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Constructing series approximate solutions of ordinary differential equations using the limit residual function method

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Abstract

This article offers analytical solutions in power series form for ordinary differential equations. It provides an effective tool for deriving accurate analytical and numerical solutions to these equations via the use of a novel analytical approach called the limit residual function method. The suggested technique demonstrates that an exact solution can be found when a pattern exists in the obtained series solution; otherwise, only rough estimates can be provided. By comparing our results with exact solutions to the problems we discussed, we conclude that the present approach is simple, easy, and effective for solving differential equations, given that the consequent series approximate solutions are in the closed form of the actual results.

Keywords: Functional analysis, ordinary differential equations, power series, residual function.

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1. Introduction

Ordinary and partial differential, integral, and integrodifferential equations are powerful tools for understanding changes and dynamic processes in various natural and engineering phenomena [4, 19, 22, 23]. Among these equations, ordinary differential equations (ODEs) have a special place. The study of ODEs has attracted the interest of many scientists over time because of its ability to model various systems surrounding us. ODEs provide a mathematical framework for describing the behavior of these systems and accurately predicting their future outcomes.

ODEs can be applied in various areas, including chemistry, where they model chemical reactions; physics, where they describe the motion of objects and analyze electrical circuits; financial sciences; and studies of population growth and environmental systems, among others.

It is important to mention some methods used to solve ODEs, including the predictor-corrector approach [7], the integral transformation method [17], the variational iteration approach [13], the Homotopy perturbation approach [2, 14], the Homotopy analysis method [24, 25], the Adomian decomposition method [1, 11, 18], the Laplace Adomian decomposition method [20, 21], the Taylor series method [15, 16]

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and the power series (PS) method [8]. However, some challenges and problems have arisen in the solution of nonhomogeneous ODEs, which led to the exploration of ways to address these issues, prompting the proposal of the residual power series (RPS) method [31]. This method extends the PS method, employing it to write the solution and using the derivatives to identify the coefficients of the series [9, 29].

The RPS method was developed by integrating the approach with the Laplace transform to solve ODEs. This process involves transferring the target equation, using the RPS method to solve it in Laplace space, then taking the inverse Laplace transform to obtain the desired results [6, 10, 30].

In this paper, a new method is introduced called the limit residual function (LRF) method [5] to solve ODEs. This approach depends on the key ideas: the limit and the residual function. When comparing the LRF technique to previous methods, we find that it reduces the time and effort required to obtain solutions for ODEs. We will introduce the theoretical framework that underpins this method, along with instructions on how to apply it to solve various ODEs. Moreover, we will provide a set of illustrative examples that cover both linear and non-linear, as well as homogeneous and non-homogeneous equations.

2. Concepts and theorems about power series

Definition 2.1 ([3])**.** If t is a variable and $a_0, a_1, a_2, \ldots,$ and t_0 are constants, then a series of the form

$$\sum_{j=0}^{\infty} a_j (t-t_0)^j = a_0 + a_1 (t-t_0) + a_2 (t-t_0)^2 + \cdots . \tag{2.1}$$

is called a PS about $t = t_0$.

All theories related to the convergence of PS (2.1) can be found in Reference [3].

Theorem 2.2 ([27])**.** *The sum of PS* (2.1) *is zero for all* t *in some interval if and only if each coefficient* a_j *is zero.*

In the following, we present the key theorem of the proposed method, which assists in determining the coefficients of the PW solutions.

Theorem 2.3 ([5])**.** *Suppose* $\omega(t) = 0$ *for all* t *in some interval* I. *If* $\omega(t) = \sum_{j=0}^{\infty} a_j (t-t_0)^j$, *then*

$$\lim_{t \to t_0} \frac{\omega_k(t)}{(t-t_0)^{k-n}} = 0, \quad k, n = 1, 2, \ldots, \quad t \neq t_0,$$

where $\omega_k(t) = \sum_{j=0}^{k} a_j (t-t_0)^j$.

3. LRF methodology for solving ODEs

In this section, we present the LRF technique for solving ODEs. This method relies on finding the limit as x approaches zero for the residual function formula. Consider the following general form of the ODE of degree n:

$$\frac{d^n\omega}{dt^n} = F\left(t, \omega, \frac{d\omega}{dt}, \frac{d^2\omega}{dt^2}, \ldots, \frac{d^{n-1}\omega}{dt^{n-1}}\right), \quad t \in [a, b], \tag{3.1}$$

subjected to the initial conditions

$$\frac{d^i\omega}{dt^i}(t_i) = \mu_i, \quad i = 0, 1, 2, \ldots, n-1, \tag{3.2}$$

where $\omega(t)$ is an unknown analytical function that will be determined, and F is an analytic function.

