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Data-driven decision-making framework for the evaluation of the traders in the stock market using cosine trigonometric single-valued neutrosophic approach

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Abstract

The cosine trigonometric single valued neutrosophic number (CT-SVNN) is a suitable expansion of the standard neutrosophic number. Single-valued neutrosophic sets (SVNSs) may effectively overcome three components: degree of truth, indeterminacy, and falsity. In recent years, the aggregation operator (AO) and its applications have undergone development. This study introduces a few new AOs for multi-attribute decision-making (MADM). We introduce a novel approach for cosine trigonometric SVNS (CT-SVNS) and CT-SVNS with normal (CT-SVNNS), which are SVNS extensions. It is also required to discuss the CT-SVNNS method fundamental features in this communication, such as idempotency, boundedness, commutativity and monotonicity. There are numerous CT-SVNNS operators that have been proposed, including CT-SVN normal weighted averaging (CT-SVNNWA), CT-SVN normal weighted geometric (CT-SVNNWG), generalized CT-SVNNWA (GCT-SVNNWA) and generalized CT-SVNNWG. A powerful strategy for solving the MADM problem is provided that makes use of new developed generalized operators. Through a case study, the value of the suggested MADM approach is demonstrated. The new strategy is shown using a market share problem, and the outcomes are contrasted and examined against an existing method. This combination of generalized AO was rated successful based on expert preferences. As a result, a varied collection of experts may be accepted.

Keywords: CT-SVNNWA, CT-SVNNWG, GCT-SVNNWA, GCT-SVNNWG, Hamming distance, aggregating operators.

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1. Introduction

The largest and most significant contributors to the expansion and development of the economy are thought to be stock markets. To understanding of the elements impacting stock market decision-making

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(DM) process while making investments. The importance of stocks to the national economy is closely linked to investment decision-making (IDM). Investors are now forced to switch from traditional methods to new, potent computer algorithms in order to maximize trading profits because of the growing complexity of stock markets and the numerous interconnected elements that influence their behavior. In order to facilitate the development of financial decision-making systems, efforts have been done for many years to improve market prediction using advanced instruments and cutting edge techniques recently found in the field of artificial intelligence. The business sector is complex, competitive and requires a wide range of multitasking skills. Most business companies nowadays employ information technology (IT) applications to improve both their operational efficiency and the quality of their goods and services. It illustrates how IT has developed into an essential element of the business sector. A number of applications have been used by the business world. E-commerce has gained recognition as a breakthrough in the business sector due to its ease of use and several advantages for individuals. It streamlines, accelerates, streamlines and increases the efficiency of business procedures. Expanding the usage of IT to other business areas, such as the stock market, is believed to offer an opportunity to look at this topic from an alternative viewpoint. Real-world structures are becoming increasingly complicated, which makes it harder for decision-makers to select the best option from a variety of possibilities. Consolidation is difficult, yet achieving a single goal is not difficult. Many businesses struggle to establish boundaries, objectives, and viewpoints. As a result, a committee or an individual must consider many objectives simultaneously when making judgments. Each decision-maker is prevented from reaching the optimal option, the optimum under all relevant criteria, by flexible responses to real-world problems. As a result, the decision maker is more likely to make the best conclusion by using practical and efficient approaches.

1.1. Related work

Almost every situation in the real world involves some degree of uncertainty. The fuzzy set (FS), intuitionistic FS (IFS) and neutrosophic set (NSS) are among the uncertain theories. A collection of elements in a given universal that have a grade or degree between 0 and 1 is called an FS, and it describes the membership degree of a particular element in that universal. The idea of an IFS and it is characterized by the sum of its MD and non-membership degree (NMD) values that do not exceed 1. We might have a DM problem. The problem arises, if the total of MD and NMD exceeds 1. Consequently, the Pythagorean FS (PFS) is an extension of IFS by [41]. It is defined by squared, where the sum of its MD and NMD does not exceed 1. The use of Pythagorean fuzzy Dombi AOs by Khan et al. [15]. Pythagorean cubic fuzzy Hamacher AOs by Abdullah et al. [2]. While "neutosophy” refers to knowledge of neutrality, FS and IFS represent distinct ways to recognizing. A NS was first conceptualized by Smarandache [37]. The degree of truth, indeterminacy and falsehood is represented by a number between 0 and 1. NS is a generalization of FS, IFS and other classical sets. For instance, if an expert discusses a particular enrollment that they believe to be true, the value is 0.75; if they say it is untrue, the value is 0.65; and if they are indeterminate, the value is 0.45. One example demonstrates that 10 ballots are utilized in the procedure because of the number pattern $(0.75, 0.65, 0.45)$. In particular, because the pattern $(0.6, 0.3, 0.1)$ is in NSS, 6 ballots had the response "yes," 3 ballots had the answer "no," and 1 ballots had no answer. The most well known method, Some of the problems listed can be resolved by NS. Deveci et al. [4] state that a number of aggregation operators based on practical contexts. Real-world applications were realized as a result of the interaction between the FSs in Memis et al. [23]. Mamis et al. [22] explored a number of similarity ideas for FSs using soft set (FSS).

Various Pythagorean AOs based on DM and their practical uses by Akram et al. [3]. The single valued NS (SVNS) is introduced by [39]. SVNSs are widely used and have many real-world applications [17]. Majumdar et al. [21] engage with the idea of SVNS using similarity and distance from one another. The concept of correlation coefficients and DM models for SVNS by [42]. Stanujkic et al. [36] discussed the WASPAS utilizing SVFNs and their real-world applications. To assess the transport service providers [19] starts an SVNS with a DM trial using an assessment laboratory model (SVNN-DEMATEL). Decision makers in challenging DM might employ the concept of linguistic neutrosophic number proposed by [26], which is utilized in the combinative distance based assessment (CODAS) model. Pamucar [27] presents the LNN CODAS and LNN MABAC (multi-attributive border approximation area comparison) and introduces the challenge in the linguistic neutrosophic numbers weighted aggregate employed in LNNWASPAS. SVNS is used in the domains of context analysis [35]. The many AOs utilized in the interval Pythagorean NS were introduced by Palanikumar et al. [25]. A few trigonometric operational rules were introduced [10]. Interval-valued fermatean NS concept explored by Broumi et al. [5]. Kalantari et al. [11] discussed the neutrosophic model for the best closed-loop, sustainable supply chain network that takes carbon emission and inflation policies. Karamasa et al. [13] interacted by weighing the elements influencing logistics outsourcing. An expanded SVNS based on AHP and MULTIMOORA concept presented by Karamasa et al. [14] to assess the best training aircraft for flying training firms.

NSS with MADM is investigated by Peng et al. [28] using TOPSIS and MABAC algorithms. Hwang et al. [38] investigated a practical use of MADM. Riaz et al. [30] used reference parameters to discuss Linear Diophantine FS (LDFS). Because it incorporates reference criteria, the LDFS is more successful and adaptable than alternative methods. Furthermore, by altering the physical importance of reference parameters, LDFS categorizes the data in MADM issues. Kannan et al. [12] have addressed the concept of the LDFS with CODAS approach for logistic specialist selection. Chakraborty et al. have looked at the practical uses of Fermatean fuzzy Bonferroni AOs [6]. The relationship between Fermatean fuzzy soft sets and their application to managing COVID-19 symptoms was examined by Zeb et al. [43]. Wei [40] introduced MADM to the novel idea of PFS interaction AOs. Single-valued trapezoidal neutrosophic number payoffs in a matrix game solution [32]. A nonlinear programming model has been developed to solve matrix games with payoffs that are single-valued neutrosophic numbers [31]. An interval neutrosophic matrix game-based solution to cyber security challenges [34]. Developments in statistical analysis and DM were made in [7]. A new CRADIS method for interval neutrosophic GDM based on triangular divergence distance. Using single-valued neutrosophic number pay-offs to solve the market share problem in matrix games [33]. Interval valued Fermatean neutrosophic super hyper soft sets and their algebraic structures: applications in health care [1]. Using the IFS based on the MCDM Method, block chain networks are assessed and ranked [24]. Resilience and sustainable urban innovation using artificial intelligence and the q-rung orthopair FS with expo logarithmic method [29].

1.2. Motivations for this research

The decision-maker can improve reliability and flexibility in the strategy-making process by utilizing SVNNs. SVNNs are superior to FS/IFS in presenting a variety of unclear circumstances. Due to its high projected return on investment, the stock market is one of the most visited locations. However, the world of business is complicated. Because price fluctuations are unpredictable, investors need to be careful when choosing their investing strategy. Making well-informed judgments should be their initial course of action when using their resources. Regular stock traders need to be up to date on the most recent stock news given the previously described circumstances. This is because the value is always changing. When the update is received, they must decide on the best course of action for the stocks. They should gain from the decision. In this situation, newbie stock traders typically make poor decisions about whether to purchase or sell the equities. This study focuses on the application of fuzzy logic and soft computing in the finance industry, which includes stocks and investment markets as well as financial organizations. Investors and decision makers are in charge of determining the location, timing, and strategy of investments.

Because this problem is unpredictable, decision-makers are always attempting to reduce risk by using sophisticated algorithms, tools, and procedures. It is anticipated that the fuzzy model developed with this method will aid in the decision-making of investment traders. Unexpected financial market events that are very difficult to forecast are the subject of this study. Soft computing has been used in the past by several academics in the business and finance domains. By comparing present market instability with known historical occurrences, it can identify them. The fuzzy model processes a large number of input parameters before generating investment recommendations. This study seeks to provide a decision-making assistance model that assists investors in identifying opportunities for both long-term critical imbalances and long-term, low-risk profit margins, as opposed to advocating for insufficient ideas. At this point, the national economy has demonstrated the significance of the stock market. The way that changes in the stock market impact the economic growth of nations in central and eastern Europe serves as evidence of this. Price volatility and capital liquidity are two examples of the two categories into which the tasks are divided.

1.3. Research gap

Data analysis of triangular NSs by Edalatpanah [9]. More generalized CT-SVNNs are introduced using cosine trigonometry for SVNN. Suppose $(0.71, 0.15, 0.57)$ be the one neutrosophic sample such as $0 \preceq \cos((\propto)\cdot 0.71) = 0.4871 \preceq 1$, $0 \preceq 1-\cos((\propto)\cdot(1-0.15)) = 0.55 \preceq 1$ and $0 \preceq 1-\cos((\propto)\cdot(1-0.57)) = 0.4604 \preceq 1$ are the results of using the AO operator. According to Mahmood et al.[20], there are several real world uses for spherical FSs and PFSs. Riaz et al. [30] describe the q-Rung orthopair FS and Linear Diophantine FS based on DM. It is critical to distinguish investor investment decisions from the way stock markets function. Around the world, research on investment decisions is still conducted utilizing a variety of backgrounds and research approaches that include a number of factors. This is due to the fact that investment activities are influenced by a wide range of factors, each of which has been the subject of numerous analyses and studies. If any investor hopes to make a sizable profit or avoid incurring a sizable loss, they must make the right choices. Thus, the goal of this research is to offer a framework for decision-making when buying or selling shares on the open market.

1.4. Contribution of the paper

This study will describe the new method for CT-SVNNS. AOs are also informed about CT-SVNNS. This research report is divided into seven parts. It is believed that Section 1 serves as an introduction. Section 2 provides a general review of CT-SVNS and associated concepts. The MADM employed in the cosine trigonometric single-valued neutrosophic normal number (CT-SVNNN) and its fundamental operations are covered in Section 3. The distance between CT-SVNNNs was the foundation for the investigation in Section 4. The MADM and a few of its AOs for CT-SVNNN are interacted with in section 5. CT-SVNN is described as an example and an innovation with comparative sections in Section 6. Finally, the purpose of Section 7 is to conclude. The goal of this research is following.

1. HDs are presented for the CT-SVNNS concept.
2. An example and a pertinence analysis of the stated definition and CT-SVNNN various operators.
3. CT-SVNNWA, CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG operator that uses two ideal values such as positive and negative.
4. Make contributions to the stock market by applying and confirming well-known models.
5. DM for using the natural number Ξ to determine the outcome.

2. Basic concepts

Our goal in this section is to review some definitions and results needed to improve our learning.

Definition 2.1. Let $\mathfrak{U}$ be the universal set. The Pythagorean IVFS (PIVFS) $\ddot{O} = \left\{\tau, \left\langle \widetilde{\sigma^{\mathcal{T}}}(\tau), \widetilde{\sigma^{\mathcal{F}}}(\tau)\right\rangle \Big| \tau \in \mathfrak{U}\right\}$, where $\widetilde{\sigma^{\mathcal{T}}} : \mathfrak{U} \to \mathrm{Int}([0,1])$ and $\widetilde{\sigma^{\mathcal{F}}} : \mathfrak{U} \to \mathrm{Int}([0,1])$ denote the MG and NMG of $\tau \in \mathfrak{U}$ to $\ddot{O}$, respectively, and $0 \preceq (\sigma^{\mathcal{T}+}(\tau))^2 + (\sigma^{\mathcal{F}+}(\tau))^2 \preceq 1$. For $\ddot{O} = \left\langle \left[\sigma^{\mathcal{T}-}, \sigma^{\mathcal{T}+}\right], \left[\sigma^{\mathcal{F}-}, \sigma^{\mathcal{F}+}\right] \right\rangle$ is called a Pythagorean interval-valued fuzzy number (PIVFN).

