J. Math. Computer Sci., 41 (2026), 244–263

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Some new approaches of integral inequalities involving Raina and Mittag-Leffler function pertaining to Atangana-Baleanu fractional integral operator

Muhammad Tariq[a,b], Sotiris K. Ntouyas[c], Waqar Afzal[d], Jessada Tariboon[e,*]

[a]*Mathematics Research Center, Near East University, Near East Boulevard, PC: 99138, Nicosia/Mersin 10, Turkey.*

[b]*Department of Mathematics, Balochistan Residential College, Loralai, Balochistan, Pakistan.*

[c]*Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece.*

[d]*Abdus Salam School of Mathematical Sciences, Government College University, Lahore, Pakistan.*

[e]*Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand.*

Abstract

Convexity and inequality, particularly as they relate to fractional analysis, have a plethora of significant applications in the applied sciences. Our goal in this manuscript is to investigate and develop a new version of the Hermite-Hadamard and Pachpatte types of integral inequality using the Atangana-Baleanu fractional integral operator in the context of generalized convex involving Raina's function. Utilizing this method, we develop a new identity for fractional integrals that is associated with Raina's functions. Additionally, some new extensions of Hermite-Hadamard type inequalities via Atangana-Baleanu fractional operator are examined with the support of Hölder inequality, power mean inequality and Young inequaity. Additionally, we present applications related to entropy measures that demonstrate the practical utility of our main findings. In terms of both outcomes and special cases, this study presents novel and noteworthy improvements over previously published findings.

Keywords: Convex function, Raina function, generalized convex involving Raina's function, AB fractional operator.

2020 MSC: 26A51, 26A33, 26D07, 26D10, 26D15.

1. Introduction

Convex functions have a fascinating history that originates back to the early nineteenth century. The popular and frequently read book "Inequalities", written by Polya et al., extensively uses and explores the concept of "convex functions". This book swiftly became a common resource for mathematicians, devoted exclusively to the subject of inequality, and a great way to get started in this intriguing area. Convex theory provides the appropriate guidelines and techniques for focusing on a broad range of

*Corresponding author

Email addresses: `captaintariq2187@gmail.com` (Muhammad Tariq), `sntouyas@uoi.gr` (Sotiris K. Ntouyas), `waqar2989@gmail.com` (Waqar Afzal), `jessada.t@sci.kmutnb.ac.th` (Jessada Tariboon)

doi: 10.22436/jmcs.041.02.07

Received: 2025-06-02 *Revised:* 2025-07-20 *Accepted:* 2025-08-30

problems in the applied sciences. In the applied sciences, convex theory gives us the right rules and methods to concentrate on a wide variety of issues. The development of many facets of mathematics and other scientific fields has been widely recognized in recent years to be influenced by mathematical inequalities. This theory has outstanding applications in engineering [16], finance [10], economics [11], and optimization [17]. This theory offers a strong foundation for the development of numerical tools for the investigation and resolution of difficult mathematical problems.

Mathematical inequalities provides a fundamental foundation for understanding the perform of functions under integration, leading to important applications in both theoretical and applied mathematics. An efficient tool in mathematical analysis are convex integral inequalities, which create relationships between the integrals of convex functions and their values at specific points. Numerous fields, including probability theory, information theory, and optimization, use them. These inequalities are crucial for numerical techniques like Simpson's rule, the trapezoidal rule, and others, especially when estimating the error bounds. For the literature, see the references [5–7, 18, 20, 21].

The subject fractional calculus has many applications in mathematical biology [13], epidemiology [3], and optimal control [8]. Because of the aforementioned pervasive perspectives and importance, readers and scholars found discussion of fractional operators to be interesting. This theory is useful when describing and analyzing statistical problems and formulas that resemble quadratures. The Atangana-Baleanu fractional integral operator (ABFIO) marks a significant breakthrough in fractional calculus, offering enhanced capabilities for modeling complex systems. The ABFIO has become as a potent technique in modeling complex systems due to its non-local and non-singular kernel involving the Mittag-Leffler function. Numerous fields where memory and hereditary effects are important find extensive use for this operator. It has been successfully applied in physics and engineering to simulate anomalous diffusion, heat transfer, and viscoelastic systems, capturing behaviors that are not possible with classical models. By taking into account long-term memory effects, it aids in the description of biological sciences processes such as drug delivery and epidemic modeling. In financial mathematics, the operator is also useful for modeling systems with memory-dependent fluctuations. Furthermore, it is essential to the establishment of generalized integral inequalities like Hermite-Hadamard, Fejér, and Pachpatte-type inequalities in the field of mathematical analysis. Overall, the ABFIO offers a more practical and adaptable framework for studying dynamic systems with memory properties.

The aim and novelty of this work are to introduce a new variant of Hermite-Hadamard (H-H) and Pachpatte-type integral inequality via generalized $\mathfrak{m}$-convex involving Raina's function $(\mathsf{G}_{\mathfrak{m}}\mathsf{CRF})$ in the frame of ABFIO. Further, we are to construct some refinements of H-H type integral inequality via ABFIO.

The order of this particular document is as follows: Firstly, in Section 2, we rehash a few familiar ideas and terminology that will help us in our investigation in the following sections. In Section 3, we introduce a new definition namely $\mathsf{G}_{\mathfrak{m}}\mathsf{CRF}$ and also present its algebraic properties. In Section 4, we introduce a novel sort of H-H-type inequality via ABFIO with some interesting corollaries and remarks. In Section 5, we explore a new integral identity, and based on this newly introduced identity, some refinements of H-H inequality are also constructed. In Section 6, we investigate a novel sort of Pachpatte-type inequality via ABFIO with some corollaries and remarks. In Section 7, we introduce entropy application via ABFIO. In the final Section 8, we offer a brief outcome and outline certain prospective possibilities for further research.

2. Preliminaries

It is best to analyze and deepen in this section due to the large number of theorems, definitions, and remarks in order to verify completeness, reader interest and quality. This section is designed to demonstrate and examine a number of well-known definitions and terminology that we will require for our investigation in sections to come. Initially, the convex, H-H inequality, Mitagg-Leffler, generalized convex set, and generalized convex function are presented. Adding Condition A, Hólder inequality and power mean inequality enhances the appeal of this portion. We sum up this portion with recalling Caputo-Fabrizio derivative operator, and ABFIO that are needed in our assessment.

Definition 2.1 ([15])**.** A real-valued function Δ is said to be convex, if

$$\Delta\left(\mathfrak{z}\mathfrak{x}_a+(1-\mathfrak{z})\mathfrak{x}_b\right) \leqslant \mathfrak{z}\Delta\left(\mathfrak{x}_a\right)+(1-\mathfrak{z})\Delta\left(\mathfrak{x}_b\right),$$

holds for all $\mathfrak{x}_a, \mathfrak{x}_b \in I$ and $\mathfrak{z} \in [0,1]$.

The most famous inequality involving convex functions is the H-H inequality [12] stated as follows.

Theorem 2.2. *If $\Delta : [\mathfrak{x}_a, \mathfrak{x}_b] \to \mathbb{R}$ is a convex function, then*

$$\Delta\left(\frac{\mathfrak{x}_a+\mathfrak{x}_b}{2}\right) \leqslant \frac{1}{\mathfrak{x}_b-\mathfrak{x}_a}\int_{\mathfrak{x}_a}^{\mathfrak{x}_b}\Delta(x)dx \leqslant \frac{\Delta(\mathfrak{x}_a)+\Delta(\mathfrak{x}_b)}{2}.$$

Raina [19] proposed a family of functions formally stated by

$$\mathcal{R}^{\rho}_{\epsilon,\sigma}(z) = \mathcal{R}^{\rho(0),\rho(1),\ldots}_{\epsilon,\sigma}(z) = \sum_{k=0}^{+\infty}\frac{\rho(\mathfrak{v})}{\Gamma(\epsilon k+\sigma)}z^k, \tag{2.1}$$

where $\rho = (\rho(0), \ldots, \rho(\mathfrak{v}), \ldots)$ and $\epsilon, \sigma > 0, |z| < R$. Equation (2.1) is the extension of classical Mittag-Leffler function. If $\epsilon = 1, \sigma = 0$, and $\rho(\mathfrak{v}) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}$ for $k = 0, 1, 2, \ldots$, where α, β, and γ are parameters, which can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \ldots$), and the symbol α_k denotes the quantity

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+k-1), \quad k = 0, 1, 2, \ldots,$$

and restricts its domain to $|z| \leqslant 1$ (with $z \in \mathbb{C}$), then we have the classical hypergeometric function, that is

$$\mathcal{R}(\alpha, \beta; \gamma; z) = \sum_{k=0}^{+\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}z^k.$$

Moreover, if $\rho = (1, 1, \ldots)$ with $\epsilon = \alpha$, $(\mathrm{Re}(\alpha) > 0), \sigma = 1$, then

$$\mathfrak{E}_{\alpha}(z) = \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(1+\alpha k)}. \tag{2.2}$$

Equation (2.2) is referred to as a classical Mittag-Leffler function. The Mittag-Leffler function appears usually in the study of fractional calculus and especially in the studies of fractional conjecture of the kinetic equation, super diffusive transport, random walks, Lévy flights, and in the studies of complicated structures. Cortez presented the generalized convex set and the convex function pertaining to Raina's function in [22, 23].

Definition 2.3 ([22])**.** Let $\rho = (\rho(0), \ldots, \rho(\mathfrak{v}), \ldots)$ and $\epsilon, \sigma > 0$. A set $X \neq \emptyset$ is said to be generalized convex, if $\mathfrak{x}_a + \mathfrak{z}\,\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - \mathfrak{x}_a) \in X$, for all $\mathfrak{x}_a, \mathfrak{x}_b \in X$ and $\mathfrak{z} \in [0,1]$.