LRF method assumes that the function $\omega(t)$ can be written in the following PS expansion:

$$\omega(t) = \sum_{j=0}^{\infty} a_j (t-t_0)^j, \qquad |t-t_0| < \rho, \tag{3.3}$$

where the radius of convergence is ρ. Since the solution of problem (3.1)-(3.2) is a series solution, truncate the series (3.3) to obtain the kth approximation:

$$\omega_k(t) = \sum_{j=0}^{k} a_j (t - t_0)^j.$$

Based on the initial conditions in (3.2), we have

$$a_0 = \mu_0, \quad a_1 = \mu_1, \quad a_2 = \frac{\mu_2}{2!}, \dots, \quad a_{n-1} = \frac{\mu_{n-1}}{(n-1)!}. \tag{3.4}$$

Thus, the kth approximation of $\omega(t)$ can be written as:

$$\omega_k(t) = \sum_{j=0}^{n-1} \frac{\mu_j}{j!} (t - t_0)^j + \sum_{j=n}^{k} a_j (t - t_0)^j. \tag{3.5}$$

The additional coefficients are determined using the LRF technique by defining the residual functions as:

$$\mathcal{R}(\omega(t)) = \frac{d^n \omega}{dt^n} - F\left(t, \omega, \frac{d\omega}{dt}, \frac{d^2\omega}{dt^2}, \dots, \frac{d^{n-1}\omega}{dt^{n-1}}\right).$$

So, the k^{th} residual function can be written as:

$$\mathcal{R}_k(\omega(t)) = \frac{d^n \omega_k}{dt^n} - F\left(t, \omega_k, \frac{d\omega_k}{dt}, \frac{d^2\omega_k}{dt^2}, \dots, \frac{d^{n-1}\omega_k}{dt^{n-1}}\right).$$

To find the k^{th} approximate solution for problem (3.1)-(3.2), we need to determine the coefficients in equation (3.5), a_j,wherej=n, n+1,...,k, by using the main tool of the LRF method, which successfully identifies the unknown coefficients and is given by

$$\lim_{t \to t_0} \frac{\mathcal{R}_j(\omega(t))}{(t - t_0)^{j-n}} = 0, \quad j = n, n+1, \dots, k.$$

4. Illustrative examples

This section tests the performance and application of the suggested method for solving ODEs. The results obtained are compared with exact solutions, and various examples are presented to demonstrate the ease and effectiveness of using LRF to find series solutions to ODEs.

Example 4.1. Consider the following homogeneous nonlinear ODEs:

$$\frac{d^5\omega}{dt^5}(t) = \omega^2(t)\, e^{-t}, \tag{4.1}$$

subjected to the following initial conditions:

$$\omega(0) = 1, \quad \omega'(0) = 1, \quad \omega''(0) = 1, \quad \omega^{(3)}(0) = 1, \quad \omega^{(4)}(0) = 1. \tag{4.2}$$

The exact solution of problem (4.1)-(4.2) is $\omega(t) = e^t$ [12]. Considering the LRF technique to create a series solution for problem (4.1)-(4.2), we begin by writing the solution on the following series expansions:

$$\omega(t) = \sum_{j=0}^{\infty} a_j t^j. \tag{4.3}$$

So, we can get the k^{th} approximate solution for problem (4.1)-(4.2) by truncating the series in (4.3) as follows:

$$\omega_k(t) = \sum_{j=0}^{k} a_j t^j.$$

Based on the construction specified in Equations (3.4) and (3.5), the k^{th} approximation of $\omega(t)$ is given by

$$\omega_k(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \sum_{j=5}^{k} a_j t^j. \tag{4.4}$$

More unknown coefficients of the series given in Equation (4.4) can be found by applying the second step of the LRF approach by defining the residual function of equation (4.1) as follows:

$$\mathcal{R}(\omega(t)) = \frac{d^5\omega}{dt^5}(t) - \omega^2(t)\, e^{-t},$$

and the k^{th} residual function as:

$$\mathcal{R}_k(\omega(t)) = \frac{d^5\omega_k}{dt^5}(t) - \omega_k^2(t)\, e^{-t}.$$

