5}(u) \cdot \begin{cases} 1, & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ \frac{\mu_B(v)}{\mu_A(u)}, & \text{else,} \end{cases} = \begin{cases} \mu_A^{0.5}(u), & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ \frac{\mu_B(v)}{\mu_A^{0.5}(u)}, & \text{else.} \end{cases} \end{aligned}$$

We choose in this case to make a graphical based demonstration. However, we can make a formulas based demonstration following the same procedure in the previous proofs. From Figure 1 (c), we see that the maximum value of the function $S_\triangle$ is $\sqrt{0.2}$ when $\mu_B(v) = 0.2$. Also, when $\mu_B(v)$ is equal to 0.8, the maximum value of the function $S_\triangle$ is $\sqrt{0.8}$. We find that the maximum value of $S_\triangle(\mu_A(u), 0.5)$ for all the values of $\mu_B(v)$ is:

$$\bigvee_{u\in U} S_\triangle(\mu_A(u), 0.5) = \mu_B^{0.5}(v).$$

We conclude that the result B' gotten from the CRI when $A' = A^{0.5}$ is equal to $\sqrt{B}$:

$$B'_\triangle = \int_V \bigvee_{u\in U} S_\triangle(\mu_A(u), 0.5)/v = \int_V \mu_B^{0.5}(v)/v.$$

□

Theorem 3.12. *The combination $(\cdot,I_\triangle)$ satisfies the criterion C4-1 of (2.7).*

Proof. The criteria C4-1 and C4-2 treat the case where A' is not A, which is represented by $A' = 1 - A$. The function $S_\triangle$ becomes:

$$\begin{aligned} S_\triangle(\mu_A(u)) &= (1-\mu_A(u)) \cdot I_\triangle(\mu_A(u), \mu_B(v)) \\ &= (1-\mu_A(u)) \cdot \begin{cases} 1, & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ \frac{\mu_B(v)}{\mu_A(u)}, & \text{else.,} \end{cases} = \begin{cases} 1-\mu_A(u), & \text{if } \mu_A(u) \leqslant \mu_B(v), \\ (1-\mu_A(u)) \cdot \frac{\mu_B(v)}{\mu_A(u)}, & \text{else.} \end{cases} \end{aligned}$$

We see, using Figure 1(d), that the maximum value of the function $S_\triangle$ is 1 when $\mu_B(v) = 0.2$. In addition to that, when we choose $\mu_B(v) = 0.8$, the maximum value of the function $S_\triangle$ is 1. The same remark is obtained with the other values of $\mu_B(v)$. We conclude that for all the values of $\mu_B(v)$, $\bigvee_{u\in U} S_\triangle$ is:

$$\bigvee_{u\in U} S_\triangle(\mu_A(u)) = 1.$$

The consequence B' obtained from the CRI when $A' = 1 - A$ is:

$$B'_\triangle = \int_V \bigvee_{u\in U} S_\triangle(\mu_A(u))/vB'_\triangle = \int_V 1/v.$$ □

In the same way, we test the satisfaction of the criteria (2.2)-(2.7) for the combinations: $(., I_m)$, $(., I_a)$, $(., I_c)$, $(., I_b)$, $(., I_\bigstar)$, $(., I_\blacktriangle)$, $(., I_\#)$, and $(., I_\square)$, where every function of these combinations is represented in the Figures 2, 3, 4, 5, 6, 7, 8, 9, respectively.

From the combinations treated in this paper, Table 7 represents all consequences obtained in the four given cases (when $A' = A$, $A' = A^2$, $A' = A^{0.5}$, and $A' = 1 - A$).