Definition 2.2 ([39])**.** The single-valued NS(SVNS) $\ddot{O} = \left\{\tau, \langle \sigma^{\mathcal{T}}(\tau), \sigma^{\mathcal{I}}(\tau), \sigma^{\mathcal{F}}(\tau)\rangle \big| \tau \in \mathfrak{U}\right\}$, $\sigma^{\mathcal{T}}(\tau)$, $\sigma^{\mathcal{I}}(\tau)$ and $\sigma^{\mathcal{F}}(\tau)$, which is belongs to $[0,1]$ denote the truth, indeterminacy and falsity MD of $\tau \in \mathfrak{U}$, respectively, and $0 \preceq \sigma^{\mathcal{T}}(\tau) + \sigma^{\mathcal{I}}(\tau) + \sigma^{\mathcal{F}}(\tau) \preceq 3$. For $\ddot{O} = \langle \sigma^{\mathcal{T}}, \sigma^{\mathcal{I}}, \sigma^{\mathcal{F}}\rangle$ is represents a single-valued neutrosophic number(SVNN).

Definition 2.3 ([39])**.** Let $\ddot{O} = \langle \sigma^{\mathscr{T}}, \sigma^{\mathscr{I}}, \sigma^{\mathscr{F}} \rangle$, $\ddot{O}_1 = \langle \sigma_1^{\mathscr{T}}, \sigma_1^{\mathscr{I}}, \sigma_1^{\mathscr{F}} \rangle$, and $\ddot{O}_2 = \langle \sigma_2^{\mathscr{T}}, \sigma_2^{\mathscr{I}}, \sigma_2^{\mathscr{F}} \rangle$ be three SVN-SNs, and $\Xi > 0$. Then

1. $\ddot{O}_1 \boxplus \ddot{O}_2 = \left[(\sigma_1^{\mathscr{T}}) + (\sigma_2^{\mathscr{T}}) - (\sigma_1^{\mathscr{T}}) \cdot (\sigma_2^{\mathscr{T}}), \sigma_1^{\mathscr{I}} \cdot \sigma_2^{\mathscr{I}}, \sigma_1^{\mathscr{F}} \cdot \sigma_2^{\mathscr{F}}\right]$;
2. $\ddot{O}_1 \otimes \ddot{O}_2 = \left[\sigma_1^{\mathscr{T}} \cdot \sigma_2^{\mathscr{T}}, (\sigma_1^{\mathscr{I}}) + (\sigma_2^{\mathscr{I}}) - (\sigma_1^{\mathscr{I}}) \cdot (\sigma_2^{\mathscr{I}}), (\sigma_1^{\mathscr{F}}) + (\sigma_2^{\mathscr{F}}) - (\sigma_1^{\mathscr{F}}) \cdot (\sigma_2^{\mathscr{F}})\right]$;
3. $\Xi \cdot \ddot{O} = \left[1 - \left(1 - (\sigma^{\mathscr{T}})\right)^{\Xi}, (\sigma^{\mathscr{I}})^{\Xi}, (\sigma^{\mathscr{F}})^{\Xi}\right]$;
4. $\ddot{O}^{\Xi} = \left[(\sigma^{\mathscr{T}})^{\Xi}, 1 - \left(1 - (\sigma^{\mathscr{I}})\right)^{\Xi}, 1 - \left(1 - (\sigma^{\mathscr{F}})\right)^{\Xi}\right]$.

Definition 2.4. The fuzzy number $M(x) = e^{-\left(\frac{x-\Psi}{\Omega}\right)^2}, (\Omega > 0)$ is known as a normal fuzzy number (NFN) if $M = (\Psi, \Omega)$, where R be the real numbers.

Definition 2.5. Let $X = (\Psi_1, \Omega_1) \in \ddot{N}$ and $Y = (\Psi_2, \Omega_2) \in \ddot{N}$, $(\Omega_1, \Omega_2 > 0)$. The distance between X and Y is $\mathbb{D}(X, Y) = \left((\Psi_1 - \Psi_2)^2 + \frac{1}{2}(\Omega_1 - \Omega_2)^2\right)^{1/2}$, where $\ddot{N}$ be the NFN.

3. New basic operations for CT-SVNNN

Some trigonometric neutrosophic numbers and NFN notions served as the foundation for the proposal of the CT-SVNNN and its functions. Here $\propto = \pi/2$.

Definition 3.1. Let $(\Psi, \Omega) \in N$, $\ddot{O} = \left\langle (\Psi, \Omega); \rho^{\mathscr{T}}, \rho^{\mathscr{I}}, \rho^{\mathscr{F}} \right\rangle$ be the CT-SVNNN is defined as

$$\cos \ddot{O} = \left\{ \cos\left(\propto \cdot (\sigma^{\mathscr{T}}(x))\right), \imath - \cos\left(\propto \cdot (\imath - (\sigma^{\mathscr{I}}(x))\right), \imath - \cos\left(\propto \cdot (\imath - \sigma^{\mathscr{F}}(x))\right) \right\}.$$

Hence, $\cos \ddot{O}$ is also CT-SVNNN, and satisfied the condition that $\cos\left(\propto \cdot \sigma^{\mathscr{T}}(x)\right) \in [0,1]$, $\cos\left(\propto \cdot \sigma^{\mathscr{I}}(x)\right) \in [0,1]$ and $\imath - \cos\left(\propto \cdot \left(\imath - \sigma^{\mathscr{F}}(x)\right)\right) \in [0,1]$. Therefore, $\cos \ddot{O} = \Big\{ \cos\left(\propto \cdot (\sigma^{\mathscr{T}}(x))\right), \imath - \cos\left(\propto \cdot (\imath - (\sigma^{\mathscr{I}}(x)))\right), \imath - \cos\left(\propto \cdot ((\imath - \sigma^{\mathscr{F}}(x)))\right) \Big\}$ is a CT-SVNNN, where $\sigma^{\mathscr{T}} = \sigma^{\mathscr{T}} e^{-\left(\frac{y-\Psi}{\Omega}\right)^2}$ and $\sigma^{\mathscr{I}} = \sigma^{\mathscr{I}} e^{-\left(\frac{y-\Psi}{\Omega}\right)^2}$ and $\sigma^{\mathscr{F}} = \sigma^{\mathscr{F}} e^{-\left(\frac{y-\Psi}{\Omega}\right)^2}$, $y \in Y$, where Y is a non-empty set.

Definition 3.2. For any CT-SVNN $\ddot{O} = \left\langle (\Psi, \Omega); \rho^{\mathscr{T}}, \rho^{\mathscr{I}}, \rho^{\mathscr{F}} \right\rangle$, then

$$S(\ddot{O}) = \frac{\Psi}{2}\left(\frac{2 + (\cos^2(\propto \cdot \sigma^{\mathscr{T}})) - (\cos^2(\propto \cdot \sigma^{\mathscr{I}})) - (\cos^2(\propto \cdot \sigma^{\mathscr{F}}))}{2}\right),$$

$S(\ddot{O}) \in [-1, 1]$, where $S(\ddot{O})$ is said to be the score function of $\ddot{O}$.

Definition 3.3. Let $\ddot{O} = \left\langle (\Psi, \Omega); \rho^{\mathscr{T}}, \rho^{\mathscr{I}}, \rho^{\mathscr{F}} \right\rangle$, $\ddot{O}_1 = \left\langle (\Psi_1, \Omega_1); \sigma_1^{\mathscr{T}}, \sigma_1^{\mathscr{I}}, \sigma_1^{\mathscr{F}} \right\rangle$, and $\ddot{O}_2 = \left\langle (\Psi_2, \Omega_2); \sigma_2^{\mathscr{T}}, \sigma_2^{\mathscr{I}}, \sigma_2^{\mathscr{F}} \right\rangle$ be three CT-SVNNNs, and $\Xi > 0$. Then

1. $\cos \ddot{O}_1 \boxplus \cos \ddot{O}_2 = \begin{bmatrix} (\Psi_1 + \Psi_2, \Omega_1 + \Omega_2); (\cos^2(\propto \cdot \sigma_1^{\mathscr{T}}))^{\Xi} + (\cos^2(\propto \cdot \sigma_2^{\mathscr{T}}))^{\Xi} \\ -(\cos^2(\propto \cdot \sigma_1^{\mathscr{T}}))^{\Xi} \cdot (\cos^2(\propto \cdot \sigma_2^{\mathscr{T}}))^{\Xi}, \\ \cos^2(\propto \cdot \sigma_1^{\mathscr{I}}) \cdot \cos^2(\propto \cdot \sigma_2^{\mathscr{I}}), \cos^2(\propto \cdot \sigma_1^{\mathscr{F}}) \cdot \cos^2(\propto \cdot \sigma_2^{\mathscr{F}}) \end{bmatrix}$;
2. $\cos \ddot{O}_1 \otimes \cos \ddot{O}_2 = \begin{bmatrix} (\Psi_1 \cdot \Psi_2, \Omega_1 \cdot \Omega_2); \cos^2(\propto \cdot \sigma_1^{\mathscr{T}}) \cdot \cos^2(\propto \cdot \sigma_2^{\mathscr{T}}), (\cos^2(\propto \cdot \sigma_1^{\mathscr{I}}))^{\Xi} + (\cos^2(\propto \cdot \sigma_2^{\mathscr{I}}))^{\Xi} \\ -(\cos^2(\propto \cdot \sigma_1^{\mathscr{I}}))^{\Xi} \cdot (\cos^2(\propto \cdot \sigma_2^{\mathscr{I}}))^{\Xi}, (\cos^2(\propto \cdot \sigma_1^{\mathscr{F}}))^{\Xi} + (\cos^2(\propto \cdot \sigma_2^{\mathscr{F}}))^{\Xi} \\ -(\cos^2(\propto \cdot \sigma_1^{\mathscr{F}}))^{\Xi} \cdot (\cos^2(\propto \cdot \sigma_2^{\mathscr{F}}))^{\Xi} \end{bmatrix}$;
3. $\Xi \cdot \cos \ddot{O} = \left[(\Xi \cdot \Psi, \Xi \cdot \Omega); \imath - \left(\imath - (\cos^2(\propto \cdot \sigma^{\mathscr{T}}))^{\Xi}\right)^{\Xi}, (\cos^2(\propto \cdot \sigma^{\mathscr{I}}))^{\Xi}, (\cos^2(\propto \cdot \sigma^{\mathscr{F}}))^{\Xi}\right]$;
4. $(\cos \ddot{O})^{\Xi} = \left[(\Psi^{\Xi}, \Omega^{\Xi}); (\cos^2(\propto \cdot \sigma^{\mathscr{T}}))^{\Xi}, \imath - \left(\imath - (\cos^2(\propto \cdot \sigma^{\mathscr{I}}))^{\Xi}\right)^{\Xi}, \imath - \left(\imath - (\cos^2(\propto \cdot \sigma^{\mathscr{F}}))^{\Xi}\right)^{\Xi}\right]$.

4. Distance for CT-SVNNNs

We discuss some mathematical properties of CT-SVNNs based on Euclidean distance (ED) and Hamming distance (HD).

Definition 4.1. Let $\ddot{O}_1 = \left\langle (\Psi_1, \Omega_1); \sigma_1^{\mathscr{T}}, \sigma_1^{\mathscr{I}}, \sigma_1^{\mathscr{F}} \right\rangle$ and $\ddot{O}_2 = \left\langle (\Psi_2, \Omega_2); \sigma_2^{\mathscr{T}}, \sigma_2^{\mathscr{I}}, \sigma_2^{\mathscr{F}} \right\rangle$ be the CT-SVNNNs. Then $\mathcal{D}_E\left(\ddot{O}_1, \ddot{O}_2\right) = \left(\left[\frac{1+P}{2}\Psi_1 - \frac{1+Q}{2}\Psi_2 \right]^2 + \frac{1}{2}\left[\frac{1+P}{2}\Omega_1 - \frac{1+Q}{2}\Omega_2 \right]^2 \right)^{1/2}$ and

$$\mathcal{D}_H\left(\ddot{O}_1, \ddot{O}_2\right) = \left[\left| \frac{1+P}{2}\Psi_1 - \frac{1+Q}{2}\Psi_2 \right| + \frac{1}{2}\left| \frac{1+P}{2}\Omega_1 - \frac{1+Q}{2}\Omega_2 \right| \right],$$

where

$$P = \cos^2(\propto \cdot \sigma_1^{\mathscr{T}}) - \cos^2(\propto \cdot \sigma_1^{\mathscr{I}}) - \cos^2(\propto \cdot \sigma_1^{\mathscr{F}}), \quad Q = \cos^2(\propto \cdot \sigma_2^{\mathscr{T}}) - \cos^2(\propto \cdot \sigma_2^{\mathscr{I}}) - \cos^2(\propto \cdot \sigma_2^{\mathscr{F}}).$$

Theorem 4.2. *If any three CT-SVNNNs* $\ddot{O}_1 = \left\langle (\Psi_1, \Omega_1); \sigma_1^{\mathscr{T}}, \sigma_1^{\mathscr{I}}, \sigma_1^{\mathscr{F}} \right\rangle$, $\ddot{O}_2 = \left\langle (\Psi_2, \Omega_2); \sigma_2^{\mathscr{T}}, \sigma_2^{\mathscr{I}}, \sigma_2^{\mathscr{F}} \right\rangle$, $\ddot{O}_3 = \left\langle (\Psi_3, \Omega_3); \sigma_3^{\mathscr{T}}, \sigma_3^{\mathscr{I}}, \sigma_3^{\mathscr{F}} \right\rangle$, *then*

1. $\mathcal{D}_E(\ddot{O}_1, \ddot{O}_2) = 0$ *if and only if* $\ddot{O}_1 = \ddot{O}_2$;
2. $\mathcal{D}_E(\ddot{O}_1, \ddot{O}_2) = \mathcal{D}_E(\ddot{O}_2, \ddot{O}_1)$;
3. $\mathcal{D}_E(\ddot{O}_1, \ddot{O}_3) \preceq \mathcal{D}_E(\ddot{O}_1, \ddot{O}_2) + \mathcal{D}_E(\ddot{O}_2, \ddot{O}_3)$.