To find the unknown coefficient a_5 in (4.4), we substitute the 5^{th} approximation,

$$\omega_5(t) = 1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4+a_5t^5,$$

into the 5^{th} residual function

$$\mathcal{R}_5(\omega(t)) = \frac{d^5\omega_5}{dt^5}(t) - \omega_5^2(t)\, e^{-t},$$

to get

$$\mathcal{R}_5(\omega(t)) = 120a_5 - \left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + a_5t^5\right)^2 e^{-t}.$$

Solving equation $\lim_{t\to 0} \mathcal{R}_5(\omega(t)) = 0$, for a_5, yields $a_5 = \frac{1}{120}$. To find the value of the coefficient a_6, we prepare the 6^{th} approximate solution $\omega_6(t) = \omega_5(t) + a_6t^6$, then substitute it into the 6th residual function, $\mathcal{R}_6(\omega(t))$, to get

$$\mathcal{R}_6(\omega(t)) = 1 + 720a_6t - \left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + a_6t^6\right)^2 e^{-t}.$$

Again, solving equation $\lim_{t\to 0} \frac{\mathcal{R}_6(\omega(t))}{t} = 0$, for a_6, then we get $a_6 = \frac{1}{720}$. Similarly, substituting the 7^{th} approximation, $\omega_7(t) = \omega_6(t) + a_7t^7$, into $\mathcal{R}_7(\omega(t))$ and then solving the equation $\lim_{t\to 0} \frac{\mathcal{R}_4(\omega(t))}{t^2} = 0$for a_7, we have $a_7 = \frac{1}{5040}$. Therefore, the 7^{th} approximate solution of problem (4.1)-(4.2) is given by

$$\omega(t) = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \frac{1}{6!}t^6 + \frac{1}{7!}t^7, \tag{4.5}$$

which is the first eight terms of the expansion of the exact solution $\omega(t) = e^t$ mentioned above [12]. Figure 1 shows the exact and 10^{th} approximate solution of problem (4.1)-(4.2) in the interval $[-8, 8]$. The curves show a full agreement between the exact and the approximate solutions in a wide interval. To evaluate the solution, some numerical data are compiled in Table 1 for the approximate solution given in equation (4.5). Table 1 includes comparisons between the 10^{th} approximate and the exact solutions over the interval $[-2, 2]$ along with the absolute and relative errors.

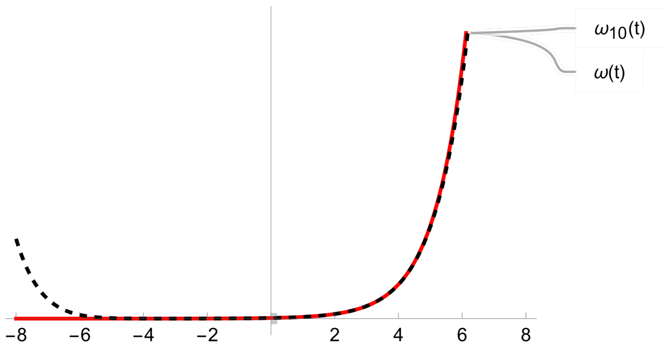

Figure 1: The curves of the exact (solid) and 10^{th} approximate (dotted) solutions of problem (4.1)-(4.2)

Table 1: Numerical data of the solution of problem (4.1)-(4.2), including exact solution, 10^{th} approximate solution, absolute error, and relative error.

t	$\omega(t)$	$\omega_{10}(t)$	Abs. err. (t)	Rel. err. (t)
-2.0	0.135335	0.135379	4.39055×10^{-5}	3.24420×10^{-4}
-1.5	0.223130	0.223132	1.92429×10^{-6}	8.62407×10^{-6}
-1.0	0.367879	0.367879	2.31143×10^{-8}	6.28311×10^{-8}
0.0	1.000000	1.000000	0	0
0.5	1.648720	1.648720	1.27625×10^{-11}	7.74082×10^{-12}
1.0	2.718280	2.718280	2.73127×10^{-8}	1.00478×10^{-8}
1.5	4.481690	4.481690	2.47279×10^{-6}	5.51753×10^{-7}
2.0	7.389060	7.388999	6.13899×10^{-5}	8.30822×10^{-6}