Definition 2.4 ([22])**.** Let ρ represent a bounded sequence then $\rho = (\rho(0), \ldots, \rho(\mathfrak{v}), \ldots)$ and $\epsilon, \sigma > 0$. If real-valued Δ holds the following inequality

$$\Delta\Big(\mathfrak{x}_a + \mathfrak{z}\,\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - \mathfrak{x}_a)\Big) \leqslant \mathfrak{z}\Delta(\mathfrak{x}_b) + (1-\mathfrak{z})\Delta(\mathfrak{x}_a),$$

for all $\mathfrak{x}_a, \mathfrak{x}_b \in X$, where $\mathfrak{x}_a < \mathfrak{x}_b$ and $\mathfrak{z} \in [0,1]$, then Δ is said to be generalized convex function.

Remark 2.5. If $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{r}_a) = \mathfrak{r}_b - \mathfrak{r}_a > 0$, then we achieve Definition 2.1.

The following Condition A first time explored by Ahmad et al. [2].

Condition A. Let X be generalized convex subset w.r.t. $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\cdot)$. For any $\mathfrak{r}_a, \mathfrak{r}_b \in X$ and $\mathfrak{z} \in [0,1]$,

$$\mathcal{R}^{\rho}_{\epsilon,\sigma}\Big(\mathfrak{r}_a - (\mathfrak{r}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{r}_a))\Big) = -\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{r}_a),$$
$$\mathcal{R}^{\rho}_{\epsilon,\sigma}\Big(\mathfrak{r}_b - \big(\mathfrak{r}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{r}_a)\big)\Big) = (1-\mathfrak{z})\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{r}_a).$$

Note that, for every $\mathfrak{r}_a$, $\mathfrak{r}_b \in X$ and for all $\mathfrak{z}_1$, $\mathfrak{z}_2 \in [0,1]$ from Condition A, we have

$$\mathcal{R}^{\rho}_{\epsilon,\sigma}\Big(\mathfrak{r}_a + \mathfrak{z}_2\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{r}_a) - (\mathfrak{r}_a + \mathfrak{z}_1\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{r}_a))\Big) = (\mathfrak{z}_2 - \mathfrak{z}_1)\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{r}_a).$$

Some well-known integral inequalities such as Hölder inequality and power-mean inequality will be used.

Theorem 2.6 ([14])**.** *Assume that* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $\Delta_1, \Delta_2 : [x_1, x_2] \to \mathbb{R}$ *are such that* $|\Delta_1|^p$ *and* $|\Delta_2|^q$ *are integrable on* $[x_1, x_2]$. *Then*

$$\int_0^1 |\Delta_1(x)\Delta_2(x)|dx \leqslant \left(\int_0^1 |\Delta_1(x)|^p dx\right)^{\frac{1}{p}} \left(\int_0^1 |\Delta_2(x)|^q dx\right)^{\frac{1}{q}}.$$

If we get $|\Delta_1||\Delta_2| = (|\Delta_1|^{\frac{1}{p}})(|\Delta_1|^{\frac{1}{q}}|\Delta_2|)$ in the Holder inequality, then we obtain the following power mean integral inequality as a simple result of the Hölder integral inequality.

Theorem 2.7 ([14])**.** *Assume that* $\Delta \geqslant 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $\Delta_1, \Delta_2 : [x_1, x_2] \to \mathbb{R}$ *are such that* $|\Delta_1|^p$ *and* $|\Delta_2|^q$ *are integrable on* $[x_1, x_2]$. *Then*

$$\int_0^1 |\Delta_1(x)\Delta_2(x)|dx \leqslant \left(\int_0^1 |\Delta_1(x)|dx\right)^{1-\frac{1}{q}} \left(\int_0^1 |\Delta_1(x)|dx \int_0^1 |\Delta_2(x)|^q dx\right)^{\frac{1}{q}}.$$

Many mathematicians with the development of fractional calculus have defined many fractional derivative and integral operators to find solutions to real-world problems. Some of them are as follows.

In Caputo-Fabrizio (C-F) derivative operator, Atangana and Baleanu utilizing the Mittag-Leffler function and investigate the new derivative operators as follows.

Definition 2.8 ([4])**.** Let $\Delta \in H^1(\mathfrak{r}_a, \mathfrak{r}_b), \mathfrak{r}_b > \mathfrak{r}_a, \omega \in [0,1)$, then the definition of the Caputo-Fabrizio derivative is given by

$${}^{ABC}_{\mathfrak{r}_a}D^{\omega}_t[\Delta(t)] = \frac{B(\omega)}{1-\omega}\int_{\mathfrak{r}_a}^t \Delta'(x)E_{\omega}\left[-\omega\frac{(t-x)^{\omega}}{(1-\omega)}\right]dx. \tag{2.3}$$

Definition 2.9 ([4])**.** Let $\Delta \in H^1(\mathfrak{r}_a, \mathfrak{r}_b), \mathfrak{r}_b > \mathfrak{r}_a, \omega \in [0,1)$, then the definition of the Caputo-Fabrizio derivative is given by

$${}^{ABR}_{\mathfrak{r}_a}D^{\omega}_t[\Delta(t)] = \frac{B(\omega)}{1-\omega}\frac{d}{dt}\int_{\mathfrak{r}_a}^t \Delta(x)E_{\omega}\left[-\omega\frac{(t-x)^{\omega}}{(1-\omega)}\right]dx. \tag{2.4}$$

Equations (2.3) and (2.4) have a non-local kernel. Also in equation (2.3) when the function is constant we get zero.

The related fractional integral operator has been defined by Atangana-Baleanu as follows.

Definition 2.10 ([4])**.** The fractional integral associated to the new fractional derivative with non-local kernel of a function $\Delta \in H^1(\mathfrak{r}_a, \mathfrak{r}_b)$ is defined as

$$ {}^{AB}_{\mathfrak{r}_a}I^{\omega}\{\Delta(t)\} = \frac{1-\omega}{B(\omega)}\Delta(t) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{\mathfrak{r}_a}^{t} \Delta(y)(t-y)^{\omega-1}dy, $$

where $b > a, \omega \in (0, 1]$.

In [1], Abdeljawad and Baleanu introduced right hand side of the integral operator as follows. The right fractional new integral with ML kernel of order $\omega \in (0, 1]$ is defined by

$$ {}^{AB}I^{\omega}_{\mathfrak{r}_b}\{\Delta(t)\} = \frac{1-\omega}{B(\omega)}\Delta(t) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{t}^{\mathfrak{r}_b} \Delta(y)(y-t)^{\omega-1}dy. $$

3. Generalized $\mathfrak{m}$-convex involving Raina's function and its properties

Here, we shall introduce and explore the new definition, i.e., $G_{\mathfrak{m}}CRF$, an interesting and useful concept for convex functions and examine some of its algebraic properties.

Definition 3.1. Let $\rho = (\rho(0), \dots, \rho(\mathfrak{v}), \dots)$ and $\epsilon, \sigma > 0$. A set $X \neq \emptyset$ is said to be generalized $\mathfrak{m}$-convex, if $\mathfrak{m}\mathfrak{r}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a) \in X$, for all $\mathfrak{r}_a, \mathfrak{r}_b \in X$ and $\mathfrak{z}, \mathfrak{m} \in [0, 1]$.

Definition 3.2. A function Δ defined on the generalized $\mathfrak{m}$-convex set X is said to be generalized $\mathfrak{m}$-convex involving Raina's function, i.e., $G_{\mathfrak{m}}CRF$, if

$$ \Delta(\mathfrak{m}\mathfrak{r}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a)) \leqslant \mathfrak{m}\,(1-\mathfrak{z})\,\Delta\,(\mathfrak{r}_a) + \mathfrak{z}\Delta\,(\mathfrak{r}_b) $$

holds for every $\mathfrak{r}_a, \mathfrak{r}_b \in X$, $\mathfrak{m} \in (0, 1]$ and $\mathfrak{z} \in [0, 1]$.

Remark 3.3. If $\mathfrak{m} = 1$ and $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a) = \mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a$, then Definition 3.2 reverts to the idea of convex function, which was investigated by Niculescu [15].

Note that every convex function is $G_{\mathfrak{m}}CRF$, but the converse does not hold in general.

Here, we are going to introduce the new condition, namely extended condition A, in the following way.

Extended Condition A. Let X be generalized $\mathfrak{m}$-convex subset w.r.t. $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\cdot)$. For any $\mathfrak{r}_a, \mathfrak{r}_b \in X$ and $\mathfrak{z} \in [0, 1]$,

$$ \begin{aligned} \mathcal{R}^{\rho}_{\epsilon,\sigma}\Big(\mathfrak{r}_a - (\mathfrak{m}\mathfrak{r}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a))\Big) &= -\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a), \\ \mathcal{R}^{\rho}_{\epsilon,\sigma}\Big(\mathfrak{r}_b - \big(\mathfrak{m}\mathfrak{r}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a)\big)\Big) &= (1-\mathfrak{z})\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a). \end{aligned} $$

Note that, for every $\mathfrak{r}_a$, $\mathfrak{r}_b \in X$ and for all $\mathfrak{z}_1$, $\mathfrak{z}_2 \in [0, 1]$ from extended condition A, we have

$$ \mathcal{R}^{\rho}_{\epsilon,\sigma}\Big(\mathfrak{m}\mathfrak{r}_a + \mathfrak{z}_2\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a) - (\mathfrak{m}\mathfrak{r}_a + \mathfrak{z}_1\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a))\Big) = (\mathfrak{z}_2 - \mathfrak{z}_1)\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a). $$

We are going to look at and develop a few properties of the recently presented concept.