Proof. The proof is provided in an appendix. □

Lemma 4.3. *If any three CT-SVNNNs* $\ddot{O}_1 = \left\langle (\Psi_1, \Omega_1); \sigma_1^{\mathscr{T}}, \sigma_1^{\mathscr{I}}, \sigma_1^{\mathscr{F}} \right\rangle$, $\ddot{O}_2 = \left\langle (\Psi_2, \Omega_2); \sigma_2^{\mathscr{T}}, \sigma_2^{\mathscr{I}}, \sigma_2^{\mathscr{F}} \right\rangle$, $\ddot{O}_3 = \left\langle (\Psi_3, \Omega_3); \sigma_3^{\mathscr{T}}, \sigma_3^{\mathscr{I}}, \sigma_3^{\mathscr{F}} \right\rangle$, *then*

1. $\mathcal{D}_H(\ddot{O}_1, \ddot{O}_2) = 0$ *if and only if* $\ddot{O}_1 = \ddot{O}_2$;
2. $\mathcal{D}_H(\ddot{O}_1, \ddot{O}_2) = \mathcal{D}_H(\ddot{O}_2, \ddot{O}_1)$;
3. $\mathcal{D}_H(\ddot{O}_1, \ddot{O}_3) \preceq \mathcal{D}_H(\ddot{O}_1, \ddot{O}_2) + \mathcal{D}_H(\ddot{O}_2, \ddot{O}_3)$.

5. Aggregation operators for CT-SVNNNs

Here, we introduced AOs based on CT-SVNNWA, CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG.

5.1. CT-SVNN weighted averaging (CT-SVNNWA)

Definition 5.1. If $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma_i^{\mathscr{T}}, \sigma_i^{\mathscr{I}}, \sigma_i^{\mathscr{F}} \right\rangle$ be the finite family of CT-SVNNNs, $W = (\kappa_1, \kappa_2, \dots, \kappa_n)$ be the weight of $\ddot{O}_i$, $\kappa_i \succeq 0$, and $\biguplus_{i \mapsto 1}^{n} \kappa_i = 1$. Then CT-SVNNWA operator is defined as CT-SVNNWA $(\ddot{O}_1, \ddot{O}_2, \dots, \ddot{O}_n) = \biguplus_{i \mapsto 1}^{n} \kappa_i \ddot{O}_i$.

Theorem 5.2. *Let* $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma_i^{\mathscr{T}}, \sigma_i^{\mathscr{I}}, \sigma_i^{\mathscr{F}} \right\rangle$ *be the finite family of CT-SVNNNs. Then*

$$\textit{CT-SVNNWA}(\ddot{O}_1, \ddot{O}_2, \dots, \ddot{O}_n) = \left[\begin{matrix} \left(\biguplus_{i \mapsto 1}^{n} \kappa_i \Psi_i, \biguplus_{i \mapsto 1}^{n} \kappa_i \Omega_i \right); \text{ı} - \bigotimes_{i \mapsto 1}^{n} \left(\text{ı} - (\cos^2(\propto \cdot \sigma_i^{\mathscr{T}}))^{\Xi} \right)^{\kappa_i}, \\ \bigotimes_{i \mapsto 1}^{n} (\cos^2(\propto \cdot \sigma_i^{\mathscr{I}}))^{\kappa_i}, \bigotimes_{i \mapsto 1}^{n} (\cos^2(\propto \cdot \sigma_i^{\mathscr{F}}))^{\kappa_i} \end{matrix} \right].$$

Proof. The proof is provided in an appendix. □

Theorem 5.3. *If all* $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma_i^{\mathscr{T}}, \sigma_i^{\mathscr{I}}, \sigma_i^{\mathscr{F}} \right\rangle$ *and* $\ddot{O}_i = \ddot{O}$, *then* CT-SVNNWA$(\ddot{O}_1, \ddot{O}_2, \dots, \ddot{O}_n) = \cos \ddot{O}$.

Proof. The proof is provided in an appendix. □

Theorem 5.4. *Let* $\ddot{O}_i = \left\langle (\Psi_{ij}, \Omega_{ij}); \sigma^{\mathscr{T}}_{ij} \sigma^{\mathscr{I}}_{ij}, \sigma^{\mathscr{F}}_{ij} \right\rangle (i \mapsto 1, 2, \dots, n); (j \mapsto 1, 2, \dots, i_j)$ *be the finite collection of CT-SVNNWA, where* $\underbrace{\Psi} = \inf \Psi_{ij}$, $\overbrace{\Psi} = \sup \Psi_{ij}$, $\underbrace{\Omega} = \sup \Omega_{ij}$, $\overbrace{\Omega} = \inf \Omega_{ij}$, $\underbrace{\sigma^{\mathscr{T}}} = \inf \sigma^{\mathscr{T}^-}_{ij}$, $\overbrace{\sigma^{\mathscr{T}}} = \sup \sigma^{\mathscr{T}^-}_{ij}$, $\underbrace{\sigma^{\mathscr{I}}} = \inf \sigma^{\mathscr{I}^-}_{ij}$, $\overbrace{\sigma^{\mathscr{I}}} = \sup \sigma^{\mathscr{I}^-}_{ij}$, $\underbrace{\sigma^{\mathscr{F}}} = \inf \sigma^{\mathscr{F}^-}_{ij}$, $\overbrace{\sigma^{\mathscr{F}}} = \sup \sigma^{\mathscr{F}^-}_{ij}$. *Then*

$$\left\langle (\underbrace{\Psi}, \underbrace{\Omega}); \underbrace{\sigma^{\mathscr{T}}}, \overbrace{\sigma^{\mathscr{I}}}, \overbrace{\sigma^{\mathscr{F}}} \right\rangle \preceq \textit{CT-SVNNWA}(\ddot{O}_1, \ddot{O}_2, \dots, \ddot{O}_n) \preceq \left\langle (\overbrace{\Psi}, \overbrace{\Omega}); \overbrace{\sigma^{\mathscr{T}}}, \underbrace{\sigma^{\mathscr{I}}}, \underbrace{\sigma^{\mathscr{F}}} \right\rangle.$$

Proof. The proof is provided in an appendix. □

Theorem 5.5. *Let* $\ddot{O}_i = \left\langle (\Psi_{t_{ij}}, \Omega_{t_{ij}}); \sigma^{\mathscr{T}}_{t_{ij}}, \sigma^{\mathscr{I}}_{t_{ij}}, \sigma^{\mathscr{F}}_{t_{ij}} \right\rangle$ *and* $\ddot{W}_i = \left\langle (\Psi_{h_{ij}}, \Omega_{h_{ij}}); \sigma^{\mathscr{T}}_{h_{ij}}, \sigma^{\mathscr{I}}_{h_{ij}}, \sigma^{\mathscr{F}}_{h_{ij}} \right\rangle$ *be the two families of CT-SVNNWAs. For any* i, *if there is* $\Psi_{t_{ij}} \preceq \Omega_{h_{ij}}$, $\left(\cos^2(\propto \cdot \sigma^{\mathscr{T}}_{t_{ij}})\right) \preceq \left(\cos^2(\propto \cdot \sigma^{\mathscr{T}}_{h_{ij}})\right)$ *and* $\left(\cos^2(\propto \cdot \sigma^{\mathscr{I}}_{t_{ij}})\right) \succeq \left(\cos^2(\propto \cdot \sigma^{\mathscr{I}}_{h_{ij}})\right)$ *and* $\left(\cos^2(\propto \cdot \sigma^{\mathscr{F}}_{t_{ij}})\right) \succeq \left(\cos^2(\propto \cdot \sigma^{\mathscr{F}}_{h_{ij}})\right)$ *or* $\ddot{O}_i \preceq \ddot{W}_i$, *then*

$$\textit{CT-SVNNWA}\left(\ddot{O}_1, \ddot{O}_2, \dots, \ddot{O}_n\right) \preceq \textit{CT-SVNNWA}\left(\ddot{W}_1, \ddot{W}_2, \dots, \ddot{W}_n\right).$$

Proof. The proof is provided in an appendix. □

5.2. CT-SVNN weighted geometric (CT-SVNNWG)

Definition 5.6. If $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma^{\mathscr{T}}_i, \sigma^{\mathscr{I}}_i, \sigma^{\mathscr{F}}_i \right\rangle$ be the family of CT-SVNNNs, the CT-SVNNWG operator is defined as CT-SVNNWG $(\ddot{O}_1, \ddot{O}_2, \dots, \ddot{O}_n) = \bigotimes_{i \mapsto 1}^{n} (\cos \ddot{O}_i)^{\kappa_i}$.

Theorem 5.7. *Let* $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma^{\mathscr{T}}_i, \sigma^{\mathscr{I}}_i, \sigma^{\mathscr{F}}_i \right\rangle$ *be the family of CT-SVNNNs. Then*

$$\text{CT-SVNNWG}(\ddot{O}_1, \ddot{O}_2, \dots, \ddot{O}_n) = \begin{bmatrix} \left(\bigotimes_{i \mapsto 1}^{n} \Psi_i^{\kappa_i}, \bigotimes_{i \mapsto 1}^{n} \Omega_i^{\kappa_i}\right); \bigotimes_{i \mapsto 1}^{n} (\cos^2(\propto \cdot \sigma^{\mathscr{T}}_i))^{\kappa_i}, \\ 1 - \bigotimes_{i \mapsto 1}^{n} \left(1 - (\cos^2(\propto \cdot \sigma^{\mathscr{I}}_i))^{\Xi}\right)^{\kappa_i}, 1 - \bigotimes_{i \mapsto 1}^{n} \left(1 - (\cos^2(\propto \cdot \sigma^{\mathscr{F}}_i))^{\Xi}\right)^{\kappa_i} \end{bmatrix}.$$

Proof. The proof based on Theorem 5.2. □

Theorem 5.8. *If all* $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma^{\mathscr{T}}_i, \sigma^{\mathscr{I}}_i, \sigma^{\mathscr{F}}_i \right\rangle$ *and* $\ddot{O}_i = \ddot{O}$, *then* $\textit{CT-SVNNWG}(\ddot{O}_1, \ddot{O}_2, \dots, \ddot{O}_n) = \ddot{O}$.

Proof. The proof based on Theorem 5.3. □

Remark 5.9. Boundedness and monotonicity are satisfied by the CT-SVNNWG operator.

Proof. The proof based on Theorems 5.4 and 5.5. □

5.3. Generalized CT-SVNNWA (GCT-SVNNWA)

Definition 5.10. Let $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma^{\mathscr{T}}_i, \sigma^{\mathscr{I}}_i, \sigma^{\mathscr{F}}_i \right\rangle$ be the family of CT-SVNNN. Then

$$\text{GCT-SVNNWA}(\ddot{O}_1, \ddot{O}_2, \dots, \ddot{O}_n) = \left(\biguplus_{i \mapsto 1}^{n} \kappa_i (\cos \ddot{O}_i)^{\Xi} \right)^{1/\Xi}.$$

Theorem 5.11. *Let* $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma^{\mathscr{T}}_i, \sigma^{\mathscr{I}}_i, \sigma^{\mathscr{F}}_i \right\rangle$ *be the family of CT-SVNNNs. Then*

$$\begin{aligned} &\text{GCT-SVNNWA}(\ddot{O}_1, \ddot{O}_2, \dots, \ddot{O}_n) \\ &= \begin{bmatrix} \left(\left(\biguplus_{i \mapsto 1}^{n} \kappa_i \Psi_i^{\Xi}\right)^{1/\Xi}, \left(\biguplus_{i \mapsto 1}^{n} \kappa_i \Omega_i^{\Xi}\right)^{1/\Xi}\right); \left(1 - \bigotimes_{i \mapsto 1}^{n} \left(1 - \left((\cos^2(\propto \cdot \sigma^{\mathscr{T}}_i))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}\right)^{1/\Xi}, \\ 1 - \left(1 - \left(\bigotimes_{i \mapsto 1}^{n} \left(1 - \left(1 - (\cos^2(\propto \cdot \sigma^{\mathscr{I}}_i))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}, \right)^{\Xi}\right)^{1/\Xi}, \\ 1 - \left(1 - \left(\bigotimes_{i \mapsto 1}^{n} \left(1 - \left(1 - (\cos^2(\propto \cdot \sigma^{\mathscr{F}}_i))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}\right)^{\Xi}\right)^{1/\Xi} \end{bmatrix}. \end{aligned}$$

Proof. The proof is provided in an appendix. □

The GCT-SVNNWA is transformed to the CT-SVNNWA in the case of $\Xi \mapsto 1$.

Theorem 5.12. *If all* $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma_i^{\mathscr{T}}, \sigma_i^{\mathscr{I}}, \sigma_i^{\mathscr{F}} \right\rangle$ *and* $\ddot{O}_i = \ddot{O}$*, then GCT-SVNNWA*$(\ddot{O}_1, \ddot{O}_2, \ldots, \ddot{O}_n) = \ddot{O}$.

Proof. The proof can be found in Theorem 5.3. □

The boundedness and monotonicity properties are satisfied in the case of the GCT-SVNNWA operator, as shown by Theorems 5.4 and 5.5.

5.4. Generalized CT-SVNNWG (GCT-SVNNWG)

Definition 5.13. Let $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma_i^{\mathscr{T}}, \sigma_i^{\mathscr{I}}, \sigma_i^{\mathscr{F}} \right\rangle$ be the family of CT-SVNNNs. Then GCT-SVNNWG $(\ddot{O}_1, \ddot{O}_2, \ldots, \ddot{O}_n) = \frac{1}{\Xi}\left(\bigotimes_{i\mapsto 1}^{n} (\Xi \cos \ddot{O}_i)^{\kappa_i} \right)$.