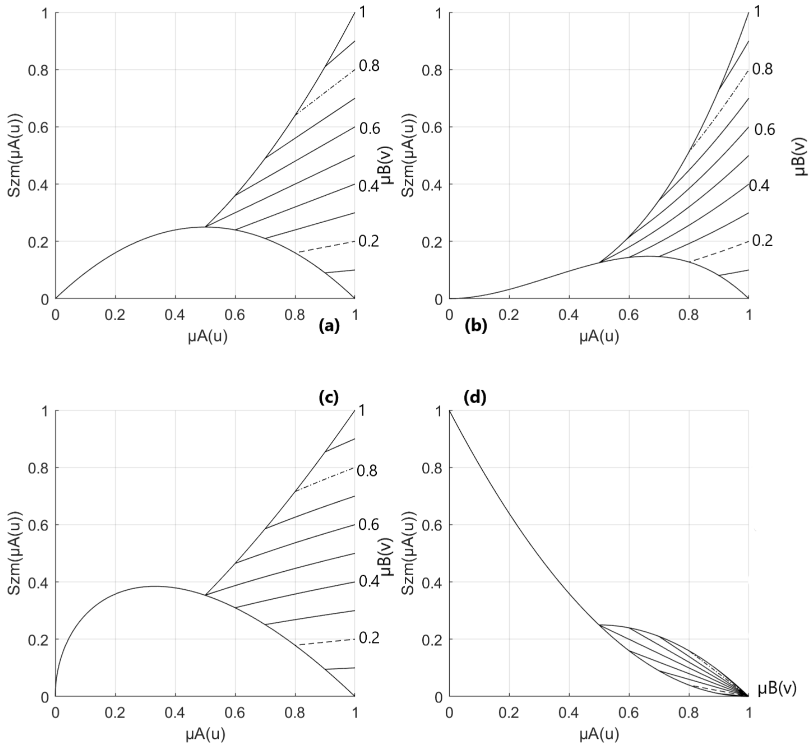

Figure 2: Representation of the result $S_m(\mu_A(u), \alpha)$ when (a) $\mu_{A'} = \mu_A$; (b) $\mu_{A'} = \mu_A^2$; (c) $\mu_{A'} = \mu_A^{0.5}$; (d) $\mu_{A'} = 1 - \mu_A$.

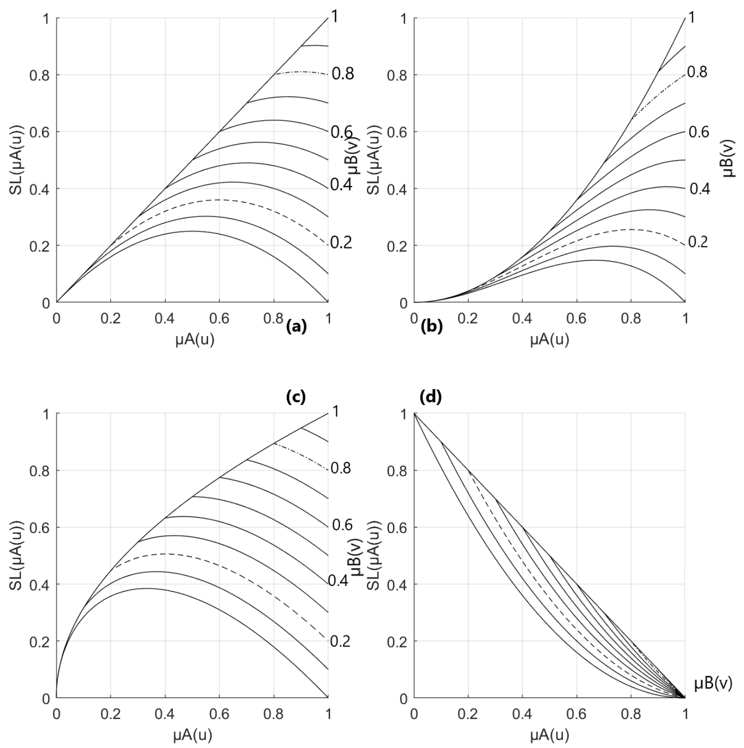

Figure 3: Representation of the result $S_a(\mu_A(u), \alpha)$ when (a) $\mu_{A'} = \mu_A$; (b) $\mu_{A'} = \mu_A^2$; (c) $\mu_{A'} = \mu_A^{0.5}$; (d) $\mu_{A'} = 1 - \mu_A$.