Example 4.2. Consider the following nonhomogeneous linear ODE:

$$\frac{d^2\omega}{dt^2}(t) + 2\frac{d\omega}{dt}(t) + 2\omega(t) = 5\sin t + 5\cos t, \tag{4.6}$$

subjected to the following initial conditions:

$$\omega(0) = 1, \quad \omega'(0) = 2. \tag{4.7}$$

We can find analytically that the exact solution is

$$\omega(t) = 2e^{-t}\cos t + e^{-t}\sin t + 3\sin t - \cos t.$$

Firstly, let us indicate the inhomogeneous term in equation (4.6) as follows:

$$h(t) = 5\sin t + 5\cos t.$$

Following the LRF technique, we assume that the solution of problem (4.6)-(4.7) has the following series expansion:

$$\omega(t) = \sum_{j=0}^{\infty} a_j t^j. \tag{4.8}$$

On the other hand, we write the function $h(t)$ in the following expansion:

$$h(t) = 5\sum_{j=0}^{\infty} (-1)^j \left(\frac{t^{2j+1}}{(2j+1)!} + \frac{t^{2j}}{(2j)!} \right).$$

So, we can get an approximate solution of problem (4.6)-(4.7) by truncating the series in (4.8) as follows:

$$\omega_k\left(t\right)=\sum_{j=0}^{k}a_jt^j.$$

Based on the initial conditions specified in equation (4.7), the k^{th} truncated series of $\omega\left(t\right)$ is given as:

$$\omega_k\left(t\right)=1+2t+\sum_{j=2}^{k}a_jt^j. \tag{4.9}$$

As mentioned in Section 3, to determine the unknown coefficients of the series in equation (4.9), define the residual function of equation (4.6) and the k^{th} residual function as follows:

$$\begin{aligned}\mathcal{R}\left(\omega\left(t\right)\right)&=\frac{d^2\omega}{dt^2}\left(t\right)+2\frac{d\omega}{dt}\left(t\right)+2\omega\left(t\right)-h\left(t\right),\\ \mathcal{R}_k\left(\omega\left(t\right)\right)&=\frac{d^2\omega_k}{dt^2}\left(t\right)+2\frac{d\omega_k}{dt}\left(t\right)+2\omega_k\left(t\right)-h_k\left(t\right),\end{aligned}$$

where

$$h_k\left(t\right)=5\left(t+1\right)-5\left(\frac{t^3}{3!}+\frac{t^2}{2!}\right)+5\sum_{j=2}^{k}\left(-1\right)^j\left(\frac{t^{2j+1}}{\left(2j+1\right)!}+\frac{t^{2j}}{\left(2j\right)!}\right).$$

To find the unknown coefficient a_2 in (4.9), we substitute the 2^{nd} approximation $\omega_2\left(t\right)=1+2t+a_2t^2$, into the 2^{nd} residual function $\mathcal{R}_2\left(\omega\left(t\right)\right)$, to get

$$\mathcal{R}_2\left(\omega\left(t\right)\right)=2a_2+2\left(2+2a_2t\right)+2\left(1+2t+a_2t^2\right)-\left(5\left(t+1\right)-5\left(\frac{t^3}{3!}+\frac{t^2}{2!}\right)+5\left(\frac{t^5}{5!}+\frac{t^4}{4!}\right)\right).$$

Employing the result in Theorem 2.3, considering k= 1, we have the equation $\lim_{t\to0}\mathcal{R}_2\left(\omega\left(t\right)\right)=0$. Solving for a_2, we get $a_2=-\frac{1}{2}$. In the same approach, to determine the value of the coefficient a_3, we need to replace the 3^{rd} approximation $\omega_3(t)=\omega_2(t)+a_3t^3$ in the 3^{rd} residual function, $\mathcal{R}_3\left(\omega\left(t\right)\right)$, to get

$$\begin{aligned}\mathcal{R}_3\left(\omega\left(t\right)\right)=&-1+6a_3t+2\left(2-t+3a_3t^2\right)+2\left(1+2t-\frac{1}{2}t^2+a_3t^3\right)\\ &-5\left[\left(t+1\right)-\left(\frac{t^3}{3!}+\frac{t^2}{2!}\right)+\left(\frac{t^5}{5!}+\frac{t^4}{4!}\right)-\left(\frac{t^7}{7!}+\frac{t^6}{6!}\right)\right].\end{aligned}$$