Theorem 5.14. *Let* $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma_i^{\mathscr{T}}, \sigma_i^{\mathscr{I}}, \sigma_i^{\mathscr{F}} \right\rangle$ *be the family of CT-SVNNNs. Then*

$$\text{GCT-SVNNWG}(\ddot{O}_1, \ddot{O}_2, \ldots, \ddot{O}_n) = \begin{bmatrix} \left(\frac{1}{\Xi} \bigotimes_{i\mapsto 1}^{n} (\Xi\Psi_i)^{\kappa_i}, \frac{1}{\Xi} \bigotimes_{i\mapsto 1}^{n} (\Xi\Omega_i)^{\kappa_i} \right); \\ \imath - \left(\imath - \left(\bigotimes\limits_{i\mapsto 1}^{n} \left(\imath - \left(\imath - (\cos^2(\propto \cdot \sigma_i^{\mathscr{T}}))^{\Xi} \right)^{\Xi} \right)^{\kappa_i} \right)^{\Xi} \right)^{1/\Xi}, \\ \left(\imath - \bigotimes\limits_{i\mapsto 1}^{n} \left(\imath - \left((\cos^2(\propto \cdot \sigma_i^{\mathscr{I}}))^{\Xi} \right)^{\Xi} \right)^{\kappa_i} \right)^{1/\Xi}, \\ \left(\imath - \bigotimes\limits_{i\mapsto 1}^{n} \left(\imath - \left((\cos^2(\propto \cdot \sigma_i^{\mathscr{F}}))^{\Xi} \right)^{\Xi} \right)^{\kappa_i} \right)^{1/\Xi}, \end{bmatrix}.$$

Proof. The proof based on Theorem 5.11. □

Since the GCT-SVNNWG operator is converted to the CT-SVNNWG operator in the case of $\Xi \mapsto 1$. In the case of the GCT-SVNNWG operator, the boundedness and monotonicity properties are satisfied, and we used Theorems 5.4 and 5.5.

Theorem 5.15. *If all* $\ddot{O}_i = \left\langle (\Psi_i, \Omega_i); \sigma_i^{\mathscr{T}}, \sigma_i^{\mathscr{I}}, \sigma_i^{\mathscr{F}} \right\rangle$ *and* $\ddot{O}_i = \ddot{O}$*, then GCT-SVNNWG*$(\ddot{O}_1, \ddot{O}_2, \ldots, \ddot{O}_n) = \ddot{O}$.

6. CT-SVNN applied for MADM

The set of n-alternatives $\ddot{O} = \{\ddot{O}_1, \ddot{O}_2, \ldots, \ddot{O}_n\}$, the set of m-attributes $A = \{A_1, A_2, \ldots, A_m\}$, and $\kappa = \{\kappa_1, \kappa_2, \ldots, \kappa_m\}$ is called the attributes of the weights, where $i \mapsto 1, 2, \ldots, n$ and $j \mapsto 1, 2, \ldots, m$, $\ddot{O}_{ij} = \left\langle (\Psi_{ij}, \Omega_{ij}); \cos^2(\propto \cdot \sigma_{ij}^{\mathscr{T}}), \cos^2(\propto \cdot \sigma_{ij}^{\mathscr{I}}), \cos^2(\propto \cdot \sigma_{ij}^{\mathscr{F}}) \right\rangle$ denote CT-SVNNN of alternative O_i in attribute A_j. Since $\cos\left(\propto \cdot (\sigma_{ij}^{\mathscr{T}}(x)\right) + \left(\imath - \cos\left(\propto \cdot \imath - (\sigma_{ij}^{\mathscr{I}}(x))\right)\right) + \left(\imath - \cos\left(\propto \cdot (\imath - \sigma_{ij}^{\mathscr{F}}(x))\right)\right) \preceq 3$ and

$$\cos\left(\propto \cdot (\sigma_{ij}^{\mathscr{T}}(x)\right) \in [0,1], \left(\imath - \cos\left(\propto \cdot \imath - (\sigma_{ij}^{\mathscr{I}}(x))\right)\right) \in [0,1], \left(\imath - \cos\left(\propto \cdot \imath - (\sigma_{ij}^{\mathscr{F}}(x))\right)\right) \in [0,1].$$

Here, $\mathcal{D} = (\ddot{O}_{ij})_{n\times m}$ is called the $n \times m$ decision matrix.

6.1. *Algorithm for CT-SVNN*

Step-1: Enter the CT-SVNN decision values.
Step-2: To demonstrate the decision values for normalization, the matrix $\mathcal{D} = (\ddot{O}_{ij})_{n\times m}$ to $\overline{\mathcal{D}} = (\overline{\ddot{O}_{ij}})_{n\times m}$, since $\ddot{O}_{ij} = \left\langle (\overline{\Psi_{ij}}, \overline{\Omega_{ij}}); \overline{\sigma^{\mathscr{T}}_{ij}}, \overline{\sigma^{\mathscr{I}}_{ij}}, \overline{\sigma^{\mathscr{F}}_{ij}} \right\rangle$ and

$$\overline{\Psi_{ij}} = \frac{\Psi_{ij}}{\sup_i(\Psi_{ij})}, \quad \overline{\Omega_{ij}} = \frac{\Omega_{ij}}{\sup_i(\Omega_{ij})} \cdot \frac{\Omega_{ij}}{\Psi_{ij}}, \quad \overline{\sigma^{\mathscr{T}}_{ij}} = \sigma^{\mathscr{T}}_{ij}.$$

Step-3: Each alternative is evaluated based on its aggregate values. $\overline{\mathcal{D}} = (\overline{\ddot{O}_{ij}})_{n\times m}$, where $\overline{\ddot{O}_{ij}} = \left\langle (\overline{\Psi_{ij}}, \overline{\Omega_{ij}}); \overline{\sigma^{\mathscr{T}}_{ij}}, \overline{\sigma^{\mathscr{I}}_{ij}}, \overline{\sigma^{\mathscr{F}}_{ij}} \right\rangle$ is aggregated into $\overline{\mathcal{D}} = (\ddot{O}_i)_{n\times m}$, where $\ddot{O}_i = \left\langle (\overline{\Psi_i}, \overline{\Omega_i}); \overline{\sigma^{\mathscr{T}}_i}, \overline{\sigma^{\mathscr{I}}_i}, \overline{\sigma^{\mathscr{F}}_i} \right\rangle$.
Step-4: Each choice should be represented by two unique ideal values, such as positive and negative,

$$\ddot{O}^+ = \left\langle \left(\sup_{1\preceq i\preceq n} (\overline{\Psi_{ij}}), \inf_{1\preceq i\preceq n} (\overline{\Omega_{ij}}) \right); 1, 0, 0 \right\rangle, \quad \ddot{O}^- = \left\langle \left(\inf_{1\preceq i\preceq n} (\overline{\Psi_{ij}}), \sup_{1\preceq i\preceq n} (\overline{\Omega_{ij}}) \right); 0, 1, 1 \right\rangle.$$

Step-5: Compute the ED and HD for each choice:

$$\mathcal{D}^+_i = \mathcal{D}_E\left(\ddot{O}_i, \ddot{O}^+\right); \quad \mathcal{D}^-_i = \mathcal{D}_E\left(\ddot{O}_i, \ddot{O}^-\right).$$

Step-6: The closest similar value should be used to compare each choice: $\mathcal{D}^*_i = \frac{\mathcal{D}^-_i}{\mathcal{D}^+_i + \mathcal{D}^-_i}$.
Step-7: The most effective solution is to provide the values of $\sup \mathcal{D}^*_i$.

Figure 1 is a flowchart illustrating the MADM process using CT-SVNN.

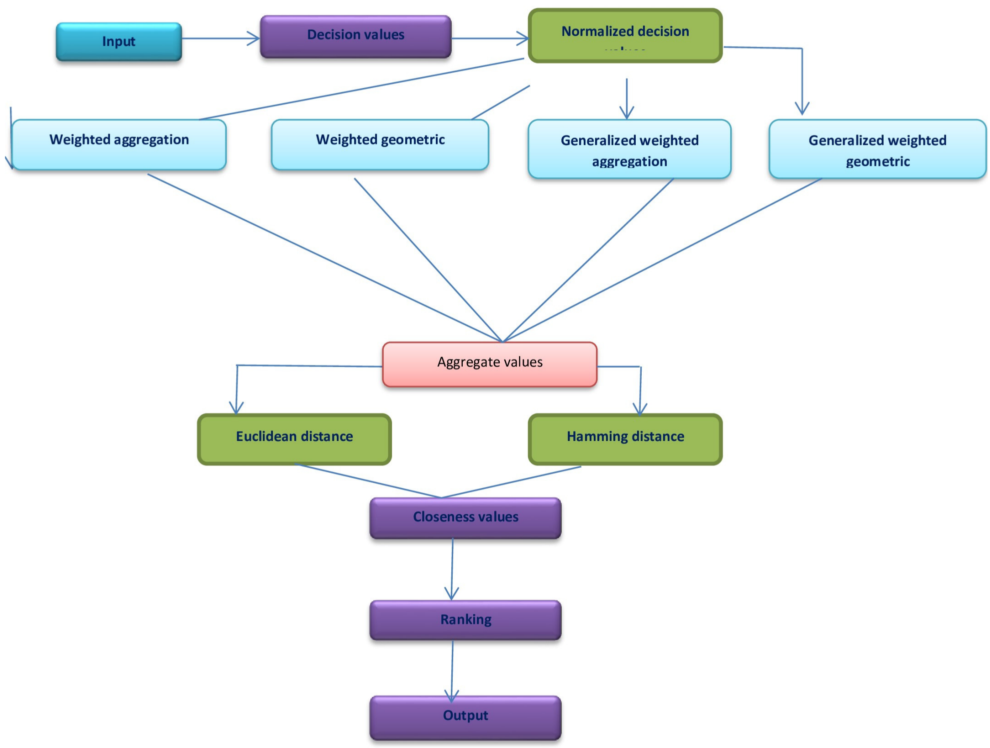

Figure 1: Structure of flowchart network

6.2. *Stock market*

Investors often invest in debt or equity assets in the hopes of making money. Given recent market events, investors could think about making changes to their investment portfolios. In a volatile and uncertain climate, investors base their decisions on trial-and-error or out-of-date rules. However, while evaluating investment prospects, emotional and cognitive aspects are really taken into account, undercutting the necessity of using reason. People may now participate in a variety of financial products thanks to the expansion of the financial sector. Behavioral finance has increased our understanding of individual investor behavior by explaining the distinct traits and psychological processes that effect people's investment intentions and actions [18]. Individuals, according to classical economic theory, make rational investment decisions and follow basic financial principles in order to maximize their wealth. Their investment approach frequently incorporates the use of technical analysis, fundamental analysis, and judgment [16]. People act irrationally due to fear of loss, regardless of their knowledge or research on investment products before to making an investment [8]. In mutual funds, a stock market share is defined as 0.1 percent of a company earnings. These financial support sectors are governed by stock market rules. When applying, following or reviewing the stock market key concepts, quality rating is the most important consideration. A company's financial growth should steadily improve over time.

Company Liquidity: The holding stock or passive stock exchange may be confirmed by the business liquidity, and the primary liquidity ratio is to determine the company value per share at a given market price.

Positive earnings: Positive profits are a significant growth signal for a firm, which is only feasible in particular phases for a high-quality stock. To achieve a single profit ratio, the stock market must close and reopen at a specific level. It also has several serious issues, such as the price-to-earnings and price-to-book ratios.

Dividents with long term: It is the best strategy to invest money. A competent stock market investor may earn thousands of dollars employing this dividend structure. He will distribute his money across dividend stocks in such a manner that it becomes a gold-level investment, which is the simplest approach to make more. Using the stock market, we should plant a seed that will develop into a stronger tree. Long-term investment is crucial for stock market investing, but most individuals are uninterested since it takes longer.

Real return calculation: If an investment is made, the majority of the return should be estimated to ensure that no significant stocks or the stock market lose their position. It might be optimistic or deceptive, but calculating true return is a stock market golden rule.

If five persons called (alternatives) trade in a firm that follows the precision and accuracy of the stock market, for example $\{\ddot{O}_a, \ddot{O}_b, \ddot{O}_c, \ddot{O}_d, \ddot{O}_e\}$.

Four attributes are considered as company liquidity (A_1), positive earnings (A_2), dividends with long term (A_3) and real return calculation $((A_4)$ and weights are $\kappa = \{0.4, 0.3, 0.2, 0.1\}$. We will choose the best option from each alternative based on an expert assessment against the criteria. Tables 1 and 2 demonstrate inputting the DM values.

Table 1: DM information.

	A_1	A_2
$\ddot{O}_a$	$\langle(0.9, 0.65); 0.28, 0.64, 0.43\rangle$	$\langle(0.65, 0.35); 0.71, 0.52, 0.45\rangle$
$\ddot{O}_b$	$\langle(0.95, 0.75); 0.28, 0.3, 0.16\rangle$	$\langle(0.55, 0.5); 0.62, 0.85, 0.44\rangle$
$\ddot{O}_c$	$\langle(0.85, 0.65); 0.2, 0.37, 0.23\rangle$	$\langle(0.5, 0.3); 0.43, 0.1, 0.45\rangle$
$\ddot{O}_d$	$\langle(0.7, 0.5, 0.29, 0.3, 0.59\rangle$	$\langle(0.5, 0.45, 0.55, 0.37, 0.44\rangle$
$\ddot{O}_e$	$\langle(0.75, 0.6, 0.27, 0.38, , 0.65\rangle$	$\langle(0.75, 0.65, 0.29, 0.52, , 0.64\rangle$

Table 2: DM information.