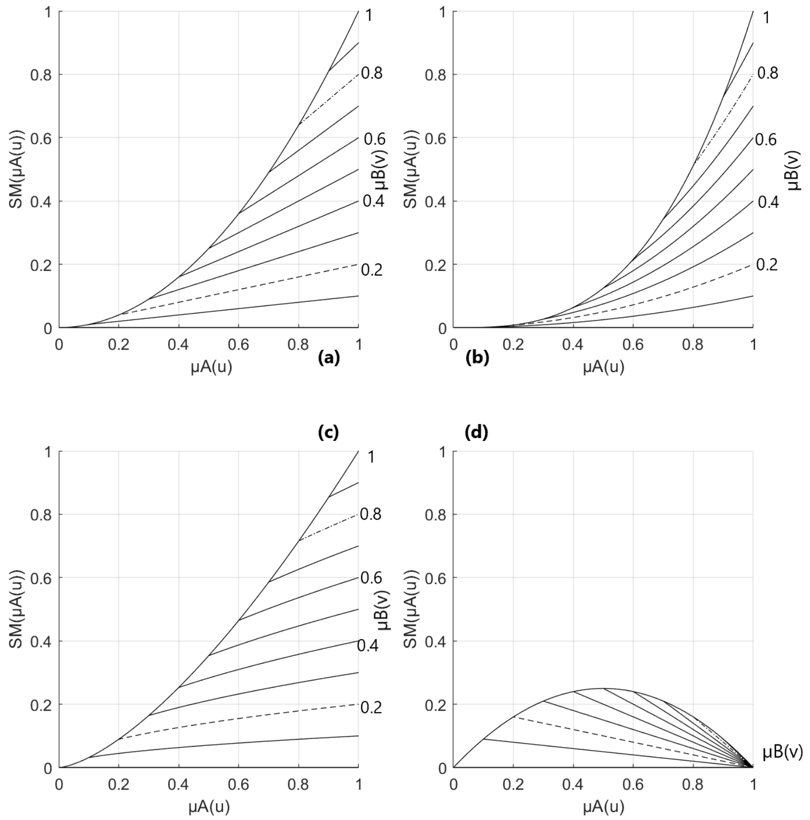

Figure 4: Representation of the result $S_c(\mu_A(u), \alpha)$ when (a) $\mu_{A'} = \mu_A$; (b) $\mu_{A'} = \mu_A^2$; (c) $\mu_{A'} = \mu_A^{0.5}$; (d) $\mu_{A'} = 1 - \mu_A$.