The algebraic equation $\lim_{t\to0}\frac{\mathcal{R}_3(\omega(t))}{t}=0$ gives $a_3=\frac{1}{2}$. Similarly, substituting the 4^{th} approximation, $\omega_4(t)=\omega_3(t)+a_4t^4$, into $\frac{\mathcal{R}_4(\omega(t))}{t^2}$, gives the following result:

$$\begin{aligned}\frac{\mathcal{R}_4(\omega(t))}{t^2}=&\,12a_4+2\left(\frac{3}{2}+4a_4t\right)+2\left(-\frac{1}{2}+\frac{1}{2}t+a_4t^2\right)\\ &+5\left[\left(\frac{t}{3!}+\frac{1}{2!}\right)-\left(\frac{t^3}{5!}+\frac{t^2}{4!}\right)+\left(\frac{t^5}{7!}+\frac{t^4}{6!}\right)-\left(\frac{t^7}{9!}+\frac{t^6}{8!}\right)\right].\end{aligned}$$

Solving the equation $\lim_{t\to0}\frac{\mathcal{R}_4(\omega(t))}{t^2}=0$ for a_4 gives the indicated coefficient $a_4=-\frac{3}{8}$. This process can be repeated multiple times to increase the order of the approximate solution. The required coefficients are summarized in Table 2.

Table 2: Other coefficients of the 10^{th} approximation of $\omega(t)$ for problem (4.6)-(4.7).

k	a_k	k	a_k
5	$\frac{7}{120}$	8	$\frac{31}{40320}$
6	$\frac{1}{80}$	9	$-\frac{13}{362880}$
7	$-\frac{3}{560}$	10	$-\frac{31}{3628800}$

Therefore, the 10^{th} approximate solution of problem (4.6)-(4.7) is given by

$$\omega_{10}(t) = 1+2t-\frac{1}{2}t^2+\frac{1}{2}t^3-\frac{3}{8}t^4+\frac{7}{120}t^5+\frac{1}{80}t^6-\frac{3}{560}t^7+\frac{31}{40320}t^8-\frac{13}{362880}t^9-\frac{31}{3628800}t^{10},$$

which is the first eleven terms of the expansion of the exact solution [28]. Figure 2 shows the exact and 10^{th} approximate solution of problem (4.6)-(4.7) in the interval [-4,4]. Figure 2 and the numerical data in Table 3 confirm the high agreement between the exact and approximate solutions. The table includes the exact solution, the 10^{th} approximate solution, as well as the absolute error and the relative error of the approximation.

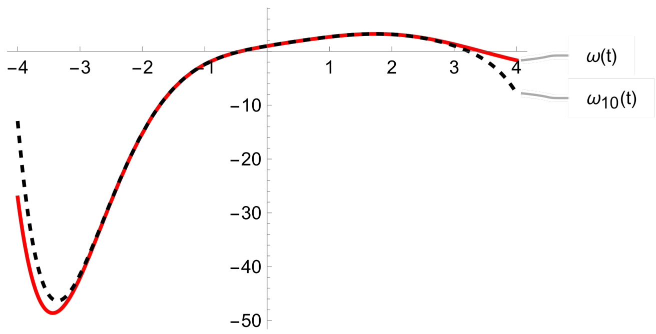

Figure 2: The curves of the exact (solid) and 10^{th} approximate (dotted) solutions of problem (4.6)-(4.7).

Table 3: Numerical data of the solution of problem (4.6)-(4.7), including exact solution, 10^{th} approximate solution, absolute error, and relative error.

t	$\omega(t)$	$\omega_{10}(t)$	Abs. err. (t)	Rel. err. (t)
-2.0	-15.1805	-15.1745	5.92715×10^{-3}	3.90446×10^{-4}
-1.5	-6.89964	-6.89940	2.37984×10^{-4}	3.44922×10^{-5}
-1.0	-2.41468	-2.41468	2.60812×10^{-6}	1.08011×10^{-6}
-0.5	-0.21252	-0.21252	1.20458×10^{-9}	5.66809×10^{-9}
0.0	1.00000	1.00000	0	0
0.5	1.91604	1.91604	1.07327×10^{-9}	5.60147×10^{-10}
1.0	2.69120	2.69120	2.07250×10^{-6}	7.70103×10^{-7}
1.5	3.17589	3.17572	1.68996×10^{-4}	5.32121×10^{-5}
2.0	3.15446	3.15069	3.77261×10^{-3}	1.19596×10^{-3}