Theorem 3.4. *If Δ_1, Δ_2 are two $G_{\mathfrak{m}}CRF$, then $(\Delta_1 + \Delta_2)$ is also an $G_{\mathfrak{m}}CRF$.*

Proof. Since given that Δ_1 and Δ_2 be two $G_{\mathfrak{m}}CRF$, then

$$ \begin{aligned} (\Delta_1 + \Delta_2)\,(\mathfrak{m}\mathfrak{r}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a)) &= \Delta_1(\mathfrak{m}\mathfrak{r}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a)) + \Delta_2(\mathfrak{m}\mathfrak{r}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{r}_b - \mathfrak{m}\mathfrak{r}_a)) \\ &\leqslant \mathfrak{m}\,(1-\mathfrak{z})\,\Delta_1\,(\mathfrak{r}_a) + \mathfrak{z}\Delta_1\,(\mathfrak{r}_b) + \mathfrak{m}\,(1-\mathfrak{z})\,\Delta_2\,(\mathfrak{r}_a) + \mathfrak{z}\Delta_2\,(\mathfrak{r}_b) \\ &= \mathfrak{m}\,(1-\mathfrak{z})\,[\Delta_1\,(\mathfrak{r}_a) + \Delta_2\,(\mathfrak{r}_a)] + \mathfrak{z}\,[\Delta_1\,(\mathfrak{r}_b) + \Delta_2\,(\mathfrak{r}_b)] \\ &= \mathfrak{m}\,(1-\mathfrak{z})\,(\Delta_1 + \Delta_2)(\mathfrak{r}_a) + \mathfrak{z}(\Delta_1 + \Delta_2)(\mathfrak{r}_b). \end{aligned} $$

This is the required proof. □

Theorem 3.5. *If Δ is G_mCRF, then $(c\Delta)$ is also an G_mCRF.*

Proof. Since Δ is G_mCRF, and c is any constant number, then

$$\begin{aligned}(c\Delta)\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right) &\leqslant c\Big(m\left(1-\mathfrak{z}\right)\Delta\left(\mathfrak{x}_a\right)+\mathfrak{z}\Delta\left(\mathfrak{x}_b\right)\Big)\\ &= m\left(1-\mathfrak{z}\right)c\Delta\left(\mathfrak{x}_a\right)+\mathfrak{z}c\Delta\left(\mathfrak{x}_b\right)= m\left(1-\mathfrak{z}\right)(c\Delta)\left(\mathfrak{x}_a\right)+\mathfrak{z}(c\Delta)\left(\mathfrak{x}_b\right).\end{aligned}$$

This completes the proof. □

Theorem 3.6. *Composition of two G_mCRF is also an G_mCRF.*

Proof.

$$\begin{aligned}(\Delta_2\circ\Delta_1)\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right) &= \Delta_2(\Delta_1(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)))\\ &\leqslant \Delta_2\Big(m\left(1-\mathfrak{z}\right)\Delta_1\left(\mathfrak{x}_a\right)+\mathfrak{z}\Delta_1\left(\mathfrak{x}_b\right)\Big)\\ &\leqslant m\left(1-\mathfrak{z}\right)\Delta_2(\Delta_1\left(\mathfrak{x}_a\right))+\mathfrak{z}\Delta_2(\Delta_1\left(\mathfrak{x}_b\right))\\ &= m\left(1-\mathfrak{z}\right)(\Delta_2\circ\Delta_1)\left(\mathfrak{x}_a\right)+\mathfrak{z}(\Delta_2\circ\Delta_1)\left(\mathfrak{x}_b\right).\end{aligned}$$

This is the required proof. □

Theorem 3.7. *Let $0<\mathfrak{x}_a<\mathfrak{x}_b$, $\Delta_j:\mathbb{X}=[\mathfrak{x}_a,\mathfrak{x}_b]\to[0,+\infty)$ be a family of G_mCRF and $\Delta(u)=\sup_j\Delta_j(u)$. Then, Δ is an G_mCRF for $m\in(0,1]$, $\mathfrak{z}\in[0,1]$, and $U=\{\Delta\in[\mathfrak{x}_a,\mathfrak{x}_b]:\Delta(\Delta_{\mathfrak{z}})<\infty\}$ is an interval.*

Proof. Let $\mathfrak{x}_a,\mathfrak{x}_b\in U$, $m\in(0,1]$ and $\mathfrak{z}\in[0,1]$, then

$$\begin{aligned}\Delta(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)) &= \sup_j\Delta_j(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a))\\ &\leqslant m\left(1-\mathfrak{z}\right)\sup_j\Delta_j\left(\mathfrak{x}_a\right)+\mathfrak{z}\sup_j\Delta_j\left(\mathfrak{x}_b\right)=m\left(1-\mathfrak{z}\right)\Delta\left(\mathfrak{x}_a\right)+\mathfrak{z}\Delta\left(\mathfrak{x}_b\right)<\infty.\end{aligned}$$

This is the required proof. □

4. Hermite-Hadamard inequality via generalized m-convex involving Raina's function pertaining to AB fractional integral operator

The main goal of this portion is to provide a new sort of the H-H-type inequality for a G_mCRF via ABFIO.

Theorem 4.1. *Let $I\subseteq\mathbb{R}$ be an open and non-empty m-convex subset and $\mathfrak{x}_a,\mathfrak{x}_b\in I$ with $m\mathfrak{x}_a<m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)$. If $\Delta:\left[m\mathfrak{x}_a,m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]\to\mathbb{R}$ is a G_mCRF, $\Delta\in L\left[m\mathfrak{x}_a,m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]$, and $\mathcal{R}^{\rho}_{\epsilon,\sigma}$ satisfies extended condition A, the following inequalities for ABFIO hold*

$$\begin{aligned}&\Delta\left(\frac{2m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}{2}\right)\\ &\leqslant\frac{B(\omega)\Gamma(\omega)}{2\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right\}+{}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right] \qquad (4.1)\\ &\quad-\frac{(1-\omega)\Gamma(\omega)}{2\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega}}\left[\Delta\left(m\mathfrak{x}_a\right)+\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right]\leqslant\frac{m\Delta\left(\mathfrak{x}_a\right)+\Delta\left(\mathfrak{x}_b\right)}{2},\end{aligned}$$

where $\omega\in(0,1]$, $B(\omega)>0$ is normalization function and $\Gamma(.)$ is Gamma function.

Proof. Since $\mathfrak{x}_a, \mathfrak{x}_b \in I$ and $I \subseteq \mathbb{R}$ be an open and non-empty m-convex subset with respect to I, for every $\mathfrak{z} \in [0,1]$, we have $m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) \in I$. By the property of G_mCRF on $\left[m\mathfrak{x}_a, \mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]$, we have for every $x, y \in \left[m\mathfrak{x}_a, m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]$ with $\mathfrak{z} = \frac{1}{2}$,

$$\Delta\left(x + \frac{\mathcal{R}^{\rho}_{\epsilon,\sigma}(y - mx)}{2}\right) \leqslant \frac{\Delta(x) + \Delta(y)}{2},$$

i.e., with $x = m\mathfrak{x}_a + (1-\mathfrak{z})\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)$ and $y = m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)$, using extended condition A, we get

$$\begin{aligned}
&2\Delta\left(m\mathfrak{x}_a + (1-\mathfrak{z})\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) + \frac{\mathcal{R}^{\rho}_{\epsilon,\sigma}(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) - m\mathfrak{x}_a + (1-\mathfrak{z})\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a))}{2}\right)\\
&= 2\Delta\left(m\mathfrak{x}_a + (1-\mathfrak{z})\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) + \frac{(2\mathfrak{z}-1)\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}{2}\right) \qquad (4.2)\\
&= 2\Delta\left(\frac{2m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}{2}\right) \leqslant \Delta\left(m\mathfrak{x}_a + (1-\mathfrak{z})\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right) + \Delta\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right).
\end{aligned}$$

Multiplying both sides of the above inequality (4.2) by $\frac{\omega}{B(\omega)\Gamma(\omega)}\mathfrak{z}^{\omega-1}$, then integrating the resulting inequality with respect to $\mathfrak{z}$ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{B(\omega)\Gamma(\omega)}\Delta\left(\frac{2m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}{2}\right)\\
&\leqslant \frac{\omega}{B(\omega)\Gamma(\omega)}\int_0^1 \mathfrak{z}^{\omega-1}\Delta\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z}\\
&\quad + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_0^1 t^{\omega-1}\Delta\left(m\mathfrak{x}_a + (1-\mathfrak{z})\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z}\\
&= \frac{\omega}{B(\omega)\Gamma(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\int_{m\mathfrak{x}_a}^{m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}(x - \mathfrak{x}_a)^{\omega-1}\Delta(x)dx\\
&\quad + \frac{\omega}{B(\omega)\Gamma(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\int_{m\mathfrak{x}_a}^{m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) - y\right)^{\omega-1}\Delta(y)dy.
\end{aligned}$$

Then we can write

$$\begin{aligned}
&\frac{2}{B(\omega)\Gamma(\omega)}\Delta\left(\frac{2m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}{2}\right)\\
&\leqslant \frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\left[\frac{\omega}{B(\omega)\Gamma(\omega)}\int_{m\mathfrak{x}_a}^{m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}(x - m\mathfrak{x}_a)^{\omega-1}\Delta(x)dx + \frac{(1-\omega)}{B(\omega)}\Delta(m\mathfrak{x}_a)\right]\\
&\quad - \frac{(1-\omega)}{B(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\Delta(m\mathfrak{x}_a)\\
&\quad + \frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\left[\frac{\omega}{B(\omega)\Gamma(\omega)}\int_{m\mathfrak{x}_a}^{\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) - y\right)^{\omega-1}\Delta(y)dy\right.\\
&\quad \left.+ \frac{(1-\omega)}{B(\omega)}\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right] - \frac{(1-\omega)}{B(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right).
\end{aligned}$$

So, using ABFIO, we get

$$\frac{2}{B(\omega)\Gamma(\omega)}\Delta\left(\frac{2m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}{2}\right)$$

$$\leqslant \frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}} \left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right]$$
$$- \frac{(1-\omega)}{B(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\left[\Delta\left(m\mathfrak{x}_a\right) + \Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right]$$

and the first inequality is proved. For the proof of the second inequality in the above inequality (4.1), we first note that if Δ is a G_mCRF, then we can write

$$\Delta\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right) \leqslant m(1-\mathfrak{z})\Delta\left(\mathfrak{x}_a\right) + \mathfrak{z}\Delta\left(\mathfrak{x}_b\right)$$

and

$$\Delta\left(m\mathfrak{x}_a + (1-\mathfrak{z})\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right) \leqslant m\mathfrak{z}\Delta\left(\mathfrak{x}_a\right) + (1-\mathfrak{z})\Delta\left(\mathfrak{x}_b\right).$$

