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Novel analytical solutions to the $(3+1)$-dimensional heat model using Lie symmetry method

M. Usman[a,b], Akhtar Hussain[a,*], Ahmed M. Zidan[c], Jorge Herrera[d]

[a] Department of Mathematics and Statistics, The University of Lahore, Lahore 54590, Pakistan.

[b] College of Electrical and Mechanical Engineering (CEME), National University of Sciences and Technology (NUST), H-12 Islamabad 44000, Pakistan.

[c] Department of Mathematics, College of Science, King Khalid University, Abha 61413, Saudi Arabia.

[d] Facultad de Ciencias Naturales e Ingenieria, Universidad de Bogota Jorge Tadeo Lozano, Bogota 110311, Colombia.

Abstract

This study focuses on applying the Lie symmetry method, to obtain exact solutions in multiple forms for the $(3+1)$-dimensional (3D) heat model This equation is a well-known model frequently used to describe numerous complex physical phenomena. Initially, the geometric vector fields for the 3D heat-type equation are determined. Using Lie symmetry reduction, we report a wide array of exact analytical solutions that encompass trigonometric and hyperbolic solitons, Lambert functions, polynomials, exponential and inverse functions, hypergeometric forms, Bessel functions, logarithmic forms, rational forms, and solitary wave solutions. These solutions include many rational forms that uncover intricate physical structures that have not been previously reported. The solutions presented in this study are original and significantly distinct from previous findings. They have significant potential for application in diverse fields, including fiber optics, plasma physics, soliton dynamics, fluid dynamics, mathematical physics, and other applied sciences. The findings demonstrated that these mathematical techniques are efficient, straightforward, and robust, making them suitable for solving other types of nonlinear equations.

Keywords: Heat-type equation, Lie symmetry method, invariant solutions, geometric vector fields, Lie algebra, conservation laws.

2020 MSC: 34C14, 34A26.

1. Introduction

In recent decades, numerous robust nonlinear models have been employed to represent various real-world phenomena across diverse fields, including plasma physics [6, 24, 38, 41], optical fiber technology [21, 25, 43, 44], chemical physics, fluid dynamics, solid-state physics, and acoustics. Given their significance, finding the exact solutions to these equations is of great importance. However, obtaining such solutions is often a challenging task because they are typically achievable only in specific cases. In recent years, considerable progress has been made in deriving exact explicit solutions for nonlinear partial

*Corresponding author

Email addresses: musman.awan112@gmail.com (M. Usman), akhtarhussain21@sms.edu.pk (Akhtar Hussain), ahmoahmed@kku.edu.sa (Ahmed M. Zidan), jorgea.herrerac@utadeo.edu.co (Jorge Herrera)

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differential equations (PDEs), leading to the development of various methodologies [4, 7–10, 12, 15, 22, 31–33, 37]. Among these methods, the Lie group method proved to be an effective, dependable, and remarkable approach for solving nonlinear PDEs. This method, which is based on the symmetries of differential equations, has inspired extensive advancements in the search for precise solutions. The Lie group approach [5, 16, 27, 28, 30] offers a foundational framework [11, 26] to identify the symmetries of nonlinear complex systems. One of its notable features is the ability to reduce the dimensionality of an equation by one through a single application. Consequently, the Lie group method is regarded as a standard, efficient, and highly versatile tool within group-theoretical techniques that enables the resolution of a wide range of equations [14, 23, 39, 42].

The nonlinear heat equation is a classical topic in the theory of PDEs and remains a fertile ground for new insights and scientific development, even after decades of research. Its relevance extends beyond mathematics owing to its wide range of practical applications. The equation models various phenomena including heat and mass transfer, combustion, explosions, filtration, chemical reactions, and biological processes. In this article, we undertake a detailed analysis of this equation

$$\nu_t - \lambda \operatorname{div}(g(\nu) \operatorname{grad} \nu) = 0, \tag{1.1}$$

where $\nu := \nu(t, \mathbf{x}) : \mathbb{R} \times \mathbb{R}^{\gamma+1} \to \mathbb{R}$, $\gamma \in \{0, 1, 2\}$. In the literature, it is known as the nonlinear heat equation or filtration equation [34], and is also recognized as a porous medium equation [36]. Depending on the physical context, the function $\nu \geqslant 0$ describes either the density of the medium or the temperature, whereas $g(\nu) \geqslant 0$ corresponds to either the filtration coefficient or the heat conductivity coefficient. The authors investigated the derivation of invariant solutions for the nonlinear heat equation (1.1), which exhibits the characteristics of a heat wave and is understood as a configuration comprising two hypersurfaces: $\nu(t, \mathbf{x}) \geqslant 0$, which represents the perturbed state, and $\nu(t, \mathbf{x}) \equiv 0$, which corresponds to the unperturbed or background state. These regions are smoothly connected along a sufficiently regular hypersurface $\Gamma(t, \mathbf{x}) = 0$ that defines the propagation front of the heat wave.

The study of heat wave propagation in a perfectly cold medium at a finite velocity, along with the first examples of heat-wave-type solutions, was introduced by Zeldovich and Kompaneets [40]. Subsequently, Barenblatt [3] obtained comparable results for filtration processes by investigating various self-similar solutions. Sidorov et al. conducted further research on the solvability of boundary value problems for heat equations with heat-wave-type solutions within the context of analytic functions. A novel approach to boundary conditions, including the Sakharov problem on heat-wave initiation, was formulated in [20] using the characteristic series method. These results were established for both one-dimensional formulations (see [19, 35]), including cases involving moving boundaries [18] and multidimensional scenarios [17]. The three dimensional case of Eq (1.1) is as follows

$$\nu_t - \lambda(g(\nu)\nu_x)_x - \lambda(g(\nu)\nu_y)_y - \lambda(g(\nu)\nu_z)_z = 0,$$

or

$$\nu_t - \lambda(g'(\nu)\nu_x^2 + g(\nu)\nu_{xx} + g'(\nu)\nu_y^2 + g(\nu)\nu_{yy} + g'(\nu)\nu_z^2 + g(\nu)\nu_{zz}) = 0. \tag{1.2}$$

The primary goal of this study was to derive new explicit solitary wave solutions for 3D heat-type equations using the Lie symmetry method [13, 29]. By applying this method, a diverse range of exact analytical solutions are obtained, including trigonometric and hyperbolic solitons, Lambert functions, polynomials, exponentials, inverse functions, hypergeometric forms, Bessel functions, logarithmic forms, rational forms, and solitary wave solutions. These solutions include many rational-form solutions that reveal intricate physical structures that have not been previously reported. The solutions presented herein are entirely novel and significantly distinct from earlier results. They have important applications in various fields such as fiber optics, plasma physics, soliton dynamics, fluid dynamics, mathematical physics, and other areas of applied science. Additionally, the dynamic behavior of these solutions was demonstrated both graphically and physically using 3D visualizations and contour plots. This approach highlights the rich and varied physical phenomena encapsulated by the obtained solutions.