	A_3	A_4
$\ddot{O}_a$	$\langle(0.85, 0.5); 0.5, 0.52, 0.45\rangle$	$\langle(0.8, 0.65); 0.71, 0.15, 0.57\rangle$
$\ddot{O}_b$	$\langle(0.75, 0.6); 0.29, 0.3, 0.44\rangle$	$\langle(0.8, 0.75); 0.5, 0.45, 0.85\rangle$
$\ddot{O}_c$	$\langle(0.55, 0.5); 0.71, 0.5, 0.31\rangle$	$\langle(0.7, 0.65); 0.27, 0.52, 0.52\rangle$
$\ddot{O}_d$	$\langle(0.6, 0.55, 0.5, 0.57, 0.72\rangle$	$\langle(0.85, 0.45, 0.28, 0.58, 0.78\rangle$
$\ddot{O}_e$	$\langle(0.7, 0.6, 0.55, 0.64, 0.45\rangle$	$\langle(0.8, 0.6, 0.34, 0.45, 0.44\rangle$

Tables 3 and 4 demonstrate that the normalized decision matrix is as shown in Table 3.

Table 3: Normalized decision matrix.

	A_1	A_2
$\ddot{O}_a$	$\langle(0.9474, 0.6259); 0.28, 0.64, 0.43\rangle$	$\langle(0.8667, 0.2899); 0.71, 0.52, 0.45\rangle$
$\ddot{O}_b$	$\langle(1, 0.7895); 0.28, 0.3, 0.16\rangle$	$\langle(0.7333, 0.6993); 0.62, 0.85, 0.44\rangle$
$\ddot{O}_c$	$\langle(0.8947, 0.6627); 0.2, 0.37, 0.23\rangle$	$\langle(0.6667, 0.2769); 0.43, 0.1, 0.45\rangle$
$\ddot{O}_d$	$\langle(0.7368, 0.4762); 0.29, 0.3, 0.59\rangle$	$\langle(0.6667, 0.6231); 0.55, 0.37, 0.44\rangle$
$\ddot{O}_e$	$\langle(0.7895, 0.64); 0.27, 0.38, , 0.65\rangle$	$\langle(1, 0.8667); 0.29, 0.52, , 0.64\rangle$

Table 4: Normalized decision matrix.

	A_3	A_4
$\ddot{O}_a$	$\langle(1, 0.4902); 0.5, 0.52, 0.45\rangle$	$\langle(0.9412, 0.7042); 0.71, 0.15, 0.57\rangle$
$\ddot{O}_b$	$\langle(0.8824, 0.8); 0.29, 0.3, 0.44\rangle$	$\langle(0.9412, 0.9375); 0.5, 0.45, 0.85\rangle$
$\ddot{O}_c$	$\langle(0.6471, 0.7576); 0.71, 0.5, 0.31\rangle$	$\langle(0.8235, 0.8048); 0.27, 0.52, 0.52\rangle$
$\ddot{O}_d$	$\langle(0.7059, 0.8403); 0.5, 0.57, 0.72\rangle$	$\langle(1, 0.3176); 0.28, 0.58, 0.78\rangle$
$\ddot{O}_e$	$\langle(0.8235, 0.8571); 0.55, 0.64, 0.45\rangle$	$\langle(9412, 0.6); 0.34, 0.45, 0.44\rangle$

Table 5 demonstrates that CT-SVNNWA utilizes AO for each option.

Table 5: CT-SVNNWA.

	CT-SVNNWA operator $(\Xi = 1)$
$\ddot{O}_a$	$\langle(0.9331, 0.5058); 0.9027, 0.488, 0.5030\rangle$
$\ddot{O}_b$	$\langle(0.8906, 0.7793); 0.9180, 0.1402, 0.0941\rangle$
$\ddot{O}_c$	$\langle(0.7697, 0.5802); 0.3594, 0.2820, 0.3313\rangle$
$\ddot{O}_d$	$\langle(0.7359, 0.5772); 0.6248, 0.1146, 0.2876\rangle$
$\ddot{O}_e$	$\langle(0.8746, 0.7474); 0.4333, 0.6864, 0.2319\rangle$

A discussion of the optimum values for each choice, both positive and negative is as $\langle(0.9331, 0.5058), 1, 0, 0\rangle$, $\langle(0.7359, 0.7793), 0, 1, 1\rangle$. Every option has an ED with the positive and negative ideal values, respectively,

$$\mathcal{D}_1^+ = 0.9181, \quad \mathcal{D}_2^+ = 0.4881, \quad \mathcal{D}_3^+ = 0.6939, \quad \mathcal{D}_4^+ = 0.9706, \quad \mathcal{D}_5^+ = 0.8741,$$

and

$$\mathcal{D}_1^- = 0.4554, \quad \mathcal{D}_2^- = 0.8855, \quad \mathcal{D}_3^- = 0.6796, \quad \mathcal{D}_4^- = 0.4029, \quad \mathcal{D}_5^- = 0.4994.$$

Calculate the relative closeness values as $\mathcal{D}_1^* = 0.3316$, $\mathcal{D}_2^* = 0.6447$, $\mathcal{D}_3^* = 0.4948$, $\mathcal{D}_4^* = 0.2934$, $\mathcal{D}_5^* = 0.3636$. Ranking of alternatives is as $\ddot{O}_b \succeq \ddot{O}_c \succeq \ddot{O}_e \succeq \ddot{O}_a \succeq \ddot{O}_d$. According to the report of $\ddot{O}_a$, $\ddot{O}_b$, $\ddot{O}_c$, $\ddot{O}_d$, $\ddot{O}_e$ traders, since the trader $\ddot{O}_b$ meets all the criteria, including sequence trading and report analysis, he will be regarded as a successful trader. In order to get superior trading, $\ddot{O}_b$ will be assigned. Consequently, $\ddot{O}_b$ is the best choice.

6.3. Comparisons between suggested and existing models

By comparing the proposed models with the existing models, this part will demonstrate their suitability and benefits for the application. The ED methodology was based on the four categories of methodologies. The following categories can be used to group distances. We use the HD-based CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG techniques in light of the aforementioned data. The various distances are displayed as in Tables 6 and 7.

Table 6: Different distances

$\Xi \mapsto 1$	CTSVNNWA	CTSVNNWG
TOPSIS	$\ddot{O}_b \geqslant \ddot{O}_c \geqslant \ddot{O}_e$	$\ddot{O}_b \geqslant \ddot{O}_d \geqslant \ddot{O}_e$
HD (proposed)	$\ddot{O}_a \geqslant \ddot{O}_d$	$\ddot{O}_a \geqslant \ddot{O}_c$
HD	$\ddot{O}_b \geqslant \ddot{O}_a \geqslant \ddot{O}_c$	$\ddot{O}_b \geqslant \ddot{O}_a \geqslant \ddot{O}_c$
[25]	$\ddot{O}_e \geqslant \ddot{O}_d$	$\ddot{O}_e \geqslant \ddot{O}_d$

Table 7: Different distances.

$\Xi \mapsto 1$	GCTSVNNWA	GCTSVNNWG
TOPSIS	$\ddot{O}_b \geqslant \ddot{O}_c \geqslant \ddot{O}_e$	$\ddot{O}_b \geqslant \ddot{O}_d \geqslant \ddot{O}_e$
HD (proposed)	$\ddot{O}_a \geqslant \ddot{O}_d$	$\ddot{O}_a \geqslant \ddot{O}_c$
HD	$\ddot{O}_b \geqslant \ddot{O}_a \geqslant \ddot{O}_c$	$\ddot{O}_b \geqslant \ddot{O}_a \geqslant \ddot{O}_c$
[25]	$\ddot{O}_e \geqslant \ddot{O}_d$	$\ddot{O}_e \geqslant \ddot{O}_d$

6.4. Effectiveness test

Reliability rates for the MADM approach vary throughout options. There are a number of prerequisites for testing. The following proximity values and rankings may be found using the CTSVNNWA technique when various Ξ values are achieved. The CTSVNNWA operator ($\Xi = 2$) must be used in order to aggregate the data for each option. This graphic illustrates the values of the aggregating operator. Lastly, a comparison with previous models is made to assess the authority and superiority of the modified model. The Ξ values are changed in accordance with CT-SVNNWA. Lastly, the rankings and relative closest values are shown below.

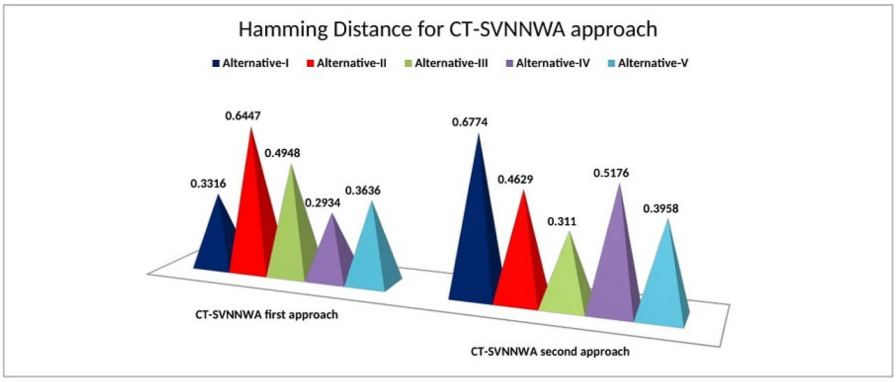

Figure 2: Graphical representation using HD for CT-SVNNWA.

We determined that the CT-SVNNWA operator utilized alternate ranking. If $\Xi = 1$, then alternative ranking is $\ddot{O}_b \succeq \ddot{O}_c \succeq \ddot{O}_e \succeq \ddot{O}_a \succeq \ddot{O}_d$. If $\Xi \mapsto 2$, then alternative ranking is $\ddot{O}_a \succeq \ddot{O}_d \succeq \ddot{O}_b \succeq \ddot{O}_e \succeq \ddot{O}_c$. As a result, the optimal alternative shifts from $\ddot{O}_b$ to $\ddot{O}_a$. Similarly, alternate rankings are generated using CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG operators. The best outcomes with the most value were achieved for each phase. The proposed AOs were tested using the available techniques to demonstrate their superiority and validity. The proposed AO's accuracy and dependability outperform those of the current technique. We suggested a novel method for determining the best option for the MADM problems.

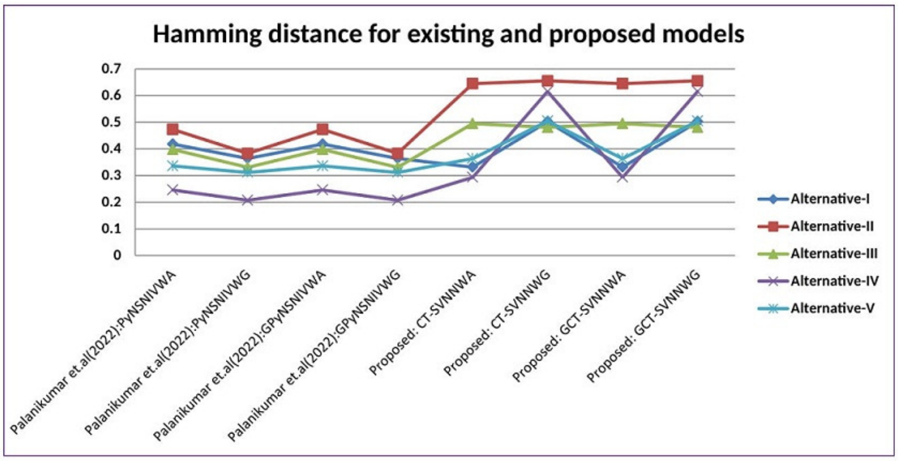

Figure 3: Graphical representation using HD for existing and proposed models.

6.5. Sensitivity Analysis

The CT-SVNNWA, CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG operators satisfy the properties of associativity, boundedness, and monotonicity. When $\Xi = 1$, this study converts the GCT-SVNNWA operator to the CT-SVNNWA operator. If $\Xi = 1$, then the GCT-SVNNWG operator is converted into the CT-SVNNWG operator. Using the above analysis, we find that the alternative ranks $\ddot{O}_b \succeq \ddot{O}_c \succeq \ddot{O}_e \succeq \ddot{O}_a \succeq \ddot{O}_d$. The new order is $\ddot{O}_a \succeq \ddot{O}_d \succeq \ddot{O}_b \succeq \ddot{O}_e \succeq \ddot{O}_c$, when $\Xi = 2$. Hence, we changed from $\ddot{O}_b$ to $\ddot{O}_a$. Alternative ranks are based on ∂ from CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG. Because it takes into account the relationships between the different features, the aforementioned strategy is beneficial. Therefore, the suggested approach yields better outcomes. This method is hence more effective in [25]. HD for CT-SVNN was established in this work. Through comparison, HD's superiority was shown. We developed a novel HD concept for CT-SVNN that aids in practical calculations and is given in an easy-to-understand mathematical manner. Consequently, a numerical example showed that the HD was superior when these two factors were coupled. The utility of HD is illustrated with a real-world example of its utilization.

6.6. Advantages

The following advantages of the suggested method are made possible by the aforementioned analysis. One well-known example of a complex system with dynamic and nonlinear behavior is the stock market. The purpose of this study was to develop a simple and acceptable methodology that would help investors choose profitable stock market chances and reduce their stress levels. Because the markets under study are unique, the model employed a neutrosophic model, which can manage a higher level of uncertainty. The created model is relatively limiting, but it does show how the NSS, which is commonly used in technical fields, may be utilized in new ways. A more sophisticated updated model might be constructed by adding more key input variables to the model now given. However, the goal of this study was to make the model and set of input variables as easy as possible so that any new investor could use them.