By adding these inequalities side by side, we have

$$\Delta\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right) + \Delta\left(m\mathfrak{x}_a + (1-\mathfrak{z})\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right) \leqslant m\Delta\left(\mathfrak{x}_a\right) + \Delta\left(\mathfrak{x}_b\right). \tag{4.3}$$

Then, multiplying both sides of the above inequality (4.3) by $\frac{\omega}{B(\omega)\Gamma(\omega)}\mathfrak{z}^{\omega-1}$ and integrating the resulting inequality with respect to $\mathfrak{z}$ over $[0,1]$, we obtain

$$\frac{\omega}{B(\omega)\Gamma(\omega)}\int_0^1 \mathfrak{z}^{\omega-1}\Delta\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z}$$
$$+ \frac{\omega}{B(\omega)\Gamma(\omega)}\int_0^1 \mathfrak{z}^{\omega-1}\Delta\left(m\mathfrak{x}_a + (1-\mathfrak{z})\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z} \leqslant \frac{\omega}{B(\omega)\Gamma(\omega)}\left[m\Delta\left(\mathfrak{x}_a\right) + \Delta\left(\mathfrak{x}_b\right)\right]\int_0^1 \mathfrak{z}^{\omega-1}d\mathfrak{z}.$$

Then, we can write

$$\frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}} \left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right]$$
$$- \frac{(1-\omega)}{B(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\left[\Delta\left(m\mathfrak{x}_a\right) + \Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right] \leqslant \frac{m\Delta\left(\mathfrak{x}_a\right) + \Delta\left(\mathfrak{x}_b\right)}{B(\omega)\Gamma(\omega)}.$$

So, the proof of this theorem is completed. □

Remark 4.2. Choosing $m = 1$ and $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) = \mathfrak{x}_b - m\mathfrak{x}_a$ in the above theorem, we have the result in [9, Proposition 2.1], inequality (13).

Remark 4.3. Considering Theorem 4.1, we establish the following new mathematical approach of H-H inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick $\rho = (1,1,\ldots)$ with $\epsilon = \alpha$ and $\sigma = 1$:

$$\Delta\left(\frac{2m\mathfrak{x}_a + \mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)}{2}\right)$$
$$\leqslant \frac{B(\omega)\Gamma(\omega)}{2\left[\mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a + \mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathfrak{E}_{\alpha}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right]$$
$$- \frac{(1-\omega)\Gamma(\omega)}{2\left[\mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\left[\Delta\left(m\mathfrak{x}_a\right) + \Delta\left(m\mathfrak{x}_a + \mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right] \leqslant \frac{m\Delta\left(\mathfrak{x}_a\right) + \Delta\left(\mathfrak{x}_b\right)}{2},$$

If in Theorem 4.1, we put $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) = \mathfrak{x}_b - m\mathfrak{x}_a$, then we get the new variant of H-H integral inequality involving convexity via ABFIO.

Theorem 4.4. *If $\Delta : [m\mathfrak{x}_a, \mathfrak{x}_b] \to \mathbb{R}$ is a m-convex function and $\Delta \in L[m\mathfrak{x}_a, \mathfrak{x}_b]$, the following inequalities for ABFIO hold*

$$\Delta\left(\frac{m\mathfrak{x}_a + \mathfrak{x}_b}{2}\right) \leqslant \frac{B(\omega)\Gamma(\omega)}{2\left[(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(\mathfrak{x}_b\right)\right\} + {}^{AB}I^{\omega}_{\mathfrak{x}_b}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right] - \frac{(1-\omega)\Gamma(\omega)}{2\left[(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\left[\Delta\left(m\mathfrak{x}_a\right) + \Delta\left(\mathfrak{x}_b\right)\right]$$
$$\leqslant \frac{m\Delta\left(\mathfrak{x}_a\right) + \Delta\left(\mathfrak{x}_b\right)}{2}.$$

5. Refinements of H-H-type inequality via AB fractional integral operator

This section's goal is to explore and offer a novel equality. We derive some novel enhancements of H-H-type inequalities using an ABFIO based on this recently studied equality. To improve the content and grab readers' attention, we include a few remarks. First, we prove a lemma in the frame of ABFIO.

Throughout in this section, $B(\omega)$ represents the normalization function and $\Gamma(.)$ represents the Gamma function.

Lemma 5.1. *Let $I \subseteq \mathbb{R}$ be an open and non-empty m-convex subset and $\mathfrak{x}_a, \mathfrak{x}_b \in I$ with $m\mathfrak{x}_a < m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)$. Suppose that $\Delta : I \to \mathbb{R}$ be a differentiable function. If $\Delta' \in L\left[m\mathfrak{x}_a, m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]$, the following identity for AB fractional integral operators holds*

$$\begin{aligned}
&\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right] \\
&\quad - \left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega} + (1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta\left(m\mathfrak{x}_a\right) + \Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right] \\
&= \int_0^1 (1-\mathfrak{z})^{\omega}\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z} - \int_0^1 \mathfrak{z}^{\omega}\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z},
\end{aligned}$$

where $\omega \in (0,1], \mathfrak{z} \in [0,1]$.

Proof. By using integration, we have

$$\begin{aligned}
&\int_0^1 (1-\mathfrak{z})^{\omega}\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z} \\
&= \left.\frac{(1-\mathfrak{z})^{\omega}\Delta\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)}{\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\right|_0^1 + \frac{\omega}{\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\int_0^1 \Delta\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)(1-\mathfrak{z})^{\omega-1}d\mathfrak{z} \\
&= -\frac{m\Delta\left(\mathfrak{x}_a\right)}{\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)} + \frac{\omega}{\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\int_0^1 (1-\mathfrak{z})^{\omega-1}\Delta\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z} \\
&= -\frac{m\Delta\left(\mathfrak{x}_a\right)}{\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)} + \frac{\omega}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\int_{m\mathfrak{x}_a}^{m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) - x\right)^{\omega-1}\Delta(x)dx.
\end{aligned} \tag{5.1}$$

If we multiply both sides of the above inequality (5.1) by $\frac{1}{B(\omega)\Gamma(\omega)}$, we get

$$\begin{aligned}
&\frac{1}{B(\omega)\Gamma(\omega)}\int_0^1 (1-\mathfrak{z})^{\omega}\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z} \\
&\quad = -\frac{\Delta\left(m\mathfrak{x}_a\right)}{B(\omega)\Gamma(\omega)\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)} \\
&\qquad + \frac{\omega}{B(\omega)\Gamma(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\int_{m\mathfrak{x}_a}^{m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) - x\right)^{\omega-1}\Delta(x)dx.
\end{aligned}$$

Then, we can write

$$\begin{aligned}
&\frac{1}{B(\omega)\Gamma(\omega)}\int_0^1 (1-\mathfrak{z})^{\omega}\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z} \\
&\quad = -\frac{\Delta\left(m\mathfrak{x}_a\right)}{B(\omega)\Gamma(\omega)\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)} \\
&\qquad + \frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\left[\frac{\omega}{B(\omega)\Gamma(\omega)}\int_{m\mathfrak{x}_a}^{m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) - x\right)^{\omega-1}\Delta(x)dx\right.
\end{aligned}$$

$$+\frac{(1-\omega)}{B(\omega)}\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\Big]-\frac{(1-\omega)}{B(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right).$$

Using ABFIO, we have

$$\begin{aligned}&\frac{1}{B(\omega)\Gamma(\omega)}\int_0^1(1-\mathfrak{z})^{\omega}\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)d\mathfrak{z}\\&=-\frac{\Delta(m\mathfrak{x}_a)}{B(\omega)\Gamma(\omega)\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}+\frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right\}\right]\\&\quad-\frac{(1-\omega)}{B(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right).\end{aligned}\tag{5.2}$$

Similarly, using integration, we get

$$\begin{aligned}&\int_0^1\mathfrak{z}^{\omega}\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)d\mathfrak{z}\\&=\frac{\mathfrak{z}^{\omega}\Delta\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)}{\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\Bigg|_0^1-\frac{\omega}{\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\int_0^1\Delta\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\mathfrak{z}^{\omega-1}d\mathfrak{z}\\&=\frac{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)}{\Omega(\mathfrak{x}_b,\mathfrak{x}_a)}-\frac{\omega}{\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\int_0^1\mathfrak{z}^{\omega-1}\Delta\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)d\mathfrak{z}\\&=\frac{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)}{\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}-\frac{\omega}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\int_{m\mathfrak{x}_a}^{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}(u-m\mathfrak{x}_a)^{\omega-1}\Delta(u)du.\end{aligned}\tag{5.3}$$

If we multiply both sides of the above inequality (5.3) by $-\frac{1}{B(\omega)\Gamma(\omega)}$, we have

$$\begin{aligned}&-\frac{1}{B(\omega)\Gamma(\omega)}\int_0^1\mathfrak{z}^{\omega}\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)d\mathfrak{z}\\&=-\frac{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)}{B(\omega)\Gamma(\omega)\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\\&\quad+\frac{\omega}{B(\omega)\Gamma(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\int_{m\mathfrak{x}_a}^{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}(u-m\mathfrak{x}_a)^{\omega-1}\Delta(u)du.\end{aligned}$$

Then we can write

$$\begin{aligned}&-\frac{1}{B(\omega)\Gamma(\omega)}\int_0^1\mathfrak{z}^{\omega}\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)d\mathfrak{z}\\&=-\frac{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)}{B(\omega)\Gamma(\omega)\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\\&\quad+\frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\Bigg[\frac{\omega}{B(\omega)\Gamma(\omega)}\int_{m\mathfrak{x}_a}^{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}(u-m\mathfrak{x}_a)^{\omega-1}\Delta(u)du\\&\quad+\frac{(1-\omega)}{B(\omega)}\Delta(m\mathfrak{x}_a)\Bigg]-\frac{(1-\omega)}{B(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\Delta(m\mathfrak{x}_a).\end{aligned}$$

Using ABFIO, we have

$$\begin{aligned}
&-\frac{1}{B(\omega)\Gamma(\omega)}\int_0^1 \mathfrak{z}^{\omega}\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)d\mathfrak{z}\\
&=-\frac{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)}{B(\omega)\Gamma(\omega)\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}+\frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right]\\
&\quad-\frac{(1-\omega)}{B(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\Delta\left(m\mathfrak{x}_a\right).
\end{aligned}\tag{5.4}$$

By adding identities (5.2) and (5.4), we obtain the proof of Lemma 5.1. □

Remark 5.2. If we put $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)=\mathfrak{x}_b-m\mathfrak{x}_a$ and $m=1$ in Lemma 5.1, then we have the result in [9, Theorem 3.1], equality (29).