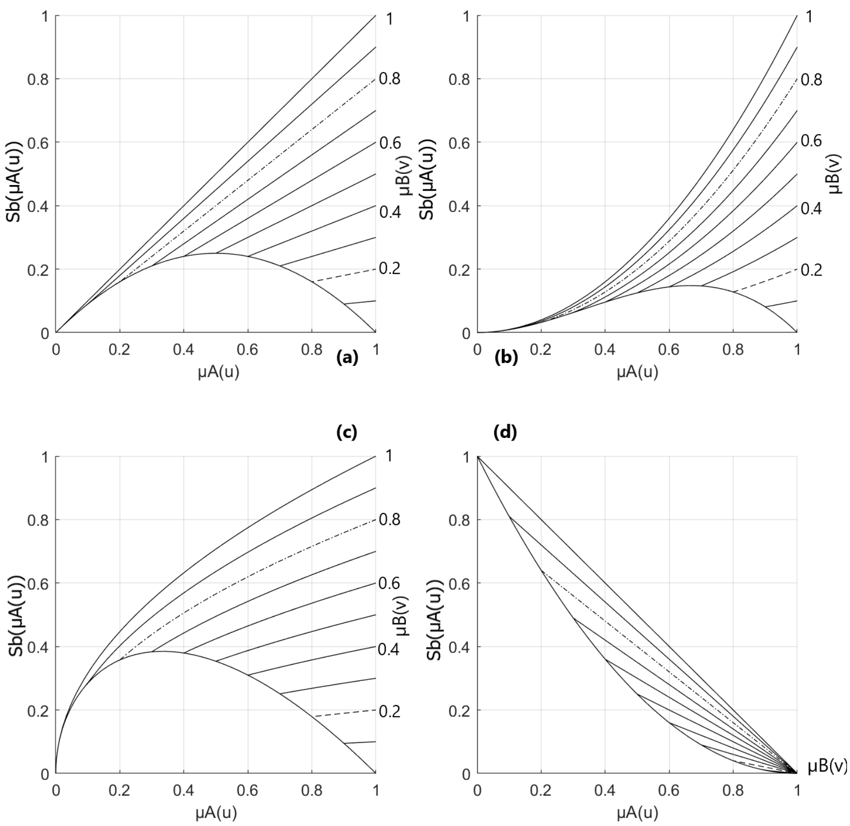

Figure 5: Representation of the result $S_b(\mu_A(u), \alpha)$ when (a) $\mu_{A'} = \mu_A$; (b) $\mu_{A'} = \mu_A^2$; (c) $\mu_{A'} = \mu_A^{0.5}$; (d) $\mu_{A'} = 1 - \mu_A$.

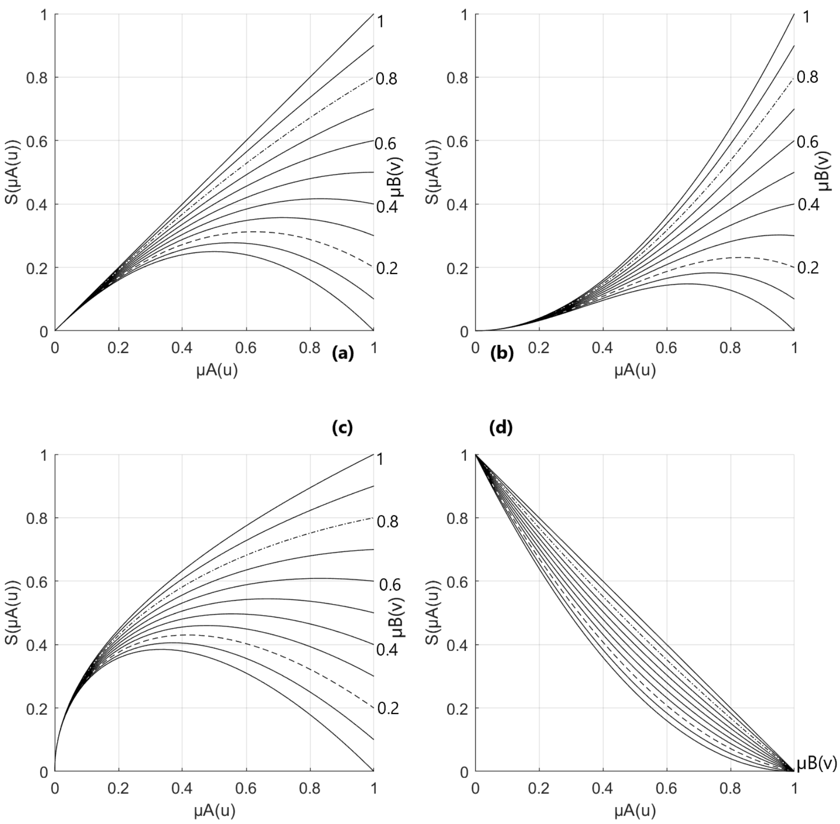

Figure 6: Representation of the result $S_{\bigstar}(\mu_A(u), \alpha)$ when (a) $\mu_{A'} = \mu_A$; (b) $\mu_{A'} = \mu_A^2$; (c) $\mu_{A'} = \mu_A^{0.5}$; (d) $\mu_{A'} = 1 - \mu_A$.