The previously given analysis emphasizes the multiple benefits of the applications. CT-SVNNS integrates FVNS principles. CT-SVNNS is used to study natural phenomena and human behavior in the real world that follow a normal distribution. The sum of TMG, IMG, and FMG exceeds 1, and it conveys complicated information. The TMG, IMG, and FMG squared values are all less than one. The decision-maker is free to choose the outcome based on q and personal preferences. Different ranking outcomes may be created dynamically for each approach employing operators like CT-SVNNWA, CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG. As a consequence, using the suggested DM approach, we can determine the optimal alternative for using CT-SVNN-based decision support systems. To assess the efficacy and dependability of the suggested algorithm, we compared it to certain other algorithms that are already in use. The recommended technique may have limited applicability in some domains or decision contexts. A proposed strategy should be assessed in light of the constraints and conditions that may make it most successful. To facilitate analysis, the proposed technique includes some assumptions and simplifications. As a result, the outcomes may not always correlate to real-world events, limiting their usefulness.

6.7. Limitations

The daily changes of the financial markets are influenced by many different types of variables. The model dependability would be much reduced if it were used to short-term investments. The reliability of investing in a single company is significantly diminished because events like partnerships, unanticipated management changes, changes in the company's focus, and other similar changes have a major effect on the stock price of the individual company but not the entire list, which consists of hundreds of companies. Using a certain set of input variables is the aim of this simple model. Due to significant variations in conditions, its suggestions would be far less dependable when applied to other indices. To confirm the efficacy of the suggested strategy, we conduct a comparison analysis. We mostly compare and examine from two angles. The comparison is conducted from the perspective of AOs as they form the foundation of our proven MADM technique. Nonetheless, the comparison is conducted from the perspective of MADM techniques. A proposed method ought to be assessed in view of the constraints and circumstances that might maximize its effectiveness. The proposed technique simplifies some aspects and makes some assumptions to facilitate analysis. Because the findings would not always match actual events, their usefulness would be limited.

7. Conclusion:

The SVNN is a significant tool for addressing uncertainty in DM difficulties. It goes without saying that various decision-makers may give different possibilities varying weights. These days, people frequently talk about the process of choosing assets. The application of CT-SVNNS logic to the problem of investment decision-making is presented in this work. Despite the high levels of non-linearity, volatility, and unpredictability in financial markets, CT-SVNNS penetration in this sector is still minimal. In other words, there is ample opportunity to explore the results of a higher level of neutrosophic thought, and the financial markets in particular offer this opportunity. As interest in this area of study has grown, a number of well-known papers containing state-of-the-art data have surfaced, providing a strong scientific basis for further investigation. Investors who want to lower risk while managing their long-term assets may find the neutrosophic model-based investment decision-making tool to be of great use. Naturally, lowering risk and identifying safe solutions are the goals of this research. This research opposes making hasty, short-term predictions. The results of a thorough testing and evaluation of a model constructed using historical data are presented in this study, showing that the model is capable of producing accurate investment suggestions with a high statistical likelihood.

For CT-SVNNWA, CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG, we also proposed AOs. An example was produced by applying the CT-SVNNS operator to the MADM problem-solving process utilizing SVNSS data. Using examples, we described the characteristics of these operators. Applying the MADM to each alternative allows individuals to create the appropriate DM in situations that are unclear and inconsistent. Thus, Ξ used the CT-SVNNWA, CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG operators to overcome MADM problems. The CT-SVNNWA, CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG operators on Ξ provide many alternative rankings. Finally, the discussion concluded that generalized values using Ξ had the greatest desirable impact on alternative rankings. Finally, the decision maker determines the DM to get at Ξ. The HD in the case of SVNNS provides a variety of applications, including data analysis. Solving a market share problem demonstrates the viability and efficacy of the recommended method. The ideal solutions are found in the SVNN form, which is preferable. The acquired findings are compared to the results produced by [25] and found that the best methods for both players are relatively comparable. This indicates the consistency of our approach. As a result of the present study, the researcher believes the discussions will be beneficial to modern scholars interested in this type of research. We will talk about the following subjects in further detail. (1) Soft sets, vague sets and complex FS utilizing AOs are all connected to the CT-SVNNWA, CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG operators. (2) We analyze the IVFS and cubic FS using the CT-SVNNWA, CT-SVNNWG, GCT-SVNNWA, and GCT-SVNNWG operators. (3) By analyzing (p, q, r)-rung CT-SVNN as well as complex (p, q, r)-rung CT-SVNN based on similarity measures. (4) In the future, further novel approaches to solving matrix games with SVNN payoffs could be studied.

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Ethical approval

This article does not contain any studies with human participants or animals preformed by any of the authors.

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Appendix

The proof of Theorem 4.2. Proofs (1) and (2) are the simplest. Now, we have to prove the other part (3). Now,

$$\Big(\mathcal{D}_E(\ddot{O}_1,\ddot{O}_2)+\mathcal{D}_E(\ddot{O}_2,\ddot{O}_3)\Big)^2=\begin{bmatrix}\Big((\Upsilon_1\Psi_1-\Upsilon_2\Psi_2)^2+\frac{1}{2}(\Upsilon_1\Omega_1-\Upsilon_2\Omega_2)^2\Big)^{1/2}\\+\Big((\Upsilon_2\Psi_2-\Upsilon_3\Psi_3)^2+\frac{1}{2}(\Upsilon_2\Omega_2-\Upsilon_3\Omega_3)^2\Big)^{1/2}\end{bmatrix}^2$$

implies

$$\begin{aligned}&\Big((\Upsilon_1\Psi_1-\Upsilon_2\Psi_2)^2+\frac{1}{2}(\Upsilon_1\Omega_1-\Upsilon_2\Omega_2)^2\Big)+\Big((\Upsilon_2\Psi_2-\Upsilon_3\Psi_3)^2+\frac{1}{2}(\Upsilon_2\Omega_2-\Upsilon_3\Omega_3)^2\Big)\\&\quad+2\bigg((\Upsilon_1\Psi_1-\Upsilon_2\Psi_2)^2+\frac{1}{2}(\Upsilon_1\Omega_1-\Upsilon_2\Omega_2)^2\bigg)^{1/2}\times\bigg((\Upsilon_2\Psi_2-\Upsilon_3\Psi_3)^2+\frac{1}{2}(\Upsilon_2\Omega_2-\Upsilon_3\Omega_3)^2\bigg)^{1/2}\end{aligned}$$

where $\Upsilon_1=\frac{1+P}{2}$, $\Upsilon_2=\frac{1+Q}{2}$, and $\Upsilon_3=\frac{1+R}{2}$,

$$\begin{aligned}P&=\cos^2(\propto\cdot\sigma_1^{\mathscr{T}})-\cos^2(\propto\cdot\sigma_1^{\mathscr{I}})-\cos^2(\propto\cdot\sigma_1^{\mathscr{F}}),\\Q&=\cos^2(\propto\cdot\sigma_2^{\mathscr{T}})-\cos^2(\propto\cdot\sigma_2^{\mathscr{I}})-\cos^2(\propto\cdot\sigma_2^{\mathscr{F}}),\\R&=\cos^2(\propto\cdot\sigma_3^{\mathscr{T}})-\cos^2(\propto\cdot\sigma_3^{\mathscr{I}})-\cos^2(\propto\cdot\sigma_3^{\mathscr{F}}).\end{aligned}$$

Now,

$$
\begin{aligned}
&\Big(\mathcal{D}_E(\ddot{O}_1,\ddot{O}_2)+\mathcal{D}_E(\ddot{O}_2,\ddot{O}_3)\Big)^2\\
&\quad\succeq \Big((\Upsilon_1\Psi_1-\Upsilon_2\Psi_2)^2+\frac{1}{2}(\Upsilon_1\Omega_1-\Upsilon_2\Omega_2)^2\Big)+\Big((\Upsilon_2\Psi_2-\Upsilon_3\Psi_3)^2+\frac{1}{2}(\Upsilon_2\Omega_2-\Upsilon_3\Omega_3)^2\Big)\\
&\qquad+2\Big((\Upsilon_1\Psi_1-\Upsilon_2\Psi_2)\times(\Upsilon_2\Psi_2-\Upsilon_3\Psi_3)+\frac{1}{2}(\Upsilon_1\Omega_1-\Upsilon_2\Omega_2)\times(\Upsilon_2\Omega_2-\Upsilon_3\Omega_3)\Big)\\
&\quad=(\Upsilon_1\Psi_1-\Upsilon_2\Psi_2+\Upsilon_2\Psi_2-\Upsilon_3\Psi_3)^2+\frac{1}{2}(\Upsilon_1\Omega_1-\Upsilon_2\Omega_2+\Upsilon_2\Omega_2-\Upsilon_3\Omega_3)^2\\
&\quad=(\Upsilon_1\Psi_1-\Upsilon_3\Psi_3)^2+\frac{1}{2}(\Upsilon_1\Omega_1-\Upsilon_3\Omega_3)^2=\mathcal{D}_E(\ddot{O}_1,\ddot{O}_3).
\end{aligned}
$$

□

The proof of Theorem 5.2. We prove the theorem by mathematical induction. If $n=2$, then CT-SVNNWA $(\ddot{O}_1,\ddot{O}_2)=\kappa_1\cos\ddot{O}_1\boxplus\kappa_2\cos\ddot{O}_2$, where

$$
\kappa_1\cos\ddot{O}_1=\left[\begin{array}{c}\Big(\kappa_1\Psi_1,\kappa_1\Omega_1\Big);\imath-\Big(\imath-(\cos^2(\propto\cdot\sigma_1^{\mathscr{T}}))^{\Xi}\Big)^{\kappa_1},\\(\cos^2(\propto\cdot\sigma_1^{\mathscr{I}}))^{\kappa_1},(\cos^2(\propto\cdot\sigma_1^{\mathscr{F}}))^{\kappa_1}\end{array}\right]
$$

and

$$
\kappa_2\cos\ddot{O}_2=\left[\begin{array}{c}\Big(\kappa_2\Psi_2,\kappa_2\Omega_2\Big);\imath-\Big(\imath-(\cos^2(\propto\cdot\sigma_2^{\mathscr{T}}))^{\Xi}\Big)^{\kappa_2},\\(\cos^2(\propto\cdot\sigma_2^{\mathscr{I}}))^{\kappa_2},(\cos^2(\propto\cdot\sigma_2^{\mathscr{F}}))^{\kappa_2}\end{array}\right].
$$

Applying to Definition 3.3,

$$
\begin{aligned}
\kappa_1\cos\ddot{O}_1\boxplus\kappa_2\cos\ddot{O}_2&=\left[\begin{array}{c}\Big(\kappa_1\Psi_1+\kappa_2\Psi_2,\kappa_1\Omega_1+\kappa_2\Omega_2\Big);\\
\Big(\imath-\Big(\imath-(\cos^2(\propto\cdot\sigma_1^{\mathscr{T}}))^{\Xi}\Big)^{\kappa_1}\Big)+\Big(\imath-\Big(\imath-(\cos^2(\propto\cdot\sigma_2^{\mathscr{T}}))^{\Xi}\Big)^{\kappa_2}\Big)\\
-\Big(\imath-\Big(\imath-(\cos^2(\propto\cdot\sigma_1^{\mathscr{T}}))^{\Xi}\Big)^{\kappa_1}\Big)\cdot\Big(\imath-\Big(\imath-(\cos^2(\propto\cdot\sigma_2^{\mathscr{T}}))^{\Xi}\Big)^{\kappa_2}\Big),\\
(\cos^2(\propto\cdot\sigma_1^{\mathscr{I}}))^{\kappa_1}\cdot(\cos^2(\propto\cdot\sigma_2^{\mathscr{I}}))^{\kappa_2},(\cos^2(\propto\cdot\sigma_1^{\mathscr{F}}))^{\kappa_1}\cdot(\cos^2(\propto\cdot\sigma_2^{\mathscr{F}}))^{\kappa_2}\end{array}\right]\\
&=\left[\begin{array}{c}\Big(\kappa_1\Psi_1+\kappa_2\Psi_2,\kappa_1\Omega_1+\kappa_2\Omega_2\Big);\\
\imath-\Big(\imath-(\cos^2(\propto\cdot\sigma_1^{\mathscr{T}}))^{\Xi}\Big)^{\kappa_1}\cdot\Big(\imath-(\cos^2(\propto\cdot\sigma_2^{\mathscr{T}}))^{\Xi}\Big)^{\kappa_2},\\
(\cos^2(\propto\cdot\sigma_1^{\mathscr{I}}))^{\kappa_1}\cdot(\cos^2(\propto\cdot\sigma_2^{\mathscr{I}}))^{\kappa_2},(\cos^2(\propto\cdot\sigma_1^{\mathscr{F}}))^{\kappa_1}\cdot(\cos^2(\propto\cdot\sigma_2^{\mathscr{F}}))^{\kappa_2}\end{array}\right],\\
\text{CT-SVNNWA}(\ddot{O}_1,\ddot{O}_2)&=\left[\begin{array}{c}\Big(\biguplus_{i\mapsto1}\kappa_i\Psi_i,\biguplus_{i\mapsto1}\kappa_i\Omega_i\Big);\imath-\bigotimes_{i\mapsto1}\Big(\imath-(\cos^2(\propto\cdot\sigma_i^{\mathscr{T}}))^{\Xi}\Big)^{\kappa_i},\\
\bigotimes_{i\mapsto1}(\cos^2(\propto\cdot\sigma_i^{\mathscr{I}}))^{\kappa_i},\bigotimes_{i\mapsto1}(\cos^2(\propto\cdot\sigma_i^{\mathscr{F}}))^{\kappa_i}\end{array}\right].
\end{aligned}
$$