Theorem 5.3. *Let $I\subseteq\mathbb{R}$ be an open and non-empty m-convex set and $\mathfrak{x}_a,\mathfrak{x}_b\in I$ with $m\mathfrak{x}_a<m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)$. Suppose that $\Delta: I\to\mathbb{R}$ is a differentiable function and $\Delta'\in L\left[m\mathfrak{x}_a, m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]$. If $|\Delta'|$ is G_mCRF, then we have the following inequality for ABFIO*

$$\begin{aligned}
&\Bigg|\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right\}+{}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right]\\
&\quad-\left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega}+(1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta\left(m\mathfrak{x}_a\right)+\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right]\Bigg|\\
&\leqslant\frac{m\left|\Delta'\left(\mathfrak{x}_a\right)\right|+\left|\Delta'\left(\mathfrak{x}_b\right)\right|}{\omega+1},
\end{aligned}$$

where $\omega\in(0,1]$.

Proof. By using the identity given in Lemma 5.1 and the properties of modulus, we can write

$$\begin{aligned}
&\Bigg|\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right\}+{}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right]\\
&\quad-\left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega}+(1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta\left(m\mathfrak{x}_a\right)+\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right]\Bigg|\\
&=\left|\int_0^1(1-\mathfrak{z})^{\omega}\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)d\mathfrak{z}-\int_0^1\mathfrak{z}^{\omega}\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)d\mathfrak{z}\right|\\
&\leqslant\int_0^1(1-\mathfrak{z})^{\omega}\left|\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right|d\mathfrak{z}+\int_0^1\mathfrak{z}^{\omega}\left|\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right|d\mathfrak{z}.
\end{aligned}$$

Since $|\Delta'|$ is G_mCRF, we obtain

$$\begin{aligned}
&\Bigg|\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right\}+{}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right]\\
&\quad-\left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega}+(1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta\left(m\mathfrak{x}_a\right)+\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right]\Bigg|\\
&\leqslant\int_0^1(1-\mathfrak{z})^{\omega}\left[m(1-\mathfrak{z})\left|\Delta'\left(\mathfrak{x}_a\right)\right|+\mathfrak{z}\left|\Delta'\left(\mathfrak{x}_b\right)\right|\right]d\mathfrak{z}+\int_0^1\mathfrak{z}^{\omega}\left[m(1-\mathfrak{z})\left|\Delta'\left(\mathfrak{x}_a\right)\right|+\mathfrak{z}\left|\Delta'\left(\mathfrak{x}_b\right)\right|\right]d\mathfrak{z}\\
&=\frac{m\left|\Delta'\left(\mathfrak{x}_a\right)\right|+\left|\Delta'\left(\mathfrak{x}_b\right)\right|}{\omega+1}.
\end{aligned}$$

So, the proof is completed. □

Remark 5.4. Considering Theorem 5.3, we establish the following new mathematical approach of H-H inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick $\rho = (1, 1, \ldots)$ with $\epsilon = \alpha$ and $\sigma = 1$:

$$\left|\frac{B(\omega)\Gamma(\omega)}{[\mathfrak{E}_\alpha(\mathfrak{x}_b - m\mathfrak{x}_a)]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\{\Delta(m\mathfrak{x}_a + \mathfrak{E}_\alpha(\mathfrak{x}_b - m\mathfrak{x}_a))\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a + \mathfrak{E}_\alpha(\mathfrak{x}_b - m\mathfrak{x}_a)}\{\Delta(m\mathfrak{x}_a)\}\right]\right.$$
$$\left. - \left(\frac{[\mathfrak{E}_\alpha(\mathfrak{x}_b - m\mathfrak{x}_a)]^{\omega} + (1-\omega)\Gamma(\omega)}{[\mathfrak{E}_\alpha(\mathfrak{x}_b - m\mathfrak{x}_a)]^{\omega+1}}\right)[\Delta(m\mathfrak{x}_a) + \Delta(m\mathfrak{x}_a + \mathfrak{E}_\alpha(\mathfrak{x}_b - m\mathfrak{x}_a))]\right|$$
$$\leqslant \frac{m|\Delta'(\mathfrak{x}_a)| + |\Delta'(\mathfrak{x}_b)|}{\omega+1}.$$

Corollary 5.5. *In the above Theorem 5.3, if we choose $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) = \mathfrak{x}_b - m\mathfrak{x}_a$, then we obtain*

$$\left|\frac{B(\omega)\Gamma(\omega)}{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\{\Delta(\mathfrak{x}_b)\} + {}^{AB}I^{\omega}_{\mathfrak{x}_b}\{\Delta(m\mathfrak{x}_a)\}\right]\right.$$
$$\left. - \left(\frac{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega} + (1-\omega)\Gamma(\omega)}{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega+1}}\right)[\Delta(m\mathfrak{x}_a) + \Delta(\mathfrak{x}_b)]\right| \leqslant \frac{m|\Delta'(\mathfrak{x}_a)| + |\Delta'(\mathfrak{x}_b)|}{\omega+1}.$$

Theorem 5.6. *Let $I \subseteq \mathbb{R}$ be an open and non-empty m-covex set and $\mathfrak{x}_a, \mathfrak{x}_b \in I$ with $m\mathfrak{x}_a < m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)$. Suppose that $\Delta : I \to \mathbb{R}$ is a differentiable function and $\Delta' \in L\left[m\mathfrak{x}_a, m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]$. If $|\Delta'|^q$ is a G_mCRF, then we have the following inequality for ABFIO:*

$$\left|\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\{\Delta(m\mathfrak{x}_a)\}\right]\right.$$
$$\left. - \left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega} + (1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta(m\mathfrak{x}_a) + \Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right]\right|$$
$$\leqslant 2\left(\frac{1}{\omega p + 1}\right)^{\frac{1}{p}}\left(\frac{m|\Delta'(\mathfrak{x}_a)|^q + |\Delta'(\mathfrak{x}_b)|^q}{2}\right)^{\frac{1}{q}},$$

where $p^{-1} + q^{-1} = 1, q > 1, \omega \in (0, 1]$.

Proof. By using Lemma 5.1, we get

$$\left|\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\{\Delta(m\mathfrak{x}_a)\}\right]\right.$$
$$\left. - \left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega} + (1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta(m\mathfrak{x}_a) + \Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right]\right|$$
$$\leqslant \int_0^1 (1-\mathfrak{z})^{\omega}\left|\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right| d\mathfrak{z} + \int_0^1 \mathfrak{z}^{\omega}\left|\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right| d\mathfrak{z}.$$

By applying Hölder inequality, we get

$$\left|\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\{\Delta(m\mathfrak{x}_a)\}\right]\right.$$
$$\left. - \left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega} + (1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta(m\mathfrak{x}_a) + \Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right]\right|$$

$$\leqslant \left(\int_0^1 (1-\mathfrak{z})^{\omega p} d\mathfrak{z}\right)^{\frac{1}{p}} \left(\int_0^1 \left|\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right|^q d\mathfrak{z}\right)^{\frac{1}{q}}$$
$$+ \left(\int_0^1 \mathfrak{z}^{\omega p} d\mathfrak{z}\right)^{\frac{1}{p}} \left(\int_0^1 \left|\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right|^q d\mathfrak{z}\right)^{\frac{1}{q}}.$$

By using G_mCRF of $|\Delta'|^q$, we obtain

$$\left|\frac{B(\omega)\Gamma(\omega)}{[\Omega(\mathfrak{x}_b,\mathfrak{x}_a)]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\{\Delta(m\mathfrak{x}_a)\}\right]\right.$$
$$\left. - \left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega} + (1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta(m\mathfrak{x}_a) + \Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right]\right|$$
$$\leqslant \left(\int_0^1 (1-\mathfrak{z})^{\omega p} d\mathfrak{z}\right)^{\frac{1}{p}} \left(\int_0^1 \left[m(1-\mathfrak{z})\left|\Delta'(\mathfrak{x}_a)\right|^q + \mathfrak{z}\left|\Delta'(\mathfrak{x}_b)\right|^q\right] d\mathfrak{z}\right)^{\frac{1}{q}}$$
$$+ \left(\int_0^1 \mathfrak{z}^{\omega p} d\mathfrak{z}\right)^{\frac{1}{p}} \left(\int_0^1 \left[m(1-\mathfrak{z})\left|\Delta'(\mathfrak{x}_a)\right|^q + \mathfrak{z}\left|\Delta'(\mathfrak{x}_b)\right|^q\right] d\mathfrak{z}\right)^{\frac{1}{q}}.$$

By calculating the integrals in the above inequality, we get the desired result. □