This paper is organized into three main sections. Section 1 provides an overview of the historical background and motivation behind the study. Section 2 introduces the fundamental concepts of Lie symmetry analysis, focusing on its application to heat-type equation. Section 3 explores the derivation of infinitesimals and the application of Lie group transformation methods to derive exact solutions. Section 4 focuses on the conserved quantities of the examined model. Finally, Section 5 presents an analysis of the findings and outlines the potential future research directions.

2. Symmetry classification of the heat Eq. (1.2)

In this section, we focus on the symmetry classification based on various forms of the heat conductivity coefficient $g(\nu)$ for the 3D heat equation (1.2). By the point form of Lie group, we have ([13, 29])

$$\begin{aligned}
&\bar{\nu} \approx \nu + \beta\rho(\nu,x,y,z,t) + O(\beta^2), \qquad \bar{t} \approx t + \beta\Phi^4(\nu,x,y,z,t) + O(\beta^2),\\
&\bar{x} \approx x + \beta\Phi^1(\nu,x,y,z,t) + O(\beta^2), \quad \bar{y} \approx y + \beta\Phi^2(\nu,x,y,z,t) + O(\beta^2),\\
&\bar{z} \approx z + \beta\Phi^3(\nu,x,y,z,t) + O(\beta^2).
\end{aligned}$$

The vector field form for the 3D heat type equation (1.2) is given by [29]

$$\mathcal{X} = \psi^x\frac{\partial}{\partial x} + \psi^y\frac{\partial}{\partial y} + \psi^z\frac{\partial}{\partial z} + \psi^t\frac{\partial}{\partial t} + \eta^\nu\frac{\partial}{\partial \nu}.$$

The objective is to determine the infinitesimal functions $\psi^x, \psi^y, \psi^z, \psi^t$, and η^ν, while the operator $\mathcal{X}$ satisfies invariance criterion ([29])

$$\mathcal{X}^{[2]}(\nu_t - \lambda(g'(\nu){\nu_x}^2 + g(\nu)\nu_{xx} + g'(\nu){\nu_y}^2 + g(\nu)\nu_{yy} + g'(\nu){\nu_z}^2 + g(\nu)\nu_{zz}))|_{(1.2)} = 0, \tag{2.1}$$

where $\mathcal{X}^{[2]}$ denotes the second prolongation of $\mathcal{X}$.

Case A: $\mathbf{g(\nu)}$ is arbitrary function, and $\mathcal{X}_1 = \frac{\partial}{\partial t}$, $\mathcal{X}_2 = \frac{\partial}{\partial x}$, $\mathcal{X}_3 = \frac{\partial}{\partial y}$, $\mathcal{X}_4 = \frac{\partial}{\partial z}$, $\mathcal{X}_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, $\mathcal{X}_6 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$, $\mathcal{X}_7 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$, $\mathcal{X}_8 = \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}$.

Case B: $\mathbf{g(\nu) = b\nu}$, where b is a constant. In this case, the solution of (2.1) provides that, system (1.2) admits the generators $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4, \mathcal{X}_5, \mathcal{X}_6, \mathcal{X}_7$ and the following

$$\mathcal{X}_8 = t\frac{\partial}{\partial t} - \nu\frac{\partial}{\partial \nu}, \quad \mathcal{X}_9 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2\nu\frac{\partial}{\partial \nu}.$$

Case C: $\mathbf{g(\nu) = b\nu^2}$, where b is a constant. In this case, the solution of (2.1) provides that, system (1.2) admits the generators $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4, \mathcal{X}_5, \mathcal{X}_6, \mathcal{X}_7$ and the following

$$\mathcal{X}_8 = t\frac{\partial}{\partial t} - \frac{\nu}{2}\frac{\partial}{\partial \nu}, \quad \mathcal{X}_9 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \nu\frac{\partial}{\partial \nu}.$$

3. Group invariant solutions and their dynamics

3.1. Symmetry reductions for Case 1: $g(\nu)$ is arbitrary

(i) $\mathcal{X}_2 = \frac{\partial}{\partial x}$. The symmetry generator $\frac{\partial}{\partial x}$ provides

$$\nu(x,y,z,t) = p(j,k,l), \quad j = y, \quad k = z, \quad l = t. \tag{3.1}$$

Eq (1.2) with the variables (3.1) becomes

$$p_l - \lambda g'{p_j}^2 - \lambda g p_{jj} - \lambda g'{p_k}^2 - \lambda g p_{kk} = 0. \tag{3.2}$$

For equation (3.2), the infinitesimal operators are outlined by

$$\mathcal{Y}_1 = \frac{\partial}{\partial l}, \ \mathcal{Y}_2 = \frac{\partial}{\partial j}, \ \mathcal{Y}_3 = \frac{\partial}{\partial k}, \ \mathcal{Y}_4 = k\frac{\partial}{\partial j} - j\frac{\partial}{\partial k}, \ \mathcal{Y}_5 = \frac{j}{2}\frac{\partial}{\partial j} + \frac{k}{2}\frac{\partial}{\partial k} + l\frac{\partial}{\partial l}.$$

$\mathcal{Y}_2 = \frac{\partial}{\partial j}$ gives $p(j,k,l) = q(\sigma,\tau)$, where $\sigma = k, \tau = l$, which ensures the reduction of (3.2) into the equation

$$q_\tau - \lambda g' {q_\sigma}^2 - \lambda g q_{\sigma\sigma} = 0. \tag{3.3}$$

For further reduction of (3.3), we use the symmetries stated below

$$\mathcal{Z}_1 = \frac{\partial}{\partial \sigma}, \ \mathcal{Z}_2 = \frac{\partial}{\partial \tau}, \ \mathcal{Z}_3 = \frac{\sigma}{2}\frac{\partial}{\partial \sigma} + \tau\frac{\partial}{\partial \tau}.$$

$\mathcal{Z}_2 = \frac{\partial}{\partial \tau}$ yields $q(\sigma,\tau) = \theta(s)$, where $s = \sigma$. Using the invariants of $\frac{\partial}{\partial \tau}$, we acquire the ODE

$$g' {\theta_s}^2 + g\theta_{ss} = 0.$$

If $g = \theta e^{b\theta}$, then

$$\theta(s) = \frac{\text{LambertW}\left(b^2(c_1 s + c_2)e^{-1}\right) + 1}{b},$$

and the solution of (1.2) is

$$v(x,y,z,t) = \frac{\text{LambertW}\left(b^2(c_1 z + c_2)e^{-1}\right) + 1}{b}.$$

(ii) $\mathcal{X}_3 = \frac{\partial}{\partial y}$. The symmetry generator $\frac{\partial}{\partial y}$ gives

$$v(x,y,z,t) = p(j,k,l), \quad j = x, \quad k = z, \quad l = t. \tag{3.4}$$

Eq. (1.2) with the variables (3.4) becomes

$$p_l - \lambda g' {p_j}^2 - \lambda g p_{jj} - \lambda g' {p_k}^2 - \lambda g p_{kk} = 0. \tag{3.5}$$