Also, valid for $n\succeq3$ and

$$
CT-SVNNWA(\ddot{O}_1,\ddot{O}_2,\ldots,\ddot{O}_l)=\left[\begin{array}{c}\Big(\biguplus_{i\mapsto1}^{l}\kappa_i\Psi_i,\biguplus_{i\mapsto1}^{l}\kappa_i\Omega_i\Big);\imath-\bigotimes_{i\mapsto1}^{l}\Big(\imath-(\cos^2(\propto\cdot\sigma_i^{\mathscr{T}}))^{\Xi}\Big)^{\kappa_i},\\
\bigotimes_{i\mapsto1}^{l}(\cos^2(\propto\cdot\sigma_i^{\mathscr{I}}))^{\kappa_i},\bigotimes_{i\mapsto1}^{l}(\cos^2(\propto\cdot\sigma_i^{\mathscr{F}}))^{\kappa_i}\end{array}\right].
$$

If $n = l+1$, then

$$\begin{aligned}
&\mathrm{CT\text{-}SVNNWA}(\ddot{O}_1,\ddot{O}_2,\ldots,\ddot{O}_l,\ddot{O}_{l+1})\\
&= \left[\begin{array}{c}
\left(\biguplus_{i\mapsto1}^{l}\kappa_i\Psi_i+\kappa_{l+1}\Psi_{l+1},\biguplus_{i\mapsto1}^{l}\kappa_i\Omega_i+\kappa_{l+1}\Omega_{l+1}\right);\\
\biguplus_{i\mapsto1}^{l}\left(1-\left(1-(\cos^2(\propto\cdot\sigma_i^{\mathscr{T}}))^{\Xi}\right)^{\kappa_i}\right)+\left(1-\left(1-(\cos^2(\propto\cdot\sigma_{l+1}^{\mathscr{T}}))^{\Xi}\right)^{\kappa_{l+1}}\right)\\
-\bigotimes_{i\mapsto1}^{l}\left(1-\left(1-(\cos^2(\propto\cdot\sigma_i^{\mathscr{T}}))^{\Xi}\right)^{\kappa_i}\right)\cdot\left(1-\left(1-(\cos^2(\propto\cdot\sigma_{l+1}^{\mathscr{T}}))^{\Xi}\right)^{\kappa_{l+1}}\right),\\
\bigotimes_{i\mapsto1}^{l}(\cos^2(\propto\cdot\sigma_i^{\mathscr{I}}))^{\kappa_i}\cdot(\cos^2(\propto\cdot\sigma_{l+1}^{\mathscr{I}}))^{\kappa_{l+1}},\bigotimes_{i\mapsto1}^{l}(\cos^2(\propto\cdot\sigma_i^{\mathscr{F}}))^{\kappa_i}\cdot(\cos^2(\propto\cdot\sigma_{l+1}^{\mathscr{F}}))^{\kappa_{l+1}}
\end{array}\right]\\
&= \left[\begin{array}{c}
\left(\biguplus_{i\mapsto1}^{l+1}\kappa_i\Psi_i,\biguplus_{i\mapsto1}^{l+1}\kappa_i\Omega_i\right);1-\bigotimes_{i\mapsto1}^{l+1}\left(1-(\cos^2(\propto\cdot\sigma_i^{\mathscr{T}}))^{\Xi}\right)^{\kappa_i},\\
\bigotimes_{i\mapsto1}^{l+1}(\cos^2(\propto\cdot\sigma_i^{\mathscr{I}}))^{\kappa_i},\bigotimes_{i\mapsto1}^{l+1}(\cos^2(\propto\cdot\sigma_i^{\mathscr{F}}))^{\kappa_i}
\end{array}\right].
\end{aligned}$$

□

The proof of Theorem 5.3. Since, $(\Psi_i,\Omega_i)=(\Psi,\Omega)$, $\cos(\propto\cdot\sigma_i^{\mathscr{T}})=\cos(\propto\cdot\sigma^{\mathscr{T}})$, $\cos(\propto\cdot\sigma_i^{\mathscr{I}})=\cos(\propto\cdot\sigma^{\mathscr{I}})$, and $\cos(\propto\cdot\sigma_i^{\mathscr{F}})=\cos(\propto\cdot\sigma^{\mathscr{F}})$, for $i\mapsto 1,2,\ldots,n$, and $\biguplus_{i\mapsto1}^{n}\kappa_i=1$. Now,

$$\begin{aligned}
\mathrm{CT\text{-}SVNNWA}(\ddot{O}_1,\ddot{O}_2,\ldots,\ddot{O}_n) &= \left[\begin{array}{c}
\left(\biguplus_{i\mapsto1}^{n}\kappa_i\Psi_i,\biguplus_{i\mapsto1}^{n}\kappa_i\Omega_i\right);1-\bigotimes_{i\mapsto1}^{n}\left(1-(\cos(\propto\cdot\sigma_i^{\mathscr{T}}))^{\Xi}\right)^{\kappa_i},\\
\bigotimes_{i\mapsto1}^{n}(\cos(\propto\cdot\sigma_i^{\mathscr{I}}))^{\kappa_i},\bigotimes_{i\mapsto1}^{n}(\cos(\propto\cdot\sigma_i^{\mathscr{F}}))^{\kappa_i}
\end{array}\right]\\
&= \left[\begin{array}{c}
\left(\Psi\biguplus_{i\mapsto1}^{n}\kappa_i,\Omega\biguplus_{i\mapsto1}^{n}\kappa_i\right);1-\left(1-(\cos(\propto\cdot\sigma^{\mathscr{T}}))^{\Xi}\right)^{\biguplus_{i\mapsto1}^{n}\kappa_i},\\
(\cos(\propto\cdot\sigma^{\mathscr{I}}))^{\biguplus_{i\mapsto1}^{n}\kappa_i},(\cos(\propto\cdot\sigma^{\mathscr{F}}))^{\biguplus_{i\mapsto1}^{n}\kappa_i}
\end{array}\right]\\
&= \left[(\Psi,\Omega);1-\left(1-(\cos(\propto\cdot\sigma^{\mathscr{T}}))^{\Xi}\right),(\cos(\propto\cdot\sigma^{\mathscr{I}})),(\cos(\propto\cdot\sigma^{\mathscr{F}}))\right]=\cos\ddot{O}.
\end{aligned}$$

□

The proof of Theorem 5.4. Since, $\underbrace{\sigma^{\mathscr{T}}}=\inf\sigma_{ij}^{\mathscr{T}^-}$, $\overbrace{\sigma^{\mathscr{T}}}=\sup\sigma_{ij}^{\mathscr{T}^-}$, and $\underbrace{\sigma^{\mathscr{T}}}\preceq\sigma_{ij}^{\mathscr{T}^-}\preceq\overbrace{\sigma^{\mathscr{T}}}$. We have,

$$\begin{aligned}
\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{T}})} &= 1-\bigotimes_{i\mapsto1}^{n}\left(1-(\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{T}})})^{\Xi}\right)^{\kappa_i}\\
&\preceq 1-\bigotimes_{i\mapsto1}^{n}\left(1-(\cos^2\cdot\propto\cdot\sigma_{ij}^{\mathscr{T}^-})^{\Xi}\right)^{\kappa_i}\preceq 1-\bigotimes_{i\mapsto1}^{n}\left(1-(\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{T}})})^{\Xi}\right)^{\kappa_i}=\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{T}})}.
\end{aligned}$$

Since $\underbrace{\sigma^{\mathscr{I}}}=\inf\sigma_{ij}^{\mathscr{I}^-}$, $\overbrace{\sigma^{\mathscr{I}}}=\sup\sigma_{ij}^{\mathscr{I}^-}$, $\underbrace{\sigma^{\mathscr{I}}}\preceq\sigma_{ij}^{\mathscr{I}^-}\preceq\overbrace{\sigma^{\mathscr{I}}}$, we have

$$\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{I}})}=\bigotimes_{i\mapsto1}^{n}(\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{I}})})^{\kappa_i}\preceq\bigotimes_{i\mapsto1}^{n}(\cos^2\cdot\propto\cdot\sigma_{ij}^{\mathscr{I}^-})^{\kappa_i}\preceq\bigotimes_{i\mapsto1}^{n}(\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{I}})})^{\kappa_i}=\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{I}})}.$$

Since, $\underbrace{\sigma^{\mathscr{F}}}=\inf\sigma_{ij}^{\mathscr{F}^-}$, $\overbrace{\sigma^{\mathscr{F}}}=\sup\sigma_{ij}^{\mathscr{F}^-}$, and $\underbrace{\sigma^{\mathscr{F}}}\preceq\sigma_{ij}^{\mathscr{F}^-}\preceq\overbrace{\sigma^{\mathscr{F}}}$, we have

$$\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{F}})}=\bigotimes_{i\mapsto1}^{n}(\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{F}})})^{\kappa_i}\preceq\bigotimes_{i\mapsto1}^{n}(\cos^2\cdot\propto\cdot\sigma_{ij}^{\mathscr{F}^-})^{\kappa_i}\preceq\bigotimes_{i\mapsto1}^{n}(\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{F}})})^{\kappa_i}=\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{F}})}.$$

Since, $\underbrace{\Psi}=\inf\Psi_{ij}$, $\overbrace{\Psi}=\sup\Psi_{ij}$, $\underbrace{\Omega}=\sup\Omega_{ij}$, $\overbrace{\Omega}=\inf\Omega_{ij}$ and $\underbrace{\Psi}\preceq\Psi_{ij}\preceq\overbrace{\Psi}$, and $\overbrace{\Omega}\preceq\Omega_{ij}\preceq\underbrace{\Omega}$, hence,

$$\biguplus_{i\mapsto1}^{n}\kappa_i\underbrace{\Psi}\preceq\biguplus_{i\mapsto1}^{n}\kappa_i\Psi_{ij}\preceq\biguplus_{i\mapsto1}^{n}\kappa_i\overbrace{\Psi}\quad\text{and}\quad\biguplus_{i\mapsto1}^{n}\kappa_i\overbrace{\Omega}\preceq\biguplus_{i\mapsto1}^{n}\kappa_i\Omega_{ij}\preceq\biguplus_{i\mapsto1}^{n}\kappa_i\underbrace{\Omega}.$$

Therefore,

$$\frac{\biguplus_{i\mapsto 1}^{n}\kappa_i\underbrace{\Psi}}{2}\left[\frac{\left(1-\otimes_{i\mapsto 1}^{n}\left(1-(\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{T}})})^{\Xi}\right)^{\kappa_i}\right)}{2}-\frac{\left(\otimes_{i\mapsto 1}^{n}(\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{I}})})^{\kappa_i}\right)}{2}-\frac{2+\left(\otimes_{i\mapsto 1}^{n}(\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{F}})})^{\kappa_i}\right)}{2}\right]$$
$$\preceq\frac{\biguplus_{i\mapsto 1}^{n}\kappa_i\Psi_{ij}}{2}\left[\frac{\left(1-\otimes_{i\mapsto 1}^{n}\left(1-(\cos^2\cdot\propto\cdot\sigma_{ij}^{\mathscr{T}-})^{\Xi}\right)^{\kappa_i}\right)}{2}-\frac{\left(\otimes_{i\mapsto 1}^{n}(\cos^2\cdot\propto\cdot\sigma_{ij}^{\mathscr{I}-})^{\kappa_i}\right)}{2}-\frac{2+\left(\otimes_{i\mapsto 1}^{n}(\cos^2\cdot\propto\cdot\sigma_{ij}^{\mathscr{F}-})^{\kappa_i}\right)}{2}\right]$$
$$\preceq\frac{\biguplus_{i\mapsto 1}^{n}\kappa_i\overbrace{\Psi}}{2}\left[\frac{\left(1-\otimes_{i\mapsto 1}^{n}\left(1-(\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{T}})})^{\Xi}\right)^{\kappa_i}\right)}{2}-\frac{\left(\otimes_{i\mapsto 1}^{n}(\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{I}})})^{\kappa_i}\right)}{2}-\frac{2+\left(\otimes_{i\mapsto 1}^{n}(\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{F}})})^{\kappa_i}\right)}{2}\right].$$

Hence,

$$\left\langle(\underbrace{\Psi},\underbrace{\Omega});\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{T}})},\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{I}})},\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{F}})}\right\rangle$$
$$\preceq \text{CT-SVNNWA}(\ddot{O}_1,\ddot{O}_2,\ldots,\ddot{O}_n)\preceq\left\langle(\overbrace{\Psi},\overbrace{\Omega});\overbrace{\cos^2(\propto\cdot\sigma^{\mathscr{T}})},\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{I}})},\underbrace{\cos^2(\propto\cdot\sigma^{\mathscr{F}})}\right\rangle.$$