Remark 5.7. Considering Theorem 5.6, we establish the following new mathematical approach of H-H inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick $\rho = (1, 1, \ldots)$ with $\epsilon = \alpha$ and $\sigma = 1$:

$$\left|\frac{B(\omega)\Gamma(\omega)}{[\mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\{\Delta(m\mathfrak{x}_a + \mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a))\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathfrak{E}_{\alpha}(\mathfrak{x}_b-m\mathfrak{x}_a)}\{\Delta(m\mathfrak{x}_a)\}\right]\right.$$
$$\left. - \left(\frac{[\mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)]^{\omega} + (1-\omega)\Gamma(\omega)}{[\mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)]^{\omega+1}}\right)[\Delta(m\mathfrak{x}_a) + \Delta(m\mathfrak{x}_a + \mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a))]\right|$$
$$\leqslant 2\left(\frac{1}{\omega p + 1}\right)^{\frac{1}{p}} \left(\frac{m\left|\Delta'(\mathfrak{x}_a)\right|^q + \left|\Delta'(\mathfrak{x}_b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.8. *In the above Theorem, if we choose* $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) = \mathfrak{x}_b - m\mathfrak{x}_a$, *then we obtain*

$$\left|\frac{B(\omega)\Gamma(\omega)}{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\{\Delta(\mathfrak{x}_b)\} + {}^{AB}I^{\omega}_{\mathfrak{x}_b}\{\Delta(m\mathfrak{x}_a)\}\right]\right.$$
$$\left. - \left(\frac{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega} + (1-\omega)\Gamma(\omega)}{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega+1}}\right)[\Delta(m\mathfrak{x}_a) + \Delta(\mathfrak{x}_b)]\right| \leqslant 2\left(\frac{1}{\omega p + 1}\right)^{\frac{1}{p}} \left(\frac{\left|\Delta'(m\mathfrak{x}_a)\right|^q + \left|\Delta'(\mathfrak{x}_b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 5.9. *Let* $I \subseteq \mathbb{R}$ *be an open and non-empty* m*-convex set and* $\mathfrak{x}_a, \mathfrak{x}_b \in I$ *with* $m\mathfrak{x}_a < m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)$. *Suppose that* $\Delta : I \to \mathbb{R}$ *is a differentiable function and* $\Delta' \in L\left[m\mathfrak{x}_a, m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]$. *If* $|\Delta'|^q$ *is a* G_mCRF, *then we have the following inequality for ABFIO:*

$$\left|\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\{\Delta(m\mathfrak{x}_a)\}\right]\right.$$
$$\left. - \left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega} + (1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta(m\mathfrak{x}_a) + \Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right]\right|$$

$$\leqslant\left(\frac{1}{\omega+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{m\left|\Delta'\left(\mathfrak{x}_a\right)\right|^q}{\omega+2}+\frac{\left|\Delta'\left(\mathfrak{x}_b\right)\right|^q}{(\omega+1)(\omega+2)}\right)^{\frac{1}{q}}+\left(\frac{m\left|\Delta'\left(\mathfrak{x}_a\right)\right|^q}{(\omega+1)(\omega+2)}+\frac{\left|\Delta'\left(\mathfrak{x}_b\right)\right|^q}{\omega+2}\right)^{\frac{1}{q}}\right],$$

where $\omega\in(0,1]$, $q\geqslant 1$.

Proof. Employing Lemma 5.1 and utilizing the power mean inequality, we get

$$\begin{aligned}
&\Bigg|\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right\}+{}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right]\\
&\quad-\left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega}+(1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta\left(m\mathfrak{x}_a\right)+\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right]\Bigg|\\
&\leqslant\int_0^1(1-\mathfrak{z})^{\omega}\left|\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right|d\mathfrak{z}+\int_0^1\mathfrak{z}^{\omega}\left|\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right|d\mathfrak{z}\\
&\leqslant\left(\int_0^1(1-\mathfrak{z})^{\omega}d\mathfrak{z}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\mathfrak{z})^{\omega}\left|\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right|^q d\mathfrak{z}\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\mathfrak{z}^{\omega}d\mathfrak{z}\right)^{1-\frac{1}{q}}\left(\int_0^1\mathfrak{z}^{\omega}\left|\Delta'\left(m\mathfrak{x}_a+\mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right|^q d\mathfrak{z}\right)^{\frac{1}{q}}.
\end{aligned}$$

By using G_mCRF of $|\Delta'|^q$, we have

$$\begin{aligned}
&\Bigg|\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right\}+{}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right]\\
&\quad-\left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega}+(1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta\left(m\mathfrak{x}_a\right)+\Delta\left(m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right]\Bigg|\\
&\leqslant\left(\int_0^1(1-\mathfrak{z})^{\omega}d\mathfrak{z}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\mathfrak{z})^{\omega}\left[m(1-\mathfrak{z})\left|\Delta'\left(\mathfrak{x}_a\right)\right|^q+\mathfrak{z}\left|\Delta'\left(\mathfrak{x}_b\right)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\mathfrak{z}^{\omega}d\mathfrak{z}\right)^{1-\frac{1}{q}}\left(\int_0^1\mathfrak{z}^{\omega}\left[m(1-\mathfrak{z})\left|\Delta'\left(\mathfrak{x}_a\right)\right|^q+\mathfrak{z}\left|\Delta'\left(\mathfrak{x}_b\right)\right|^q\right]d\mathfrak{z}\right)^{\frac{1}{q}}\\
&=\left(\frac{1}{\omega+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{m\left|\Delta'\left(\mathfrak{x}_a\right)\right|^q}{\omega+2}+\frac{\left|\Delta'\left(\mathfrak{x}_b\right)\right|^q}{(\omega+1)(\omega+2)}\right)^{\frac{1}{q}}+\left(\frac{m\left|\Delta'\left(\mathfrak{x}_a\right)\right|^q}{(\omega+1)(\omega+2)}+\frac{\left|\Delta'\left(\mathfrak{x}_b\right)\right|^q}{\omega+2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Th proof is completed. □

Remark 5.10. Considering Theorem 5.9, we establish the following new mathematical approach of H-H inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick $\rho=(1,1,\dots)$ with $\epsilon=\alpha$ and $\sigma=1$:

$$\begin{aligned}
&\Bigg|\frac{B(\omega)\Gamma(\omega)}{\left[\mathfrak{E}_{\alpha}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a+\mathfrak{E}_{\alpha}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right\}+{}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathfrak{E}_{\alpha}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta\left(m\mathfrak{x}_a\right)\right\}\right]\\
&\quad-\left(\frac{\left[\mathfrak{E}_{\alpha}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega}+(1-\omega)\Gamma(\omega)}{\left[\mathfrak{E}_{\alpha}(\mathfrak{x}_b-m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta\left(m\mathfrak{x}_a\right)+\Delta\left(m\mathfrak{x}_a+\mathfrak{E}_{\alpha}(\mathfrak{x}_b-m\mathfrak{x}_a)\right)\right]\Bigg|\\
&\leqslant\left(\frac{1}{\omega+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{m\left|\Delta'\left(\mathfrak{x}_a\right)\right|^q}{\omega+2}+\frac{\left|\Delta'\left(\mathfrak{x}_b\right)\right|^q}{(\omega+1)(\omega+2)}\right)^{\frac{1}{q}}+\left(\frac{m\left|\Delta'\left(\mathfrak{x}_a\right)\right|^q}{(\omega+1)(\omega+2)}+\frac{\left|\Delta'\left(\mathfrak{x}_b\right)\right|^q}{\omega+2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 5.11. *In the above Theorem, if we choose* $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) = \mathfrak{x}_b - m\mathfrak{x}_a$, *we obtain*

$$\begin{aligned}&\Bigg|\frac{B(\omega)\Gamma(\omega)}{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\{\Delta(\mathfrak{x}_b)\} + {}^{AB}I^{\omega}_{\mathfrak{x}_b}\{\Delta(m\mathfrak{x}_a)\}\right]\\ &\quad - \left(\frac{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega} + (1-\omega)\Gamma(\omega)}{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega+1}}\right)[\Delta(m\mathfrak{x}_a) + \Delta(\mathfrak{x}_b)]\Bigg|\\ &\leqslant \left(\frac{1}{\omega+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{|\Delta'(m\mathfrak{x}_a)|^q}{\omega+2} + \frac{|\Delta'(\mathfrak{x}_b)|^q}{(\omega+1)(\omega+2)}\right)^{\frac{1}{q}} + \left(\frac{|\Delta'(m\mathfrak{x}_a)|^q}{(\omega+1)(\omega+2)} + \frac{|\Delta'(\mathfrak{x}_b)|^q}{\omega+2}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Theorem 5.12. *Let* $I \subseteq \mathbb{R}$ *be an open and non-empty* m*-convex set and* $\mathfrak{x}_a, \mathfrak{x}_b \in I$ *with* $m\mathfrak{x}_a < m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)$. *Suppose that* $\Delta : I \to \mathbb{R}$ *is a differentiable function and* $\Delta' \in L\left[m\mathfrak{x}_a, m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]$. *If* $|\Delta'|^q$ *is a* G_mCRF*, then we have the following inequality for ABFIO:*

$$\begin{aligned}&\Bigg|\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\{\Delta(m\mathfrak{x}_a)\}\right]\\ &\quad - \left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega} + (1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta(m\mathfrak{x}_a) + \Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right]\Bigg|\\ &\leqslant \frac{2}{p(\omega p+1)} + \frac{m|\Delta'(\mathfrak{x}_a)|^q + |\Delta'(\mathfrak{x}_b)|^q}{q},\end{aligned}$$

where $p^{-1} + q^{-1} = 1, q > 1, \omega \in (0, 1]$.