For equation (3.5), the infinitesimal operators are

$$\mathcal{Y}_1 = \frac{\partial}{\partial l}, \ \mathcal{Y}_2 = \frac{\partial}{\partial j}, \ \mathcal{Y}_3 = \frac{\partial}{\partial k}, \ \mathcal{Y}_4 = k\frac{\partial}{\partial j} - j\frac{\partial}{\partial k}, \ \mathcal{Y}_5 = \frac{j}{2}\frac{\partial}{\partial j} + \frac{k}{2}\frac{\partial}{\partial k} + l\frac{\partial}{\partial l}.$$

$\mathcal{Y}_2 = \frac{\partial}{\partial j}$ gives $p(j,k,l) = q(\sigma,\tau)$, where $\sigma = k, \tau = l$, which ensures the reduction of (3.5) into the equation

$$q_\tau - \lambda g' {q_\sigma}^2 - \lambda g q_{\sigma\sigma} = 0. \tag{3.6}$$

For further reduction of (3.6), we use the symmetries stated below

$$\mathcal{Z}_1 = \frac{\partial}{\partial \sigma}, \ \mathcal{Z}_2 = \frac{\partial}{\partial \tau}, \ \mathcal{Z}_3 = \frac{\sigma}{2}\frac{\partial}{\partial \sigma} + \tau\frac{\partial}{\partial \tau}.$$

$\mathcal{Z}_2 = \frac{\partial}{\partial \tau}$ yields $q(\sigma,\tau) = \theta(s)$, where $s = \sigma$. Using the invariants of $\frac{\partial}{\partial \tau}$, we acquire the ODE

$$g' {\theta_s}^2 + g\theta_{ss} = 0.$$

If $g = \theta^2 \ln(b\theta)$, then

$$\theta(s) = \frac{e^{\frac{\text{LambertW}\left(-9c_1(c_2+s)e^{-1}b^3\right)}{3} + \frac{1}{3}}}{b},$$

and the solution of (1.2) is

$$v(x,y,z,t) = \frac{e^{\frac{\text{LambertW}\left(-9c_1(c_2+z)e^{-1}b^3\right)}{3} + \frac{1}{3}}}{b}.$$

3.2. Symmetry reductions for Case 2: $g(v) = bv$

(i) $\mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. The symmetry generator $\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ gives

$$v(x,y,z,t) = p(j,k,l), \quad j = z, \quad k = t - x, \quad l = y - x. \tag{3.7}$$

Eq. (1.2) with the variables (3.7) becomes

$$-\lambda b p_k^2 - 2p_k p_l b\lambda - 2b\lambda p_l^2 - 2b\lambda p p_{ll} - 2b\lambda p p_{kl} - p p_{jj} b\lambda - p p_{kk} b\lambda - p_j^2 b\lambda + p_k = 0. \tag{3.8}$$

For equation (3.8), the infinitesimal operators are

$$\mathcal{Y}_1 = \frac{\partial}{\partial j},\ \mathcal{Y}_2 = \frac{\partial}{\partial k},\ \mathcal{Y}_3 = \frac{\partial}{\partial l},\ \mathcal{Y}_4 = j\frac{\partial}{\partial j} + k\frac{\partial}{\partial k} + l\frac{\partial}{\partial l} + p\frac{\partial}{\partial p},\ \mathcal{Y}_5 = l\frac{\partial}{\partial j} - j\frac{\partial}{\partial k} - 2j\frac{\partial}{\partial l}.$$

The linear combination $\mathcal{Y}_1 + \mathcal{Y}_3 = \frac{\partial}{\partial j} + \frac{\partial}{\partial l}$ gives $p(j,k,l) = q(\sigma,\tau)$, where $\sigma = k, \tau = l - j$, which ensures the reduction of (3.8) into the equation

$$-\lambda b q_\sigma^2 - 2q_\sigma q_\tau b\lambda - 3\lambda b q_\tau^2 - 3\lambda b q q_{\tau\tau} - \lambda b q q_{\sigma\sigma} - 2\lambda b q q_{\sigma\tau} + q_\sigma = 0. \tag{3.9}$$

For further reduction of (3.9), we use the symmetries stated below

$$\mathcal{Z}_1 = \frac{\partial}{\partial \sigma},\ \mathcal{Z}_2 = \frac{\partial}{\partial \tau},\ \mathcal{Z}_3 = \sigma\frac{\partial}{\partial \sigma} + \tau\frac{\partial}{\partial \tau} + q\frac{\partial}{\partial q}.$$

$\mathcal{Z}_1 + \mathcal{Z}_2 = \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau}$ yields $q(\sigma,\tau) = \theta(s)$, where $s = \tau - \sigma$. Using the invariants of $\frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau}$, we acquire the ODE

$$-2\lambda b\theta_s^2 - 2\lambda b\theta\theta_{ss} - \theta_s = 0,$$

and its solution is

$$\theta(s) = e^{-\frac{4\,\mathrm{LambertW}\left(\frac{e^{-1} e^{-\frac{c_2}{4c_1 b^2\lambda^2}} e^{-\frac{s}{4c_1 b^2\lambda^2}}}{2b\lambda c_1}\right) c_1 b^2\lambda^2 + 4c_1 b^2\lambda^2 + c_2 + s}{4c_1 b^2\lambda^2}} + 2c_1 b\lambda.$$

Therefore, the solution of (1.2) is

$$v(x,y,z,t) = e^{-\frac{4\,\mathrm{LambertW}\left(\frac{e^{-1} e^{-\frac{c_2}{4c_1 b^2\lambda^2}} e^{-\frac{y-z-t}{4c_1 b^2\lambda^2}}}{2b\lambda c_1}\right) c_1 b^2\lambda^2 + 4c_1 b^2\lambda^2 + c_2 + y - z - t}{4c_1 b^2\lambda^2}} + 2c_1 b\lambda.$$

(ii) $\mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_4 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial z}$.

The symmetry generator $\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial z}$ provides

$$v(x,y,z,t) = p(j,k,l), \quad j = y, \quad k = t - x, \quad l = z - x. \tag{3.10}$$

Eq. (1.2) with the variables (3.10) becomes

$$-\lambda b p_k^2 - 2p_k p_l b\lambda - 2b\lambda p_l^2 - 2b\lambda p p_{ll} - 2b\lambda p p_{kl} - p p_{jj} b\lambda - p p_{kk} b\lambda - p_j^2 b\lambda + p_k = 0. \tag{3.11}$$

For equation (3.11), the infinitesimal operators are

$$\mathcal{Y}_1 = \frac{\partial}{\partial j},\ \mathcal{Y}_2 = \frac{\partial}{\partial k},\ \mathcal{Y}_3 = \frac{\partial}{\partial l},\ \mathcal{Y}_4 = j\frac{\partial}{\partial j} + k\frac{\partial}{\partial k} + l\frac{\partial}{\partial l} + p\frac{\partial}{\partial p},\ \mathcal{Y}_5 = l\frac{\partial}{\partial j} - j\frac{\partial}{\partial k} - 2j\frac{\partial}{\partial l}.$$