Example 4.3. Consider the following nonhomogeneous nonlinear singular ODE:

$$\frac{d^2\omega}{dt^2}(t)+\frac{2}{t}\frac{d\omega}{dt}(t)-t\sin(\omega(t))=e^{2t},\ t\in(0,2), \tag{4.10}$$

subjected to the initial conditions:

$$\omega(0) = \pi, \quad \omega'(0) = 0. \tag{4.11}$$

Firstly, expand the inhomogeneous term in equation (4.10) as follows:

$$h(t) = e^{2t} = \sum_{j=0}^{\infty} \frac{(2t)^j}{j!}.$$

Following the LRF approach to construct an analytical series solution for problem (4.10)-(4.11), we write the solution as the following series expansions:

$$\omega(t) = \sum_{j=0}^{\infty} a_j t^j. \tag{4.12}$$

So, the k^{th} approximate solution of problem (4.10)-(4.11) can be obtained by truncating the series in (4.12) as:

$$\omega_k(t) = \sum_{j=0}^{k} a_j t^j.$$

Based on the initial conditions specified in equation (4.11), the k^{th} approximation of $\omega(t)$ will be as follows:

$$\omega_k(t) = \pi + \sum_{j=2}^{k} a_j t^j. \tag{4.13}$$

To identify the additional coefficients of the series given in equation (4.13), we write the residual function of equation (4.10) as:

$$\mathcal{R}(\omega(t)) = \frac{d^2\omega}{dt^2}(t) + \frac{2}{t}\frac{d\omega}{dt}(t) - t\sin(\omega(t)) - h(t),$$

and the k^{th} residual function as:

$$\mathcal{R}_k(\omega(t)) = \frac{\partial^2 \omega_k}{\partial t^2}(t) + \frac{2}{t}\frac{\partial \omega_k}{\partial t}(t) - t\sin(\omega_k(t)) - h_k(t),$$

where

$$h_k(t) = \sum_{j=0}^{k} \frac{(2t)^j}{j!}.$$

Substitute the 2^{nd} approximation $\omega_2(t) = \pi + a_2 t^2$, into the 2^{nd} residual function, $\mathcal{R}_2(\omega(t))$, we get

$$\mathcal{R}_2(\omega(t)) = 6a_2 - t\sin\left(\pi + a_2 t^2\right) - 1 - 2t - 2t^2.$$

Solving the equation $\lim_{t\to 0} \mathcal{R}_2(\omega(t)) = 0$, for a_2, we obtain $a_2 = \frac{1}{6}$. To determine the value of the coefficient a_3, we need to find the 3^{rd} approximation $\omega_3(t) = \omega_2(t) + a_3 t^3$, then substituting it into the 3^{rd} residual function, $\mathcal{R}_3(\omega(t))$, we get

$$\mathcal{R}_3(\omega(t)) = 12a_3 t - t\sin\left(\pi + \frac{1}{6}t^2 + a_3 t^3\right) - 2t - 2t^2 - \frac{4}{3}t^3.$$

Solving the equation $\lim_{t\to 0} \frac{\mathcal{R}_3(\omega(t))}{t} = 0$, for a_3, we get $a_3 = \frac{1}{6}$. Similarly, to set the coefficient a_4, we substitute $\omega_4(t) = \omega_3(t) + a_4 t^4$ in $\mathcal{R}_4(\omega(t))$ to obtain

$$\mathcal{R}_4(\omega(t)) = 20a_4 t^2 - t\sin\left(\pi + \frac{1}{6}t^2 + \frac{1}{6}t^3 + a_4 t^4\right) - 2t^2 - \frac{4}{3}t^3 - \frac{2}{3}t^4.$$

Using the fact $\lim_{t\to 0} \frac{\mathcal{R}_4(\omega(t))}{t^2} = 0$ supplies the coefficient $a_4 = \frac{1}{10}$. The process may be repeated several times to find an approximate solution. Table 4 outlines the necessary coefficients.