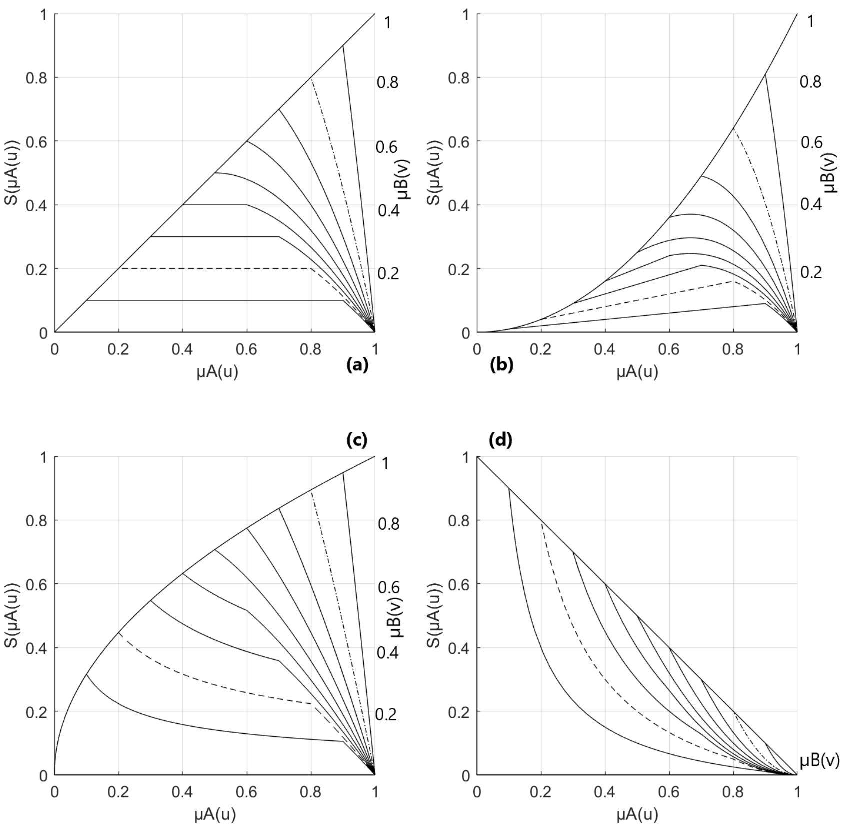

Figure 7: Representation of the result $S_{\blacktriangle}(\mu_A(u), \alpha)$ when (a) $\mu_{A'} = \mu_A$; (b) $\mu_{A'} = \mu_A^2$; (c) $\mu_{A'} = \mu_A^{0.5}$; (d) $\mu_{A'} = 1 - \mu_A$.