$\mathcal{Y}_5 = l\frac{\partial}{\partial j} - j\frac{\partial}{\partial k} - 2j\frac{\partial}{\partial l}$ gives $p(j,k,l) = q(\sigma,\tau)$, where $\sigma = -\frac{l}{2} + k, \tau = 2j^2 + l^2$, which ensures the reduction of (1.2) into the equation

$$-b\lambda q q_{\sigma\sigma} - 16 q q_{\tau,\tau} b\lambda\tau - \lambda b q_\sigma^2 + 2q_\sigma - 16b\lambda q_\tau \left(\tau q_\tau + q\right) = 0. \tag{3.12}$$

For further reduction of (3.12), we use the symmetries stated below

$$\mathcal{Z}_1 = \frac{\partial}{\partial \sigma},\ \mathcal{Z}_2 = \sigma\frac{\partial}{\partial \sigma} + 2\tau\frac{\partial}{\partial \tau} + q\frac{\partial}{\partial q}.$$

$\mathcal{Z}_1 = \frac{\partial}{\partial \sigma}$ yields $q(\sigma,\tau) = \theta(s)$, where $s = \tau$. Using the invariants of $\frac{\partial}{\partial \sigma}$, we acquire the ODE

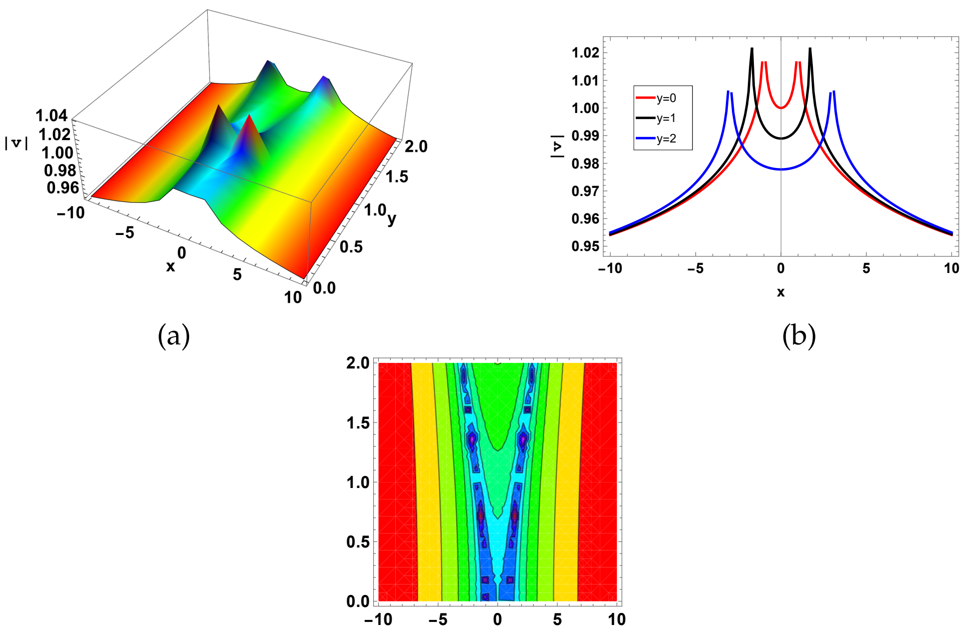
$$-16b\lambda\left(s\theta\theta_{ss} + \theta_s^2 s + \theta_s\theta\right) = 0,$$

and its solution is

$$\theta(s) = \pm\sqrt{2c_1\ln(s) + 2c_2}.$$

For (1.2), the solution takes the form

$$v(x,y,z,t) = \pm\sqrt{2c_1\ln(2y^2 + (z-x)^2) + 2c_2}. \tag{3.13}$$



(a) (b) (c)

Figure 1: Visual analysis of the solution (3.13): (a) three-dimensional dynamics; (b) two-dimensional dynamics for the values $y = 1.0$, $y = 2.0$, $y = 3.0$; and (c) contour dynamics.

(iii) $\mathcal{X}_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. The symmetry generator $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ provides

$$v(x,y,z,t) = p(j,k,l),\quad j = t,\quad k = z,\quad l = x^2 + y^2. \tag{3.14}$$

Eq. (1.2) with the variables (3.14) becomes

$$-4p_l^2 b\lambda l - 4p p_{ll} b\lambda l - 4b\lambda p p_l - p p_{kk} b\lambda - \lambda b p_k^2 + p_j = 0. \tag{3.15}$$

For equation (3.15), the infinitesimal operators are

$$\mathcal{Y}_1 = \frac{\partial}{\partial j},\ \mathcal{Y}_2 = \frac{\partial}{\partial k},\ \mathcal{Y}_3 = j\frac{\partial}{\partial j} - p\frac{\partial}{\partial p},\ \mathcal{Y}_4 = k\frac{\partial}{\partial k} + 2l\frac{\partial}{\partial l} + 2p\frac{\partial}{\partial p}.$$

$\mathcal{Y}_1 = \frac{\partial}{\partial j}$ gives $p(j,k,l) = q(\sigma,\tau)$, where $\sigma = k, \tau = l$, which ensures the reduction of (3.15) into the equation

$$-4b\left(qq_{\tau\tau}\tau + q_\tau^2\tau + qq_\tau + \frac{qq_{\sigma\sigma}}{4} + \frac{q_\sigma^2}{4}\right)\lambda = 0. \tag{3.16}$$

For further reduction of (3.16), we use the symmetries stated below

$$\mathcal{Z}_1 = \frac{\partial}{\partial\sigma},\ \mathcal{Z}_2 = q\frac{\partial}{\partial q},\ \mathcal{Z}_3 = \sigma\frac{\partial}{\partial\sigma} + 2\tau\frac{\partial}{\partial\tau},\ \mathcal{Z}_4 = (\tau - \sigma^2)\frac{\partial}{\partial\sigma} - 4\sigma\tau\frac{\partial}{\partial\tau} + \frac{\sigma q}{2}\frac{\partial}{\partial q}.$$

$\mathcal{Z}_1 = \frac{\partial}{\partial\sigma}$ gives $q(\sigma,\tau) = \theta(s)$, where $s = \tau$. Using the invariants of $\frac{\partial}{\partial\sigma}$, we acquire the ODE

$$-4b\lambda\left(s\theta\theta_{ss} + \theta_s^2 s + \theta_s\theta\right) = 0,$$