Table 4: Other coefficients of the 10^{th} approximation of $\omega(t)$ for problem (4.10)-(4.11).

k	a_k	k	a_k
5	$\frac{7}{180}$	8	$\frac{1}{1440}$
6	$\frac{1}{84}$	9	$\frac{647}{4082400}$
7	$\frac{1}{336}$	10	$\frac{43}{831600}$

Therefore, the solution of problem (4.10)-(4.11) will be in the form

$$\omega(t) = \pi + \frac{1}{6}t^2 + \frac{1}{6}t^3 + \frac{1}{10}t^4 + \frac{7}{180}t^5 + \frac{1}{84}t^6 + \frac{1}{336}t^7 + \frac{1}{1440}t^8 + \frac{647}{4082400}t^9 + \frac{43}{831600}t^{10} + \cdots .$$

Figure 3 shows the 10^{th} and 20^{th} approximate LRf solutions to problem (4.10)-(4.11). Table 5 provides numerical data for the two approximate solutions plotted in Figure 3, as well as the two types of errors: consecutive and relative errors.

The consecutive and relative errors are defined, respectively, as follows:

$$\text{Con. err}(t) = |\omega_{20}(t) - \omega_{10}(t)|, \quad \text{Rel. err}(t) = \left|\frac{\omega_{20}(t) - \omega_{10}(t)}{\omega_{20}(t)}\right|.$$

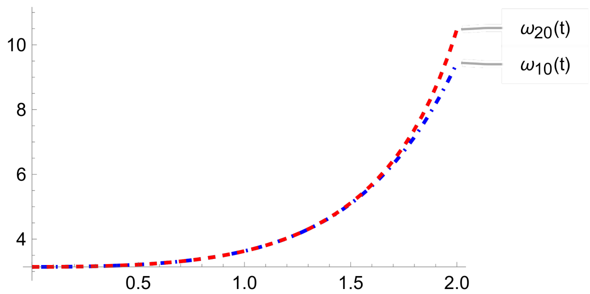

Figure 3: The 10^{th} and 20^{th} approximate LRF solution of $\omega(t)$ of problem (4.10)-(4.11).

Table 5: Numerical data of the solution of problem (4.10)-(4.11), including 10th, and 20^{th} approximate solutions, consecutive error, and relative error.

t	$\omega_{20}(t)$	$\omega_{10}(t)$	Con. err. (t)	Rel. err. (t)
0.0	3.14159	3.14159	0	0
0.2	3.14977	3.14977	8.14016×10^{-13}	2.58437×10^{-13}
0.4	3.18194	3.18194	2.02295×10^{-9}	6.35761×10^{-10}
0.6	3.25423	3.25423	2.17499×10^{-7}	6.68358×10^{-8}
0.8	3.39119	3.39118	6.56717×10^{-6}	1.93654×10^{-6}
1.0	3.62970	3.62960	1.00053×10^{-4}	2.75651×10^{-5}
1.2	4.02505	4.02406	9.96985×10^{-4}	2.47695×10^{-4}
1.4	4.66238	4.65494	7.44276×10^{-3}	1.59634×10^{-3}
1.6	5.68510	5.64009	4.50077×10^{-2}	7.91679×10^{-3}

Example 4.4. Consider the following nonhomogeneous nonlinear ODE:

$$\frac{d\omega}{dt}(t) - 1 = \omega^2(t), \tag{4.14}$$

subjected to the initial condition:

$$\omega(0) = 0. \tag{4.15}$$

We can check that the exact solution is $\omega(t) = \tan(t)$. According to the LRF technique, we write the solution in the following series expansion:

$$\omega(t) = \sum_{j=0}^{\infty} a_j t^j.$$

So, the k^{th} approximate solution of problem (4.14)-(4.15) can be considered as:

$$\omega_k(t) = \sum_{j=1}^{k} a_j t^j.$$

The k^{th} residual function of equation (4.14) is as follows:

$$\mathcal{R}_k\left(\omega(t)\right) = \frac{d\omega_k}{dt}(t) - 1 - \omega_k^2(t).$$

Substituting the 1^{st} approximation $\omega_1(t) = a_1 t$ into the 1^{st} residual function, $\mathcal{R}_1\left(\omega\left(t\right)\right)$, to get

$$\mathcal{R}_1\left(\omega(t)\right) = a_1 - 1 - \left(a_1 t\right)^2.$$

So, solving the equation $\lim_{t\to 0} \mathcal{R}_1\left(\omega\left(t\right)\right) = 0$, for a_1, gives $a_1 = 1$. The second coefficient a_2 is obtained by substituting $\omega_2(t) = \omega_1(t) + a_2 t^2$ into the 2^{nd} residual function, $\mathcal{R}_2\left(\omega\left(t\right)\right)$, as:

$$\mathcal{R}_2\left(\omega(t)\right) = 2a_2 t - \left(t + a_2 t^2\right)^2.$$

The equation $\lim_{t\to 0} \frac{\mathcal{R}_2(\omega(t))}{t} = 0$ yields that $a_2 = 0$. In the same way, set the 3^{rd} approximation, $\omega_3(t) = \omega_2(t) + a_3 t^3$, into $\mathcal{R}_3\left(\omega\left(t\right)\right)$. The equation $\lim_{t\to 0} \frac{\mathcal{R}_3(\omega(t))}{t^2} = 0$ gives $a_3 = \frac{1}{3}$. Similarly, we can obtain $a_4 = 0$. The process can be repeated multiple times to increase the order of the approximate solution. The needed coefficients for the 10^{th} approximation are summarized in Table 6.

Table 6: Other coefficients of the 10^{th} approximation of $\omega(t)$ for problem (4.14)-(4.15).

k	a_k	k	a_k
5	$\frac{2}{15}$	8	0
6	0	9	$\frac{62}{2835}$
7	$\frac{17}{315}$	10	0

Therefore, the solution of problem (4.14)-(4.15) will be in the form:

$$\omega(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \cdots,$$

which is the expansion of the exact solution $\omega\left(t\right) = \tan\left(t\right)$ [26]. Figure 4 shows the exact and 10^{th} approximate solution in the interval $[-1.5, 1.5]$ of problem (4.14)-(4.15). The figure clearly illustrates full agreement between the two solutions. In Table 7, we compare approximate and exact solutions numerically. Table 7 presents the results of this comparison, including the absolute and relative errors of the approximate solution. The results are obtained within the range of $[-0.8, 0.8]$, which indicates a strong level of approximation.

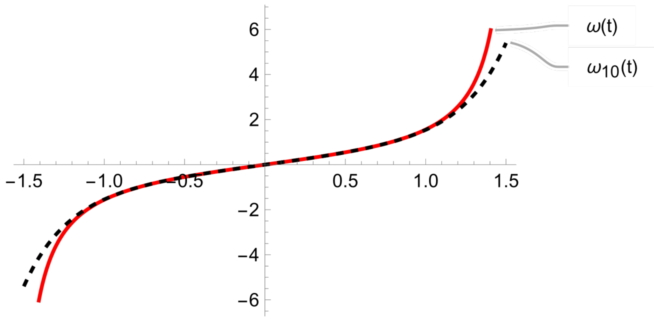

Figure 4: The curves of the exact (solid) and and 10^{th} approximate (dotted) solutions of problem (4.14)-(4.15).

Table 7: Numerical data of the solution of problem (4.14)-(4.15), including 10th, and 20^{th} approximate solutions, consecutive error, and relative error.

t	$\omega(t)$	$\omega_{10}(t)$	Abs. err. (t)	Rel. err. (t)
−0.8	−1.029640	−1.028610	1.02799×10^{-3}	9.98396×10^{-4}
−0.6	−0.684137	−0.684099	3.76486×10^{-5}	5.50308×10^{-5}
−0.4	−0.422793	−0.422793	3.97529×10^{-7}	9.40244×10^{-7}
−0.2	−0.202710	−0.202710	1.84510×10^{-10}	9.10218×10^{-10}
0.0	0	0	0	0
0.2	0.202710	0.202710	1.84510×10^{-10}	9.10218×10^{-10}
0.4	0.422793	0.422793	3.97529×10^{-7}	9.40244×10^{-7}
0.6	0.684137	0.684099	3.76486×10^{-5}	5.50308×10^{-5}
0.8	1.029640	1.028610	1.02799×10^{-3}	9.98396×10^{-4}

5. Conclusion

The article aims to employ an efficient technique called the LRF method to get analytical series solutions of linear and nonlinear ODEs. Utilizing a key property of the residual function, a straightforward and effective algorithm is provided, which can be easily implemented using software packages such as Mathematica. The main objective is to evaluate the effectiveness of this suggested method in finding series solutions for challenging ODEs. For the general class of linear ODEs with analytic coefficients, this method effectively provides exact PS solutions. Furthermore, the LRF method achieves high-accuracy approximations for nonlinear ODEs. Certainly, the new method can be used to solve other sets of equations that were not investigated in earlier research, including partial differential equations, integral equations, and integrodifferential equations. Furthermore, since the LRF method has not been applied to solve differential equations with boundary conditions, all these topics will be investigated in the next research.

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