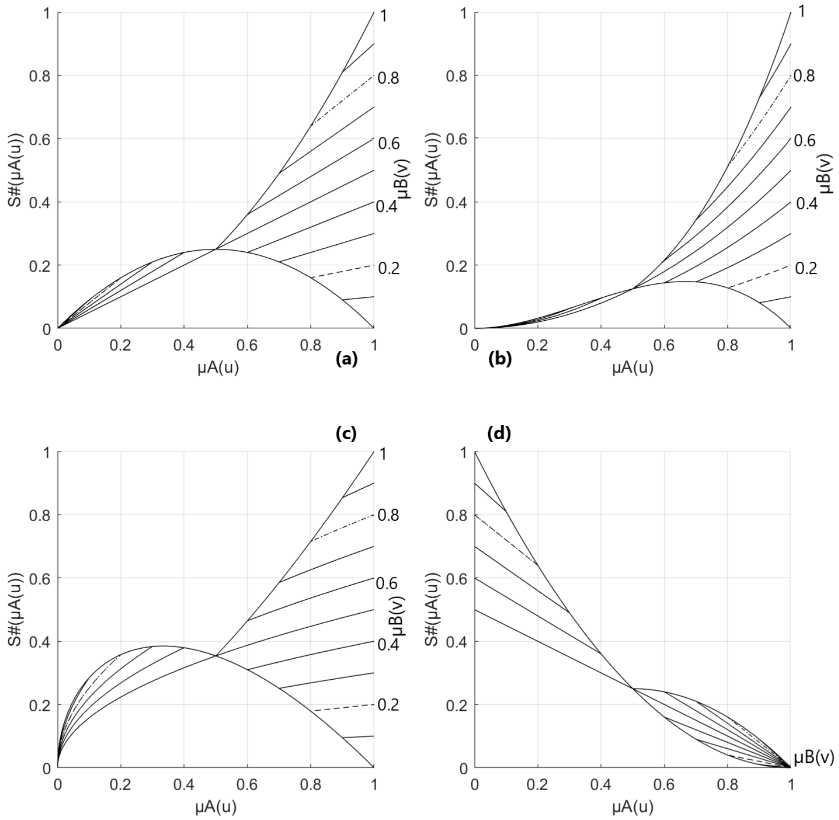

Figure 8: Representation of the result $S_{\#}(\mu_A(u), \alpha)$ when (a) $\mu_{A'} = \mu_A$; (b) $\mu_{A'} = \mu_A^2$; (c) $\mu_{A'} = \mu_A^{0.5}$; (d) $\mu_{A'} = 1 - \mu_A$.

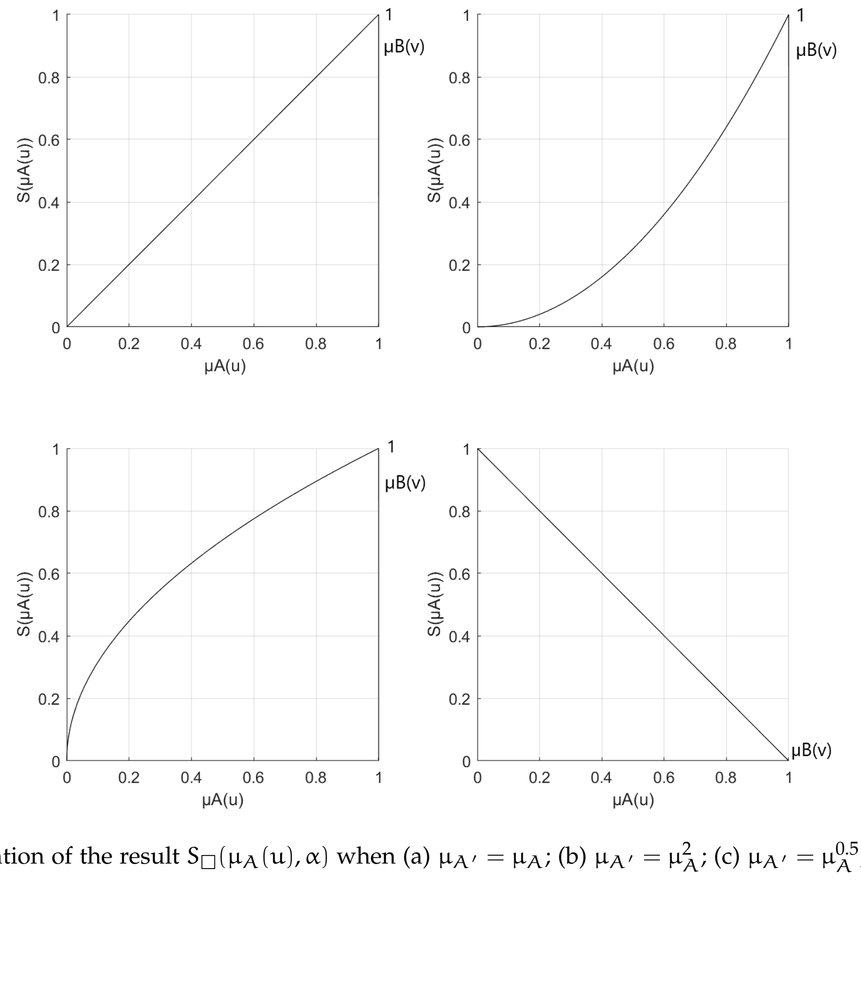

Figure 9: Representation of the result $S_{\square}(\mu_A(u), \alpha)$ when (a) $\mu_{A'} = \mu_A$; (b) $\mu_{A'} = \mu_A^2$; (c) $\mu_{A'} = \mu_A^{0.5}$; (d) $\mu_{A'} = 1 - \mu_A$.