and its solution is

$$\theta(s) = \pm\sqrt{2c_1\ln(s) + 2c_2}.$$

For (1.2), the solution takes the form

$$v(x,y,z,t) = \pm\sqrt{2c_1\ln(x^2+y^2) + 2c_2}. \tag{3.17}$$

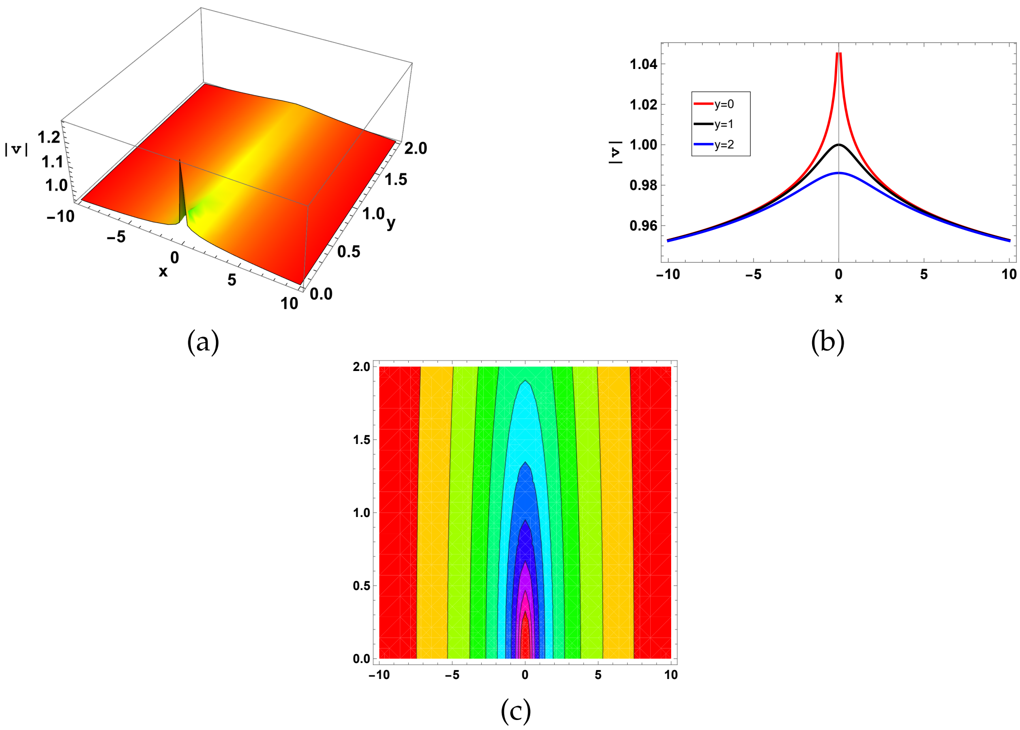


Figure 2: Visual analysis of the solution (3.17): (a) three-dimensional dynamics; (b) two-dimensional dynamics for the values $y = 1.0$, $y = 2.0$, $y = 3.0$; and (c) contour dynamics.

(iv) $\mathcal{X}_9 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2v\frac{\partial}{\partial v}$. The symmetry generator $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2v\frac{\partial}{\partial v}$ provides

$$v(x,y,z,t) = x^2 p(j,k,l),\quad j = t,\quad k = \frac{y}{x},\quad l = \frac{z}{x}. \tag{3.18}$$

Eq. (1.2) with the variables (3.18) becomes

$$\begin{aligned}&-p_{kk}pb\lambda k^2 - 2p_{kl}pb\lambda kl - b\lambda pp_{ll}l^2 - \lambda bp_k^2k^2 - 2\lambda bp_k kp_l l - p_l^2 b\lambda l^2\\&+6\lambda bp_k kp + 6b\lambda pp_l l - pp_{kk}b\lambda - 6\lambda bp^2 - b\lambda pp_{ll} - \lambda bp_k^2 - b\lambda p_l^2 + p_j = 0.\end{aligned} \tag{3.19}$$

For equation (3.19), the infinitesimal operators are $\mathcal{Y}_1 = \frac{\partial}{\partial j}$, $\mathcal{Y}_2 = -l\frac{\partial}{\partial k} + k\frac{\partial}{\partial l}$, $\mathcal{Y}_3 = j\frac{\partial}{\partial j} - p\frac{\partial}{\partial p}$, $\mathcal{Y}_4 =$

$(k^2+1)\frac{\partial}{\partial k}+kl\frac{\partial}{\partial l}+2pk\frac{\partial}{\partial p}$, $\mathcal{Y}_5=\frac{kl}{2}\frac{\partial}{\partial k}+\left(\frac{l^2}{2}+\frac{1}{2}\right)\frac{\partial}{\partial l}+pl\frac{\partial}{\partial p}$. $\mathcal{Y}_2=-l\frac{\partial}{\partial k}+k\frac{\partial}{\partial l}$ gives $p(j,k,l)=q(\sigma,\tau)$, where $\sigma=j, \tau=k^2+l^2$, which ensures the reduction of (3.19) into the equation

$$-4b\tau\lambda q\left(\tau+1\right)q_{\tau\tau}-4b\tau\lambda\left(\tau+1\right)q_{\tau}^2+10b\left(\tau-\frac{2}{5}\right)q\lambda q_{\tau}-6\lambda bq^2+q_{\sigma}=0. \tag{3.20}$$

For further reduction of (3.20), we use the symmetries stated below

$$\mathcal{Z}_1=\frac{\partial}{\partial\tau},\quad \mathcal{Z}_2=\sigma\frac{\partial}{\partial\sigma}-q\frac{\partial}{\partial q}.$$

$\mathcal{Z}_1=\frac{\partial}{\partial\tau}$ yields $q(\sigma,\tau)=\theta(s)$, where $s=\tau$. Using the invariants of $\frac{\partial}{\partial\tau}$, we acquire the ODE

$$-4b\lambda\left(s\left(s+1\right)\theta\theta_{ss}+\left(s^2+s\right)\theta_s^2+\theta\left(-\frac{5s}{2}+1\right)\theta_s+\frac{3\theta^2}{2}\right)=0,$$

and its solution is

$$\begin{aligned}\theta(s)=\pm\frac{1}{24\sqrt{s+1}}\Big(&-6\sqrt{s+1}(9\sqrt{s+1}\ln(\sqrt{s+1}-1)c_2s^2-9\sqrt{s+1}\ln(\sqrt{s+1}+1)c_2s^2\\&-72\sqrt{s+1}\ln(\sqrt{s+1}-1)c_2s+72\sqrt{s+1}\ln(\sqrt{s+1}+1)c_2s+3c_1s^2\sqrt{s+1}\\&+24\sqrt{s+1}\ln(\sqrt{s+1}-1)c_2-24\sqrt{s+1}\ln(\sqrt{s+1}+1)c_2-24\sqrt{s+1}c_1s-110c_2s^2\\&+8c_1\sqrt{s+1}-10c_2s+100c_2)\Big)^{\frac{1}{2}}.\end{aligned}$$

For equation (1.2), the solution takes the form

$$\begin{aligned}v(x,y,z,t)=\pm\frac{x^2}{24\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}}\Bigg(&-6\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}\,\Bigg(9\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}\\&\times\ln\left(\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}-1\right)c_2\left(\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right)^2-9\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}\\&\times\ln\left(\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}+1\right)c_2\left(\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right)^2-72\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}\\&\times\ln\left(\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}-1\right)c_2\left(\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right)+72\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}\\&\times\ln\left(\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}+1\right)c_2\left(\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right)+3c_1\left(\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right)^2\\&\times\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}+24\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}\ln\left(\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}-1\right)c_2\\&-24\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}\ln\left(\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}+1\right)c_2-24\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}\\&\times c_1\left(\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right)-110c_2\left(\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right)^2+8c_1\sqrt{\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2+1}\\&-10c_2\left(\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right)+100c_2\Bigg)\Bigg)^{\frac{1}{2}}.\end{aligned}$$