□

The proof of Theorem 5.5. For any i, $\Psi_{t_{ij}}\preceq\Omega_{h_{ij}}$. Therefore, $\biguplus_{i\mapsto 1}^{n}\Psi_{t_{ij}}\preceq\biguplus_{i\mapsto 1}^{n}\Omega_{h_{ij}}$. For any i, $\left(\cos^2(\propto\cdot\sigma_{t_{ij}}^{\mathscr{T}})\right)\preceq\left(\cos^2(\propto\cdot\sigma_{h_{ij}}^{\mathscr{T}})\right)$. Therefore, $1-\left(\cos^2(\propto\cdot\sigma_{t_i}^{\mathscr{T}})\right)\succeq 1-\left(\cos^2(\propto\cdot\sigma_{h_i}^{\mathscr{T}})\right)$. Hence,

$$\bigotimes_{i\mapsto 1}^{n}\left(1-\left(\cos^2(\propto\cdot\sigma_{t_i}^{\mathscr{T}})\right)\right)^{\kappa_i}\succeq\bigotimes_{i\mapsto 1}^{n}\left(1-\left(\cos^2(\propto\cdot\sigma_{h_i}^{\mathscr{T}})\right)\right)^{\kappa_i}$$

and

$$1-\bigotimes_{i\mapsto 1}^{n}\left(1-\left(\cos^2(\propto\cdot\sigma_{t_i}^{\mathscr{T}})\right)^{\Xi}\right)^{\kappa_i}\preceq 1-\bigotimes_{i\mapsto 1}^{n}\left(1-\left(\cos^2(\propto\cdot\sigma_{h_i}^{\mathscr{T}})\right)^{\Xi}\right)^{\kappa_i}.$$

For any i, $\left(\cos^2(\propto\cdot\sigma_{t_{ij}}^{\mathscr{I}})\right)^{\Xi}\succeq\left(\cos^2(\propto\cdot\sigma_{h_{ij}}^{\mathscr{I}})\right)^{\Xi}$. Therefore, $-\frac{\left(\otimes_{i\mapsto 1}^{n}\cos^2(\propto\cdot\sigma_{t_{ij}}^{\mathscr{I}})\right)}{2}\preceq-\frac{\left(\otimes_{i\mapsto 1}^{n}\cos^2(\propto\cdot\sigma_{h_{ij}}^{\mathscr{I}})\right)}{2}$. For any i, $\left(\cos^2(\propto\cdot\sigma_{t_{ij}}^{\mathscr{F}})\right)\succeq\left(\cos^2(\propto\cdot\sigma_{h_{ij}}^{\mathscr{F}})\right)$. Therefore, $-\frac{2+\left(\otimes_{i\mapsto 1}^{n}\cos^2(\propto\cdot\sigma_{t_{ij}}^{\mathscr{F}})\right)}{2}\preceq-\frac{2+\left(\otimes_{i\mapsto 1}^{n}\cos^2(\propto\cdot\sigma_{h_{ij}}^{\mathscr{F}})\right)}{2}$. Hence,

$$\frac{\biguplus_{i\mapsto 1}^{n}\Psi_{tij}}{2}\times\left[\begin{array}{c}\frac{\left(1-\bigotimes_{i\mapsto 1}^{n}\left(1-(\cos^2\cdot\propto\cdot\sigma_{ti}^{\mathscr{T}-})^{\Xi}\right)^{\kappa_i}\right)}{2}\\ -\frac{\left(\otimes_{i\mapsto 1}^{n}(\cos^2\cdot\propto\cdot\sigma_{tij}^{\mathscr{I}-})\right)}{2}-\frac{2+\left(\otimes_{i\mapsto 1}^{n}(\cos^2\cdot\propto\cdot\sigma_{tij}^{\mathscr{F}-})\right)}{2}\end{array}\right]$$
$$\preceq\frac{\biguplus_{i\mapsto 1}^{n}\Psi_{hij}}{2}\times\left[\begin{array}{c}\frac{\left(1-\bigotimes_{i\mapsto 1}^{n}\left(1-(\cos^2\cdot\propto\cdot\sigma_{hi}^{\mathscr{T}-})^{\Xi}\right)^{\kappa_i}\right)}{2}\\ -\frac{\left(\otimes_{i\mapsto 1}^{n}(\cos^2\cdot\propto\cdot\sigma_{hij}^{\mathscr{I}-})\right)}{2}-\frac{2+\left(\otimes_{i\mapsto 1}^{n}(\cos^2\cdot\propto\cdot\sigma_{hij}^{\mathscr{F}-})\right)}{2}\end{array}\right].$$

Hence, $\text{CT-SVNNWA}(\ddot{O}_1,\ddot{O}_2,\ldots,\ddot{O}_n)\preceq\text{CT-SVNNWA}(\ddot{W}_1,\ddot{W}_2,\ldots,\ddot{W}_n)$. □

The proof of Theorem 5.11. Via mathematical induction, it is compulsory to show that

$$\biguplus_{i\mapsto 1}^{n} \kappa_i(\cos \ddot{O}_i)^{\Xi} = \begin{bmatrix} \left(\left(\biguplus_{i\mapsto 1}^{n} \kappa_i \Psi_i^{\Xi}\right), \left(\biguplus_{i\mapsto 1}^{n} \kappa_i \Omega_i^{\Xi}\right)\right); \\ \text{ı} - \bigotimes_{i\mapsto 1}^{n} \left(\text{ı} - \left((\cos^2(\propto \cdot \sigma_i^{\mathscr{T}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}, \\ \bigotimes_{i\mapsto 1}^{n} \left(\text{ı} - \left(\text{ı} - (\cos^2(\propto \cdot \sigma_i^{\mathscr{I}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}, \\ \bigotimes_{i\mapsto 1}^{n} \left(\text{ı} - \left(\text{ı} - (\cos^2(\propto \cdot \sigma_i^{\mathscr{F}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i} \end{bmatrix}.$$

Putting $n = 2$,

$$\kappa_1(\cos O_1)^{\Xi} \boxplus \kappa_2(\cos O_2)^{\Xi} = \begin{bmatrix} \left(\kappa_1 \Psi_1^{\Xi} + \kappa_2 \Psi_2^{\Xi}, \kappa_1 \Omega_1^{\Xi} + \kappa_2 \Omega_2^{\Xi}\right); \\ \left(\text{ı} - \left(\text{ı} - \left((\cos^2(\propto \cdot \sigma_1^{\mathscr{T}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_1}\right)^{\Xi} + \left(\text{ı} - \left(\text{ı} - \left((\cos^2(\propto \cdot \sigma_2^{\mathscr{T}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_1}\right)^{\Xi}, \\ -\left(\text{ı} - \left(\text{ı} - \left((\cos^2(\propto \cdot \sigma_1^{\mathscr{T}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_1}\right)^{\Xi} \cdot \left(\text{ı} - \left(\text{ı} - \left((\cos^2(\propto \cdot \sigma_2^{\mathscr{T}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_1}\right)^{\Xi} \\ \left(\text{ı} - \left(\text{ı} - (\cos^2(\propto \cdot \sigma_1^{\mathscr{I}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_1} \cdot \left(\text{ı} - \left(\text{ı} - (\cos^2(\propto \cdot \sigma_2^{\mathscr{I}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_1}, \\ \left(\text{ı} - \left(\text{ı} - (\cos^2(\propto \cdot \sigma_1^{\mathscr{F}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_1} \cdot \left(\text{ı} - \left(\text{ı} - (\cos^2(\propto \cdot \sigma_2^{\mathscr{F}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_1} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\biguplus_{i\mapsto 1} \kappa_i \Psi_i^{\Xi}, \biguplus_{i\mapsto 1} \kappa_i \Omega_i^{\Xi}\right); \\ \text{ı} - \bigotimes_{i\mapsto 1} \left(\text{ı} - \left((\cos^2(\propto \cdot \sigma_1^{\mathscr{T}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}, \\ \bigotimes_{i\mapsto 1} \left(\text{ı} - \left(\text{ı} - (\cos^2(\propto \cdot \sigma_i^{\mathscr{I}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}, \\ \bigotimes_{i\mapsto 1} \left(\text{ı} - \left(\text{ı} - (\cos^2(\propto \cdot \sigma_i^{\mathscr{F}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i} \end{bmatrix}.$$

In general,

$$\biguplus_{i\mapsto 1}^{l} \kappa_i(\cos \ddot{O}_i)^{\Xi} = \begin{bmatrix} \left(\biguplus_{i\mapsto 1}^{l} \kappa_i \Psi_i^{\Xi}, \biguplus_{i\mapsto 1}^{l} \kappa_i \Omega_i^{\Xi}\right); \\ \text{ı} - \bigotimes_{i\mapsto 1}^{l} \left(\text{ı} - \left((\cos^2(\propto \cdot \sigma_1^{\mathscr{T}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}, \\ \bigotimes_{i\mapsto 1}^{l} \left(\text{ı} - \left(\text{ı} - (\cos^2(\propto \cdot \sigma_i^{\mathscr{I}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}, \\ \bigotimes_{i\mapsto 1}^{l} \left(\text{ı} - \left(\text{ı} - (\cos^2(\propto \cdot \sigma_i^{\mathscr{F}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i} \end{bmatrix}.$$

If $n = l+1$, then $\biguplus_{i\mapsto 1}^{l} \kappa_i(\cos \ddot{O}_i)^{\Xi} + \kappa_{l+1}(\cos \ddot{O}_{l+1})^{\Xi} = \biguplus_{i\mapsto 1}^{l+1} \kappa_i(\cos \ddot{O}_i)^{\Xi}$. Now,

$$
\begin{aligned}
&\biguplus_{i\mapsto 1}^{l} \kappa_i(\cos \ddot{O}_i)^{\Xi} + \kappa_{l+1}(\cos \ddot{O}_{l+1})^{\Xi}\\
&= \kappa_1(\cos \ddot{O}_1)^{\Xi} \boxplus \kappa_2(\cos \ddot{O}_2)^{\Xi} \boxplus \ldots \boxplus \kappa_l(\cos \ddot{O}_l)^{\Xi} \boxplus \kappa_{l+1}(\cos \ddot{O}_{l+1})^{\Xi}\\
&= \left[\begin{array}{c}
\left(\biguplus_{i\mapsto 1}^{l} \kappa_i \Psi_i^{\Xi} + \kappa_{l+1}\Psi_{l+1}^{\Xi}, \biguplus_{i\mapsto 1}^{l} \kappa_i \Omega_i^{\Xi} + \kappa_{l+1}\Omega_{l+1}^{\Xi}\right);\\
\left(\imath - \bigotimes_{i\mapsto 1}^{l}\left(\imath - \left((\cos^2(\propto \cdot \sigma_i^{\mathscr{T}}))^{\Xi}\right)^{\kappa_i}\right)\right)^{\Xi} + \left(\imath - \left(\imath - \left((\cos^2(\propto \cdot \sigma_{l+1}^{\mathscr{T}}))^{\Xi}\right)^{\kappa_1}\right)\right)^{\Xi},\\
-\left(\imath - \bigotimes_{i\mapsto 1}^{l}\left(\imath - \left((\cos^2(\propto \cdot \sigma_i^{\mathscr{T}}))^{\Xi}\right)^{\kappa_i}\right)\right)^{\Xi} \cdot \left(\imath - \left(\imath - \left((\cos^2(\propto \cdot \sigma_{l+1}^{\mathscr{T}}))^{\Xi}\right)^{\kappa_1}\right)\right)^{\Xi}\\
\bigotimes_{i\mapsto 1}^{l}\left(\imath - \left(\imath - (\cos^2(\propto \cdot \sigma_i^{\mathscr{I}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i} \cdot \left(\imath - \left(\imath - (\cos^2(\propto \cdot \sigma_{l+1}^{\mathscr{I}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_1},\\
\bigotimes_{i\mapsto 1}^{l}\left(\imath - \left(\imath - (\cos^2(\propto \cdot \sigma_i^{\mathscr{F}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i} \cdot \left(\imath - \left(\imath - (\cos^2(\propto \cdot \sigma_{l+1}^{\mathscr{F}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_1}
\end{array}\right],
\end{aligned}
$$

$$
\biguplus_{i\mapsto 1}^{l+1} \kappa_i(\cos \ddot{O}_i)^{\Xi} = \left[\begin{array}{c}
\left(\biguplus_{i\mapsto 1}^{l+1} \kappa_i \Psi_i^{\Xi}, \biguplus_{i\mapsto 1}^{l+1} \kappa_i \Omega_i^{\Xi}\right);\\
\imath - \bigotimes_{i\mapsto 1}^{l+1}\left(\imath - \left((\cos^2(\propto \cdot \sigma_1^{\mathscr{T}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i},\\
\bigotimes_{i\mapsto 1}^{l+1}\left(\imath - \left(\imath - (\cos^2(\propto \cdot \sigma_i^{\mathscr{I}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i},\\
\bigotimes_{i\mapsto 1}^{l+1}\left(\imath - \left(\imath - (\cos^2(\propto \cdot \sigma_i^{\mathscr{F}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}
\end{array}\right],
$$

$$
\left(\biguplus_{i\mapsto 1}^{l+1} \kappa_i(\cos \ddot{O}_i)^{\Xi}\right)^{1/\Xi} = \left[\begin{array}{c}
\left(\left(\biguplus_{i\mapsto 1}^{l+1} \kappa_i \Psi_i^{\Xi}\right)^{1/\Xi}, \left(\biguplus_{i\mapsto 1}^{l+1} \kappa_i \Omega_i^{\Xi}\right)^{1/\Xi}\right);\\
\left(\imath - \bigotimes_{i\mapsto 1}^{l+1}\left(\imath - \left((\cos^2(\propto \cdot \sigma_i^{\mathscr{T}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}\right)^{1/\Xi},\\
\imath - \left(\imath - \left(\bigotimes_{i\mapsto 1}^{l+1}\left(\imath - \left(\imath - (\cos^2(\propto \cdot \sigma_i^{\mathscr{I}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}\right)\right)^{1/\Xi},\\
\imath - \left(\imath - \left(\bigotimes_{i\mapsto 1}^{l+1}\left(\imath - \left(\imath - (\cos^2(\propto \cdot \sigma_i^{\mathscr{F}}))^{\Xi}\right)^{\Xi}\right)^{\kappa_i}\right)\right)^{1/\Xi}
\end{array}\right].
$$

□

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