Table 7: The result B' with product t-norm.

Relation	A	Very A	More or less A	Not A
I_m	$\frac{1}{4} \vee \mu_B(\nu)$	$\frac{\sqrt{3}-1}{5} \vee \mu_B(\nu)$	$\frac{3-\sqrt{5}}{2} \vee \mu_B(\nu)$	1
I_a	$\frac{1+\mu_B(\nu)(2+\mu_B(\nu))}{4}$	$\begin{cases} \mu_B(\nu), & \text{if } \mu_B(\nu) \geqslant \frac{1}{2}, \\ \frac{4^{\mu_B(\nu)}\sqrt{3}-1}{5}, & \text{else,} \end{cases}$	$\begin{cases} \sqrt{\mu_B(\nu)}, & \text{if } \mu_B(\nu) \geqslant \frac{1}{2}, \\ \frac{3+\mu_B(\nu)-\sqrt{5-\mu_B(\nu)}}{2}, & \text{else,} \end{cases}$	1
I_c	$\mu_B(\nu)$	$\mu_B(\nu)$	$\mu_B(\nu)$	$\frac{1}{4} \wedge \mu_B(\nu)(1-\mu_B(\nu))$
I_s	$\mu_B(\nu)$	$\mu_B^2(\nu)$	$\sqrt{\mu_B(\nu)}$	1
I_g	$\mu_B(\nu)$	$\mu_B(\nu)$	$\sqrt{\mu_B(\nu)}$	1
I_{sg}	$\mu_B(\nu)$	$\mu_B^2(\nu)$	$\sqrt{\mu_B(\nu)}$	$1-\mu_B(\nu)$
I_{gg}	$\mu_B(\nu)$	$\mu_B(\nu)$	$\sqrt{\mu_B(\nu)}$	1-$\mu_B(\nu)$
I_{gs}	$\mu_B(\nu)$	$\mu_B(\nu)$	$\sqrt{\mu_B(\nu)}$	$1-\mu_B(\nu)$
I_{ss}	$\mu_B(\nu)$	$\mu_B^2(\nu)$	$\sqrt{\mu_B(\nu)}$	$1-\mu_B(\nu)$
I_b	$\frac{1}{4} \vee \mu_B(\nu)$	$\frac{\sqrt{3}-1}{5} \vee \mu_B(\nu)$	$\frac{3-\sqrt{5}}{2} \vee \mu_B(\nu)$	1
$I_\triangle$	$\mu_B(\nu)$	$\mu_B(\nu)$	$\sqrt{\mu_B(\nu)}$	1
$I_\blacktriangle$	$\mu_B(\nu)$	$\begin{cases} \frac{\sqrt{6}}{2}.(\mu_B^2(\nu) - \frac{\mu_B(\nu)}{2} + \frac{5}{17}), & \text{if } 0.4 \leqslant \mu_B(\nu) \leqslant 0.6, \\ \mu_B^2(\nu) \vee \mu_B(\nu)(1-\mu_B(\nu)), & \text{else,} \end{cases}$	$\sqrt{\mu_B(\nu)}$	1
$I_\bigstar$	$\begin{cases} \frac{\mu_B(\nu)+1}{4} + \mu_B^2(\nu)(\frac{1}{3\sqrt{3}} + \frac{3\mu_B(\nu)}{5}, & \text{if } \mu_B(\nu) \leqslant \frac{1}{2}) \\ \mu_B(\nu), & \text{else,} \end{cases}$	$\begin{cases} \frac{1}{4}\mu_B(\nu)(2\sqrt{3}\mu_B(\nu)+1) + \frac{\sqrt{3}-1}{5}, & \text{if } \mu_B(\nu) < 0.3, \\ \mu_B(\nu), & \text{else,} \end{cases}$	$\begin{cases} \frac{1}{4}\mu_B(\nu)(1-\mu_B(\nu)) + \frac{5}{13}(1+2\mu_B(\nu)^3), & \text{if } \mu_B(\nu) < 0.7, \\ \mu_B(\nu), & \text{else,} \end{cases}$	1
$I_\#$	$\frac{1}{4} \vee \mu_B(\nu)$	$\frac{\sqrt{3}-1}{5} \vee \mu_B(\nu)$	$\frac{3-\sqrt{5}}{2} \vee \mu_B(\nu)$	$\mu_B(\nu) \vee (1-\mu_B(\nu))$
$I_\square$	1	1	1	1