Proof. By using the identity given in Lemma 5.1 and applying the Young inequality $xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q$, we get

$$\begin{aligned}&\Bigg|\frac{B(\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\{\Delta(m\mathfrak{x}_a)\}\right]\\ &\quad - \left(\frac{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega} + (1-\omega)\Gamma(\omega)}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega+1}}\right)\left[\Delta(m\mathfrak{x}_a) + \Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right]\Bigg|\\ &\leqslant \int_0^1 (1-\mathfrak{z})^{\omega}\left|\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right| d\mathfrak{z} + \int_0^1 \mathfrak{z}^{\omega}\left|\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right| d\mathfrak{z}.\\ &\leqslant \frac{1}{p}\int_0^1 (1-\mathfrak{z})^{\omega p} d\mathfrak{z} + \frac{1}{q}\int_0^1 \left|\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right|^q d\mathfrak{z}\\ &\quad + \frac{1}{p}\int_0^1 \mathfrak{z}^{\omega p} d\mathfrak{z} + \frac{1}{q}\int_0^1 \left|\Delta'\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right|^q d\mathfrak{z}.\end{aligned}$$

By using G_mCRF of $|\Delta'|^q$ and by a simple computation, we have the desired result. □

Remark 5.13. Considering Theorem 5.12, we establish the following new mathematical approach of H-H inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick $\rho = (1, 1, \ldots)$ with $\epsilon = \alpha$ and $\sigma = 1$:

$$\begin{aligned}&\Bigg|\frac{B(\omega)\Gamma(\omega)}{[\mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)]^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\{\Delta(m\mathfrak{x}_a + \mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a))\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a+\Omega(\mathfrak{x}_b,\mathfrak{x}_a)}\{\Delta(m\mathfrak{x}_a)\}\right]\\ &\quad - \left(\frac{[\mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)]^{\omega} + (1-\omega)\Gamma(\omega)}{[\mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)]^{\omega+1}}\right)\left[\Delta(m\mathfrak{x}_a) + \Delta\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right]\Bigg|\\ &\leqslant \frac{2}{p(\omega p+1)} + \frac{m|\Delta'(\mathfrak{x}_a)|^q + |\Delta'(\mathfrak{x}_b)|^q}{q}.\end{aligned}$$

Corollary 5.14. *In the above Theorem 5.12, if we choose* $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) = \mathfrak{x}_b - m\mathfrak{x}_a$, *we obtain*

$$\left|\frac{B(\omega)\Gamma(\omega)}{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega+1}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\{\Delta(\mathfrak{x}_b)\} + {}^{AB}I^{\omega}_{\mathfrak{x}_b}\{\Delta(m\mathfrak{x}_a)\}\right] - \left(\frac{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega} + (1-\omega)\Gamma(\omega)}{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega+1}}\right)[\Delta(m\mathfrak{x}_a) + \Delta(\mathfrak{x}_b)]\right| \leqslant \frac{2}{p(\omega p+1)} + \frac{|\Delta'(m\mathfrak{x}_a)|^q + |\Delta'(\mathfrak{x}_b)|^q}{q}.$$

6. Pachpatte-type inequality via AB fractional integral operator

In this section, we study and explore the Pachpatte-type inequality via ABFIO. We enhance this section's utility through the notes that are provided.

Theorem 6.1. *Let* $I \subseteq \mathbb{R}$ *be an open and non-empty* m*-convex set and* $\mathfrak{x}_a, \mathfrak{x}_b \in I$ *with* $m\mathfrak{x}_a < m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)$. *If* $\Delta_1, \Delta_2 : [m\mathfrak{x}_a, m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)] \to \mathbb{R}$ *are* G_mCRF, $\Delta_1, \Delta_2 \in L[m\mathfrak{x}_a, m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)]$, *then the following inequality for ABFIO holds:*

$$\begin{aligned}
&\frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta_1\Delta_2\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)}\{\Delta_1\Delta_2(m\mathfrak{x}_a)\}\right] \\
&\leqslant \frac{\omega}{B(\omega)\Gamma(\omega)}\left[[\Delta_1(m\mathfrak{x}_a)\Delta_2(m\mathfrak{x}_a) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_b)]\left(\frac{2}{\omega(\omega+1)(\omega+2)} + \frac{1}{\omega+2}\right)\right. \\
&\quad \left. + 2\frac{m[\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_b) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_a)]}{(\omega+1)(\omega+2)}\right] + \frac{(1-\omega)}{B(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\Big[\Delta_1(m\mathfrak{x}_a)\Delta_2(m\mathfrak{x}_a) \\
&\quad + \Delta_1\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\Delta_2\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\Big],
\end{aligned}$$

where $\omega \in (0, 1]$.

Proof. Since Δ_1 and Δ_2 are G_mCRF on $[m\mathfrak{x}_a, m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)]$, we get

$$\Delta_1\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right) \leqslant m(1-\mathfrak{z})\Delta_1(\mathfrak{x}_a) + \mathfrak{z}\Delta_1(\mathfrak{x}_b) \tag{6.1}$$

and

$$\Delta_2\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right) \leqslant m(1-\mathfrak{z})\Delta_2(\mathfrak{x}_a) + \mathfrak{z}\Delta_2(\mathfrak{x}_b). \tag{6.2}$$

By multiplying both inequalities (6.1) and (6.2) side by side, we get

$$\begin{aligned}
&\Delta_1\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\Delta_2\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right) \\
&\quad \leqslant m^2(1-\mathfrak{z})^2\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_a) + \mathfrak{z}^2\Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_b) + m\mathfrak{z}(1-\mathfrak{z})[\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_b) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_a)].
\end{aligned} \tag{6.3}$$

By multiplying both sides of (6.3) with $(1-\mathfrak{z})^{\omega-1}$ and integrating the resulting inequality w.r.t. $\mathfrak{z}$ over $[0, 1]$, we obtain

$$\begin{aligned}
&\int_0^1 (1-\mathfrak{z})^{\omega-1}\Delta_1\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\Delta_2\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z} \\
&\leqslant \int_0^1 (1-\mathfrak{z})^{\omega-1}\left[m^2(1-\mathfrak{z})^2\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_a) + \mathfrak{z}^2\Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_b)\right. \\
&\quad \left. + m\mathfrak{z}(1-\mathfrak{z})[\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_b) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_a)]\right]d\mathfrak{z} \\
&= \frac{m^2\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_a)}{\omega+2} + 2\frac{\Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_b)}{\omega(\omega+1)(\omega+2)} + \frac{m[\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_b) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_a)]}{(\omega+1)(\omega+2)}.
\end{aligned} \tag{6.4}$$

By changing the variable $m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) = x$, we can write the inequality in (6.4) as

$$\begin{aligned} &\frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}} \int_{m\mathfrak{x}_a}^{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)} \left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) - x\right)^{\omega-1} \Delta_1(x)\Delta_2(x)dx \\ &\quad \leqslant \frac{m^2\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_a)}{\omega+2} + 2\frac{\Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_b)}{\omega(\omega+1)(\omega+2)} + \frac{m\left[\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_b) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_a)\right]}{(\omega+1)(\omega+2)}. \end{aligned} \tag{6.5}$$

By multiplying the both sides of (6.5) by $\frac{\omega}{B(\omega)\Gamma(\omega)}$ and then adding the term $\frac{(1-\omega)}{B(\omega)[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)]^{\omega}}\Delta_1(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a))\Delta_2\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)$ to both sides of (6.5) and finally using ABFIO, we get

$$\begin{aligned} &\frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}} \left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta_1\Delta_2\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\}\right] \\ &\leqslant \frac{\omega}{B(\omega)\Gamma(\omega)}\left[\frac{m^2\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_a)}{\omega+2} + 2\frac{\Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_b)}{\omega(\omega+1)(\omega+2)} + \frac{m\left[\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_b, m) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_a)\right]}{(\omega+1)(\omega+2)}\right] \\ &\quad + \frac{(1-\omega)}{B(\omega)\left[\Omega(\mathfrak{x}_b, \mathfrak{x}_a)\right]^{\omega}}\Delta_1\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\Delta_2\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right). \end{aligned} \tag{6.6}$$

Similarly, by multiplying both sides of (6.3) with $\mathfrak{z}^{\omega-1}$ and integrating the resulting inequality w.r.t. $\mathfrak{z}$ over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 \mathfrak{z}^{\omega-1}\Delta_1\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\Delta_2\left(m\mathfrak{x}_a + \mathfrak{z}\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)d\mathfrak{z} \\ &\leqslant \int_0^1 \mathfrak{z}^{\omega-1}\left[m^2(1-\mathfrak{z})^2\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_a) + \mathfrak{z}^2\Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_b) + m\mathfrak{z}(1-\mathfrak{z})\left[\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_b) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_a)\right]\right]d\mathfrak{z} \\ &= 2\frac{m^2\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_a)}{\omega(\omega+1)(\omega+2)} + \frac{\Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_b)}{\omega+2} + \frac{m\left[\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_b) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_a)\right]}{(\omega+1)(\omega+2)}. \end{aligned}$$

By making calculations similar to those in the proof of (6.6), we obtain

$$\begin{aligned} &\frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}} \left[{}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta_1\Delta_2(m\mathfrak{x}_a)\right\}\right] \\ &\leqslant \frac{\omega}{B(\omega)\Gamma(\omega)}\left[2\frac{m^2\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_a)}{\omega(\omega+1)(\omega+2)} + 2\frac{\Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_b)}{\omega+2} + \frac{m\left[\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_b) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_a)\right]}{(\omega+1)(\omega+2)}\right] \\ &\quad + \frac{(1-\omega)}{B(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\Delta_1(m\mathfrak{x}_a)\Delta_2(m\mathfrak{x}_a). \end{aligned} \tag{6.7}$$