(v) $\mathcal{X}_8=t\frac{\partial}{\partial t}-v\frac{\partial}{\partial v}$. The symmetry generator $t\frac{\partial}{\partial t}-v\frac{\partial}{\partial v}$ provides

$$v(x,y,z,t)=\frac{p(j,k,l)}{t},\quad j=x,k=y,l=z. \tag{3.21}$$

Eq. (1.2) with the variables (3.21) becomes

$$-p-\lambda b p_j^2-\lambda b p p_{jj}-\lambda b p_k^2-\lambda b p p_{kk}-\lambda b p_l^2-\lambda b p p_{ll}=0. \tag{3.22}$$

For equation (3.22), the infinitesimal operators are $\mathcal{Y}_1=\frac{\partial}{\partial j}$, $\mathcal{Y}_2=\frac{\partial}{\partial k}$, $\mathcal{Y}_3=\frac{\partial}{\partial l}$, $\mathcal{Y}_4=k\frac{\partial}{\partial j}-j\frac{\partial}{\partial k}$, $\mathcal{Y}_5=l\frac{\partial}{\partial j}-j\frac{\partial}{\partial l}$, $\mathcal{Y}_6=j\frac{\partial}{\partial j}+k\frac{\partial}{\partial k}+l\frac{\partial}{\partial l}+2p\frac{\partial}{\partial p}$. $\mathcal{Y}_3=\frac{\partial}{\partial l}$ gives $p(j,k,l)=q(\sigma,\tau)$, where $\sigma=j,\tau=k$, which ensures the reduction of (3.22) into the equation

$$-2\lambda b q^2-\lambda b q q_{\tau\tau}-\lambda b q q_{\sigma\sigma}-\lambda b q_\sigma^2-\lambda b q_\tau^2=0. \tag{3.23}$$

For further reduction of (3.23), we use the symmetries stated below

$$\mathcal{Z}_1=\frac{\partial}{\partial\sigma},\ \mathcal{Z}_2=\frac{\partial}{\partial\tau},\ \mathcal{Z}_3=q\frac{\partial}{\partial q},\ \mathcal{Z}_4=\tau\frac{\partial}{\partial\sigma}-\sigma\frac{\partial}{\partial\tau}.$$

$\mathcal{Z}_1=\frac{\partial}{\partial\sigma}$ gives $q(\sigma,\tau)=\theta(s)$, where $s=\tau$. Using the invariants of $\frac{\partial}{\partial\sigma}$, we acquire the ODE

$$-2\theta^2-\theta\theta_{ss}-\theta_s^2=0,$$

and its solution is

$$\theta(s)=\pm\sqrt{-c_1\sin(2s)+c_2\cos(2s)}.$$

For (1.2), the solution takes the form

$$v(x,y,z,t)=\pm\frac{\sqrt{-c_1\sin(2y)+c_2\cos(2y)}}{t}. \tag{3.24}$$

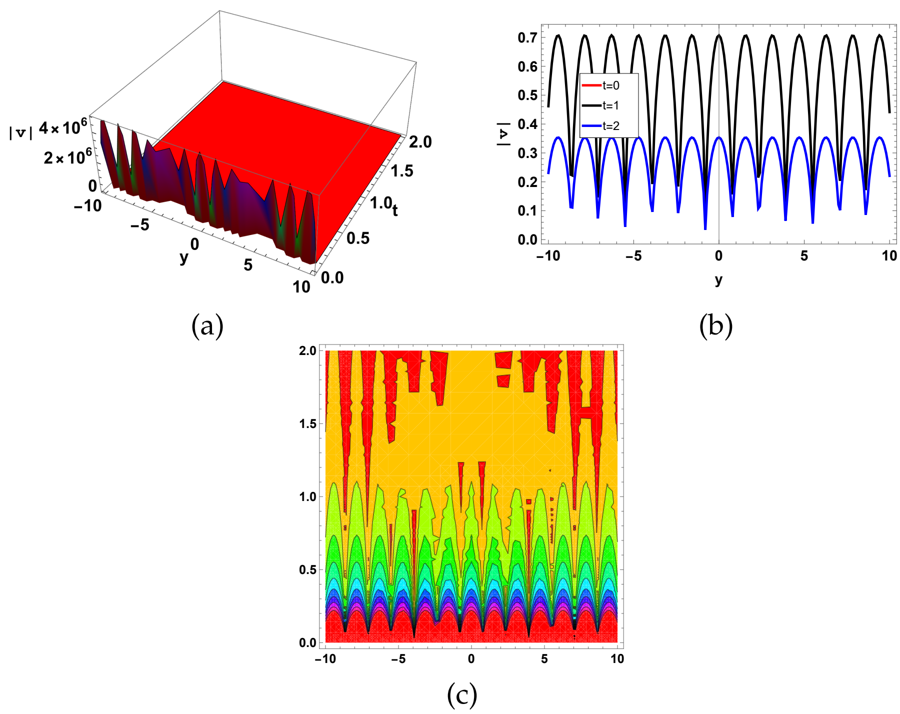


Figure 3: Visual analysis of the solution (3.24): (a) three-dimensional dynamics; (b) two-dimensional dynamics for the values $t=1.0, t=2.0, t=3.0$; and (c) contour dynamics.

(vi) $\mathcal{X}_4+\mathcal{X}_8=\frac{\partial}{\partial z}+t\frac{\partial}{\partial t}-v\frac{\partial}{\partial v}$. The symmetry generator $\frac{\partial}{\partial z}+t\frac{\partial}{\partial t}-v\frac{\partial}{\partial v}$ provides

$$v(x,y,z,t)=e^{-z}p(j,k,l),\quad j=x,k=y,l=te^{-z}. \tag{3.25}$$

Eq. (1.2) with the variables (3.25) becomes

$$-l^2\left(\lambda b p p_{jj}+\lambda b p p_{kk}+p p_{ll} b\lambda l^2+p_l^2 b\lambda l^2+(5bl\lambda p-1)\,p_l+2b\left(p^2+\frac{p_j^2}{2}+\frac{p_k^2}{2}\right)\lambda\right)=0. \tag{3.26}$$

For equation (3.26), the infinitesimal operators are $\mathcal{Y}_1=\frac{\partial}{\partial j}$, $\mathcal{Y}_2=\frac{\partial}{\partial k}$, $\mathcal{Y}_3=k\frac{\partial}{\partial j}-j\frac{\partial}{\partial k}$, $\mathcal{Y}_4=l\frac{\partial}{\partial l}-p\frac{\partial}{\partial p}$. $\mathcal{Y}_3=k\frac{\partial}{\partial j}-j\frac{\partial}{\partial k}$ gives $p(j,k,l)=q(\sigma,\tau)$, where $\sigma=l,\tau=j^2+k^2$, which ensures the reduction of (3.26) into the equation

$$-2\lambda b q^2-\lambda b q q_{\tau\tau}-\lambda b q q_{\sigma\sigma}-\lambda b q_\sigma^2-\lambda b q_\tau^2=0. \tag{3.27}$$