4. Discussion

In Table 8 and using the results from Table 7, we show whether the criteria (2.2)-(2.7) are satisfied by each combination. We can see that the implication I_s with product t-norm satisfies the criteria C1, C2-1, C3, and C4-1 and belongs to type 1. In addition, the combinations with the implications I_g and $I_\triangle$ satisfy the criteria C1, C2-2, C3, and C4-1 and they belong to type 2. We notice also that the combinations that satisfy the criteria C1, C2-1, C3, and C4-2, and belong to the type 3 are with implications I_{sg} and I_{ss}. Moreover, those that belong to the type 4 and satisfy the criteria C1, C2-2, C3, and C4-2 are with the implications I_{gg} and I_{gs}. As a consequence, the above combinations give desirable results and their implications are considered as the best with product t-norm.

There are also other combinations that satisfy some of the criteria, which is not sufficient to get an appropriate result. For example, the couple $(., I_m)$ satisfies just the criterion C4-1. The same for the combinations with implications: I_a, I_b, $I_\bigstar$, and $I_\square$. Nevertheless, the implication $I_\#$ does not satisfy any criterion.

In the other hand, there are some implications that satisfy a good number of criteria, but they do not belong to any types. We cite $I_\blacktriangle$ implication.

Referring to Tables 3, 4, 5, 7, and 8, we can note that the couples (T, I) having the same implication belong to the same type. When we take one implication for example I_s, it satisfies the criteria C1, C2-1, C3, and C4-1 under all the compositions, which means that it belongs to type 1 when combined with one of the t-norms mentioned before.

Table 8: The satisfaction of the criteria with product t-norm.

	C1	C2-1	C2-2	C3	C4-1	C4-2	Type
I_m	×	×	×	×	√	×	/
I_a	×	×	×	×	√	×	/
I_c	√	×	√	×	×	×	/
I_s	√	√	×	√	√	×	1
I_g	√	×	√	√	√	×	2
I_{sg}	√	√	×	√	×	√	3
I_{gg}	√	×	√	√	×	√	4
I_{gs}	√	×	√	√	×	√	4
I_{ss}	√	√	×	√	×	√	3
I_b	×	×	×	×	√	×	/
$I_\triangle$	√	×	√	√	√	×	2
$I_\blacktriangle$	√	×	×	√	√	×	/
$I_\bigstar$	×	×	×	×	√	×	/
$I_\#$	×	×	×	×	×	×	/
$I_\square$	×	×	×	×	√	×	/

5. Conclusion

The compositional rule of inference (CRI) is a method used in fuzzy inference systems to reason from vague knowledge. It has two operators as parameters: a t-norm T and a fuzzy implication I. The choice of the combination (T, I) is delicate because it can lead to success or failure to provide a result close to human induction. Many studies were done to find the compatible combinations, like min t-norm, drastic t-norm, and Lukasiewicz t-norm combined with various implications. From that, we chose to study the product t-norm in the CRI, and we combined it with fifteen fuzzy implications. Thus, we tested whether the considered combinations check the axiomatics of approximate reasoning. We have concluded from our study that a good number of couples give a reasonable result. With this work, in addition to previous works, the four most known and used t-norms are studied with a large set of implications.

Consequently, these studies will guide a fuzzy inference system designer in the choice of the couple (t-norm, implication). A current work focuses on testing all the combinations in a practical application in order to evaluate their performance involving a fuzzy PID controller. In another future project, we envisage studying the product t-norm in more complex cases as conjunctive and disjunctive knowledge. Moreover, it would be interesting to explore more t-norms or implication operators that were not studied.

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