Adding (6.6) and (6.7) side by side, we get

$$\begin{aligned} &\frac{1}{\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}} \left[{}^{AB}_{m\mathfrak{x}_a}I^{\omega}\left\{\Delta_1\Delta_2\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a+\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b-m\mathfrak{x}_a)}\left\{\Delta_1\Delta_2(m\mathfrak{x}_a)\right\}\right] \\ &\leqslant \frac{\omega}{B(\omega)\Gamma(\omega)}\left[\left[m^2\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_a) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_b)\right]\left(\frac{2}{\omega(\omega+1)(\omega+2)} + \frac{1}{\omega+2}\right)\right. \\ &\quad \left. + 2\frac{m\left[\Delta_1(\mathfrak{x}_a)\Delta_2(\mathfrak{x}_b) + \Delta_1(\mathfrak{x}_b)\Delta_2(\mathfrak{x}_a)\right]}{(\omega+1)(\omega+2)}\right] + \frac{(1-\omega)}{B(\omega)\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right]^{\omega}}\left[\Delta_1(m\mathfrak{x}_a)\Delta_2(m\mathfrak{x}_a)\right. \\ &\quad \left. + \Delta_1\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\Delta_2\left(m\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a)\right)\right]. \end{aligned}$$

The proof is completed. □

Remark 6.2. Considering Theorem 6.1, we establish the following new mathematical approach of Pachpatte-type inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick $\rho = (1, 1, \dots)$ with $\epsilon = \alpha$ and $\sigma = 1$:

$$\frac{1}{[\mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)]^{\omega}} \left[{}^{AB}_{m\mathfrak{x}_a} I^{\omega} \{\Delta_1\Delta_2 (m\mathfrak{x}_a + \mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a))\} + {}^{AB}I^{\omega}_{m\mathfrak{x}_a + \mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)} \{\Delta_1\Delta_2 (m\mathfrak{x}_a)\} \right]$$
$$\leqslant \frac{\omega}{B(\omega)\Gamma(\omega)} \left[[\Delta_1 (m\mathfrak{x}_a) \Delta_2 (m\mathfrak{x}_a) + \Delta_1 (\mathfrak{x}_b) \Delta_2 (\mathfrak{x}_b)] \left(\frac{2}{\omega(\omega+1)(\omega+2)} + \frac{1}{\omega+2} \right) \right.$$
$$\left. +2\frac{m [\Delta_1 (\mathfrak{x}_a) \Delta_2 (\mathfrak{x}_b) + \Delta_1 (\mathfrak{x}_b) \Delta_2 (\mathfrak{x}_a)]}{(\omega+1)(\omega+2)} \right] + \frac{(1-\omega)}{B(\omega) [\mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)]^{\omega}} \Big[\Delta_1 (m\mathfrak{x}_a) \Delta_2 (m\mathfrak{x}_a)$$
$$+ \Delta_1 (m\mathfrak{x}_a + \mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)) \Delta_2 (m\mathfrak{x}_a + \mathfrak{E}_{\alpha}(\mathfrak{x}_b - m\mathfrak{x}_a)) \Big].$$

Also, in the above Theorem 6.1, if we put $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b - m\mathfrak{x}_a) = \mathfrak{x}_b - m\mathfrak{x}_a$, then we get the new variant of Pachpatte-type integral inequality involving m-convexity via ABFIO.

Theorem 6.3. *If $\Delta_1, \Delta_2 : [m\mathfrak{x}_a, \mathfrak{x}_b] \to \mathbb{R}$ are m-convex functions, $\Delta_1, \Delta_2 \in L[m\mathfrak{x}_a, \mathfrak{x}_b]$, then the following inequality for ABFIO holds:*

$$\frac{1}{(\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega}} \left[{}^{AB}_{m\mathfrak{x}_a} I^{\omega} \{\Delta_1\Delta_2 (\mathfrak{x}_b)\} + {}^{AB}I^{\omega}_{\mathfrak{x}_b} \{\Delta_1\Delta_2 (m\mathfrak{x}_a)\} \right]$$
$$\leqslant \frac{\omega}{B(\omega)\Gamma(\omega)} \left[[\Delta_1 (m\mathfrak{x}_a) \Delta_2 (m\mathfrak{x}_a) + \Delta_1 (\mathfrak{x}_b) \Delta_2 (\mathfrak{x}_b)] \left(\frac{2}{\omega(\omega+1)(\omega+2)} + \frac{1}{\omega+2} \right) \right.$$
$$\left. +2\frac{m [\Delta_1 (\mathfrak{x}_a) \Delta_2 (\mathfrak{x}_b) + \Delta_1 (\mathfrak{x}_b) \Delta_2 (\mathfrak{x}_a)]}{(\omega+1)(\omega+2)} \right] + \frac{(1-\omega)}{B(\omega) (\mathfrak{x}_b - m\mathfrak{x}_a)^{\omega}} [\Delta_1 (m\mathfrak{x}_a) \Delta_2 (m\mathfrak{x}_a) + \Delta_1 (\mathfrak{x}_b) \Delta_2 (\mathfrak{x}_b)].$$

7. Applications to entropy

Consider the exponential random variable E_{λ} with parameter $\lambda > 0$, whose probability density function (PDF) is $f(\mu) = \lambda e^{-\lambda\mu}$, $\mu \geqslant 0$. The Shannon entropy of E_{λ} is defined by

$$H(E_{\lambda}) = -\int_0^{\infty} f(\mu) \log f(\mu) d\mu = -\int_0^{\infty} \lambda e^{-\lambda\mu} \log \left(\lambda e^{-\lambda\mu}\right) d\mu.$$

Compute the Shannon entropy explicitly using standard integration,

$$\begin{aligned} H(E_{-}) &= -\int_0^{\infty} \lambda e^{-\lambda\mu} \big(\log\lambda - \lambda\mu \big) d\mu \\ &= -\log\lambda \int_0^{\infty} \lambda e^{-\lambda\mu} dx + \lambda \int_0^{\infty} \lambda\mu e^{-\lambda\mu} dx = -\log\lambda \cdot 1 + \lambda \cdot \frac{1}{\lambda} = 1 - \log\lambda. \end{aligned}$$

Define the function Δ related to the integrand and consider

$$\Delta(\mu) := -\log f(\mu) = -\log \left(\lambda e^{-\lambda\mu}\right) = \lambda\mu - \log\lambda,$$

which is linear (hence convex) on $[0, \infty)$. Choose the interval and Mittag-Leffler deformation and restrict the domain to the compact interval

$$[\mathfrak{x}_a, \mathfrak{x}_b] := \left[0, \frac{1}{\lambda}\right].$$

Define the Mittag-Leffler type deformation

$$\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b) := \sum_{k=0}^{\infty} \frac{\rho(k)}{\Gamma(\epsilon k + \sigma)} \left(\frac{1}{\lambda}\right)^k,$$

with parameters $\rho(k) = 1$, $\epsilon = \alpha \in (0, 1)$, $\sigma = 1$, so that

$$\mathcal{R}^{(1)}_{\alpha,1}\left(\frac{1}{\lambda}\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{\lambda}\right)^k}{\Gamma(1+\alpha k)} =: \mathrm{E}_\alpha\left(\frac{1}{\lambda}\right),$$

the classical Mittag-Leffler function. Applying Theorem 4.1 to Δ. From the integral inequality theorem for generalized-convex functions and AB-fractional integrals, we have the double inequality for $m = 1$:

$$\begin{aligned}\Delta\left(\frac{2\mathfrak{x}_a + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b)}{2}\right) \leqslant \frac{B(\gamma)\Gamma(\gamma)}{2\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b)\right]^{\gamma}}&\left[0\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b)\Delta\left(\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b)\right) + \mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b)0\Delta(0)\right]\\ &- \frac{(1-\gamma)\Gamma(\gamma)}{2\left[\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b)\right]^{\gamma}}\left[\Delta(0) + \Delta\left(\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b)\right)\right] \leqslant \frac{\Delta(0) + \Delta(\mathfrak{x}_b)}{2}.\end{aligned} \tag{7.1}$$

Substitute the values $\mathfrak{x}_a = 0$, $\mathfrak{x}_b = \frac{1}{\lambda}$, $\mathcal{R}^{\rho}_{\epsilon,\sigma}(\mathfrak{x}_b) = \mathrm{E}_\alpha\left(\frac{1}{\lambda}\right)$, and recall $\Delta(\mu) = \lambda\mu - \log\lambda$. Thus,

$$\Delta\left(\frac{\mathrm{E}_\alpha\left(\frac{1}{\lambda}\right)}{2}\right) = \lambda \cdot \frac{\mathrm{E}_\alpha\left(\frac{1}{\lambda}\right)}{2} - \log\lambda,$$

and

$$\frac{\Delta(0) + \Delta(\frac{1}{\lambda})}{2} = \frac{(-\log\lambda) + (1 - \log\lambda)}{2} = \frac{1 - 2\log\lambda}{2}.$$

Therefore, inequality (7.1) becomes

$$\lambda \cdot \frac{\mathrm{E}_\alpha\left(\frac{1}{\lambda}\right)}{2} - \log\lambda \leqslant (\text{fractional AB integral expression}) \leqslant \frac{1 - 2\log\lambda}{2}.$$

This inequality provides fractional integral bounds on the entropy-related function Δ, linking fractional calculus, Mittag-Leffler functions, and information-theoretic measures.

8. Conclusions

Authors and researchers from a wide range of fields have shown a great deal of interest in fractional calculus. Convexity theory, on the other hand, has become an effective technique for creating novel numerical models that make it possible to solve challenging issues in the applied and pure sciences. As a result of continuous advancements, extensions, and applications, convex analysis and the inequalities that go along with it are seeing a rise in research interest and popularity. In this work:

(1) first, we explored a new approach of H-H inequality via ABFIO with some remarks and corollaries;
(2) we introduced a new lemma, further, we discussed some new refinements of H-H inequality based on newly constructed lemma;
(3) we introduced a new sort of Pachpatte-type inequality via a newly introduced concept in the frame of the ABFIO.

Furthermore, the discussed inequalities can be explored within the frameworks of quantum calculus and interval analysis. Notably, integral inequality is an rapidly evolving research area. The integration of interval-valued analysis and quantum calculus into the study of integral inequalities is poised to captivate scientists, offering exciting avenues for future exploration.

Acknowledgment

This research was funded by the National Science, Research and Innovation Fund (NSRF) and King Mongkut's University of Technology North Bangkok, under Contract No. KMUTNB-FF-68-B-04.

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