For further reduction of (3.27), we use the symmetries stated below

$$\mathcal{Z}_1=\frac{\partial}{\partial\sigma},\ \mathcal{Z}_2=\frac{\partial}{\partial\tau},\ \mathcal{Z}_3=q\frac{\partial}{\partial q},\ \mathcal{Z}_4=\tau\frac{\partial}{\partial\sigma}-\sigma\frac{\partial}{\partial\tau}.$$

$\mathcal{Z}_1+\mathcal{Z}_3=\frac{\partial}{\partial\sigma}+q\frac{\partial}{\partial q}$ yields $q(\sigma,\tau)=e^{\sigma}\theta(s)$, where $s=\tau$. Using the invariants of $\frac{\partial}{\partial\sigma}+q\frac{\partial}{\partial q}$, we acquire the ODE

$$-4\lambda b\theta^2-\lambda b\theta\theta_{ss}-\lambda b\theta_s^2=0,$$

and its solution is

$$\theta(s)=\pm\frac{\sqrt{-2\sqrt{2}\,c_1\sin\left(2\sqrt{2}\,s\right)+2\cos\left(2\sqrt{2}\,s\right)\sqrt{2}\,c_2}}{2}.$$

For (1.2), the solution takes the form

$$v(x,y,z,t)=\pm\frac{e^{te^{-z}-z}\sqrt{-2\sqrt{2}\,c_1\sin\left(2\sqrt{2}\,(x^2+y^2)\right)+2\cos\left(2\sqrt{2}\,(x^2+y^2)\right)\sqrt{2}\,c_2}}{2}. \tag{3.28}$$

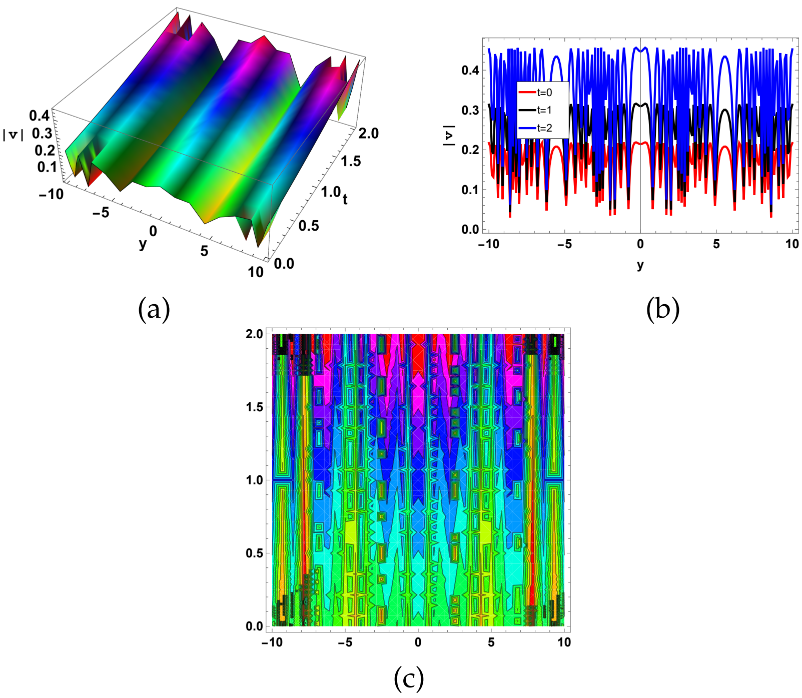


Figure 4: Visual analysis of the solution (3.28): (a) three-dimensional dynamics; (b) two-dimensional dynamics for the values $t=1.0, t=2.0, t=3.0$; and (c) contour dynamics.

3.3. Symmetry reductions for Case 3: $g(v) = bv^2$

(i) $\mathcal{X}_1 + \mathcal{X}_4 = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}$. The symmetry generator $\frac{\partial}{\partial t} + \frac{\partial}{\partial z}$ provides

$$v(x,y,z,t) = p(j,k,l), \quad j = x, \quad k = y, \quad l = t - z. \tag{3.29}$$

Eq. (1.2) with the variables (3.29) becomes

$$-b\lambda p^2 p_{ll} - b\lambda p^2 p_{j,j} - b\lambda p^2 p_{kk} - 2pp_l^2 b\lambda + p_l - 2b\lambda p\left(p_j^2 + p_k^2\right) = 0. \tag{3.30}$$

For equation (3.30), the infinitesimal operators are $\mathcal{Y}_1 = \frac{\partial}{\partial j}$, $\mathcal{Y}_2 = \frac{\partial}{\partial k}$, $\mathcal{Y}_3 = \frac{\partial}{\partial l}$, $\mathcal{Y}_4 = k\frac{\partial}{\partial j} - j\frac{\partial}{\partial k}$, $\mathcal{Y}_5 = j\frac{\partial}{\partial j} + k\frac{\partial}{\partial k} + l\frac{\partial}{\partial l} + \frac{p}{2}\frac{\partial}{\partial p}$. $\mathcal{Y}_1 + \mathcal{Y}_3 = \frac{\partial}{\partial j} + \frac{\partial}{\partial l}$ gives $p(j,k,l) = q(\sigma,\tau)$, where $\sigma = k, \tau = l - j$, which ensures the reduction of (3.30) into the equation

$$-b\lambda q^2 q_{\sigma\sigma} - 2b\lambda q^2 q_{\tau\tau} - 2qq_\sigma^2 b\lambda - 4qq_\tau^2 b\lambda + q_\tau = 0. \tag{3.31}$$

For further reduction of (3.31) we use the symmetries stated below

$$\mathcal{Z}_1 = \frac{\partial}{\partial\sigma}, \ \mathcal{Z}_2 = \frac{\partial}{\partial\tau}, \ \mathcal{Z}_3 = \sigma\frac{\partial}{\partial\sigma} + \tau\frac{\partial}{\partial\tau} + \frac{q}{2}\frac{\partial}{\partial q}.$$

$\mathcal{Z}_1 + \mathcal{Z}_2 = \frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}$ gives $q(\sigma,\tau) = \theta(s)$, where $s = \tau - \sigma$. Using the invariants of $\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}$, we acquire the ODE

$$-3b\lambda\theta^2\theta_{ss} - 6\theta\theta_s^2 b\lambda + \theta_s = 0,$$

and its solution is

$$\theta(s) = \text{InverseFunction}\left[3b\lambda\left(9b^2\lambda^2{c_1}^2\ln(3b\lambda c_1 + \#1) - 6b\lambda c_1(3b\lambda c_1 + \#1) + \frac{1}{2}(3b\lambda c_1 + \#1)^2\right)\&\right][s + c_2].$$

For (1.2), the solution takes the form

$$\begin{aligned} v(x,y,z,t) = \text{InverseFunction}&\left[3b\lambda\left(9b^2\lambda^2{c_1}^2\ln(3b\lambda c_1 + \#1)\right.\right.\\ &\left.\left.- 6b\lambda c_1(3b\lambda c_1 + \#1) + \frac{1}{2}(3b\lambda c_1 + \#1)^2\right)\&\right][t - z - x - y + c_2]. \end{aligned}$$

(ii) $\mathcal{X}_1 + \mathcal{X}_5 = \frac{\partial}{\partial t} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. The symmetry generator $\frac{\partial}{\partial t} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ provides

$$v(x,y,z,t) = p(j,k,l), \quad j = z, \quad k = -\arctan\left(\frac{x}{y}\right) + t, \quad l = x^2 + y^2. \tag{3.32}$$

Eq. (1.2) with the variables (3.32) becomes

$$-b\lambda p^2 p_{kk} - 4p^2 p_{ll} b\lambda\, l^2 - p^2 p_{jj} b\lambda l - 2pp_k^2 b\lambda + p_k l - 8\lambda lb\left(lp_l^2 + \frac{pp_l}{2} + \frac{p_j^2}{4}\right)p = 0. \tag{3.33}$$

For equation (3.33), the infinitesimal operators are $\mathcal{Y}_1 = \frac{\partial}{\partial j}$, $\mathcal{Y}_2 = \frac{\partial}{\partial k}$, $\mathcal{Y}_3 = j\frac{\partial}{\partial j} + 2l\frac{\partial}{\partial l} + p\frac{\partial}{\partial p}$. $\mathcal{Y}_3 = j\frac{\partial}{\partial j} + 2l\frac{\partial}{\partial l} + p\frac{\partial}{\partial p}$ gives $p(j,k,l) = j\ q(\sigma,\tau)$, where $\sigma = k, \tau = \frac{l}{j^2}$, which ensures the reduction of (3.33) into the equation

$$\begin{aligned} &-4\lambda bq^2 q_{\tau\tau}\tau^3 - 8qq_\tau^2 b\,\tau^3\lambda - 4\lambda bq^2 q_{\tau\tau}\tau^2 + 6\lambda bq^2\tau^2 q_\tau \\ &\quad - 8qq_\tau^2 b\,\tau^2\lambda - 2\lambda\tau bq^3 - 4\lambda bq^2\tau q_\tau - b\lambda q^2 q_{\sigma\sigma} - 2qq_\sigma^2 b\lambda + q_\sigma\tau = 0. \end{aligned} \tag{3.34}$$

For further reduction of (3.34), we use the symmetry generator $\mathcal{Z} = \frac{\partial}{\partial\sigma}$. $\mathcal{Z} = \frac{\partial}{\partial\sigma}$ yields $q(\sigma,\tau) = \theta(s)$, where $s = \tau$. Using the invariants of $\frac{\partial}{\partial\sigma}$, we acquire the ODE

$$-4\lambda\left(s\theta\left(s+1\right)\theta_{ss}+\left(2s^2+2s\right)\theta_s^2+\theta\left(-\frac{3s}{2}+1\right)\theta_s+\frac{\theta^2}{2}\right)\theta bs=0,$$

and its solution is

$$\begin{aligned}\theta(s) = \frac{-1}{4}\Bigg(&\frac{1}{\sqrt{s+1}}\Big(18\ln\left(\sqrt{s+1}+1\right)\sqrt{s+1}\,c_2 s-18\ln\left(\sqrt{s+1}-1\right)\sqrt{s+1}\,c_2 s\\&-12\ln\left(\sqrt{s+1}+1\right)\sqrt{s+1}\,c_2+12\ln\left(\sqrt{s+1}-1\right)\sqrt{s+1}\,c_2-6\sqrt{s+1}\,c_1 s-16c_2 s^2\\&+4\sqrt{s+1}\,c_1+28c_2 s+44c_2\Big)\Bigg)^{\frac{1}{3}}\pm\frac{1}{4}\Bigg(\iota\sqrt{3}\Bigg(\frac{1}{\sqrt{s+1}}\Big(18\ln\left(\sqrt{s+1}+1\right)\sqrt{s+1}\,c_2 s\\&-18\ln\left(\sqrt{s+1}-1\right)\sqrt{s+1}\,c_2 s-12\ln\left(\sqrt{s+1}+1\right)\sqrt{s+1}\,c_2\\&+12\ln\left(\sqrt{s+1}-1\right)\sqrt{s+1}\,c_2-6\sqrt{s+1}\,c_1 s-16c_2 s^2+4\sqrt{s+1}\,c_1+28c_2 s+44c_2\Big)\Bigg)^{\frac{1}{3}}\Bigg).\end{aligned}$$

For (1.2), the solution takes the form

$$\begin{aligned}v(x,y,z,t) = \frac{-x}{4}\Bigg(&\frac{1}{\sqrt{\frac{(t-z)}{x^2}+1}}\Bigg(18\ln\left(\sqrt{\frac{(t-z)}{x^2}+1}+1\right)\sqrt{\frac{(t-z)}{x^2}+1}\,c_2\frac{(t-z)}{x^2}\\&-18\ln\left(\sqrt{\frac{(t-z)}{x^2}+1}-1\right)\sqrt{\frac{(t-z)}{x^2}+1}\;c_2\frac{(t-z)}{x^2}-12\ln\left(\sqrt{\frac{(t-z)}{x^2}+1}+1\right)\\&\times\sqrt{\frac{(t-z)}{x^2}+1}\,c_2+12\ln\left(\sqrt{\frac{(t-z)}{x^2}+1}-1\right)\sqrt{\frac{(t-z)}{x^2}+1}\,c_2-6\sqrt{\frac{(t-z)}{x^2}+1}\;c_1\frac{(t-z)}{x^2}\\&-16c_2\left(\frac{(t-z)}{x^2}\right)^2+4\sqrt{\frac{(t-z)}{x^2}+1}\,c_1+28c_2\frac{(t-z)}{x^2}+44c_2\Bigg)\Bigg)^{\frac{1}{3}}\pm\frac{x}{4}\Bigg(\iota\sqrt{3}\Bigg(\frac{1}{\sqrt{\frac{(t-z)}{x^2}+1}}\\&\times\Bigg(18\ln\left(\sqrt{\frac{(t-z)}{x^2}+1}+1\right)\sqrt{\frac{(t-z)}{x^2}+1}\,c_2\frac{(t-z)}{x^2}-18\ln\left(\sqrt{\frac{(t-z)}{x^2}+1}-1\right)\\&\times\sqrt{\frac{(t-z)}{x^2}+1}\,c_2\frac{(t-z)}{x^2}-12\ln\left(\sqrt{\frac{(t-z)}{x^2}+1}+1\right)\sqrt{\frac{(t-z)}{x^2}+1}\,c_2\\&+12\ln\left(\sqrt{\frac{(t-z)}{x^2}+1}-1\right)\sqrt{\frac{(t-z)}{x^2}+1}\,c_2-6\sqrt{\frac{(t-z)}{x^2}+1}\,c_1\frac{(t-z)}{x^2}-16c_2\left(\frac{(t-z)}{x^2}\right)^2\\&+4\sqrt{\frac{(t-z)}{x^2}+1}\,c_1+28c_2\frac{(t-z)}{x^2}+44c_2\Bigg)\Bigg)^{\frac{1}{3}}\Bigg).\end{aligned}$$

(iii) $\mathcal{X}_1+\mathcal{X}_8 = \frac{\partial}{\partial t}+t\frac{\partial}{\partial t}-\frac{v}{2}\frac{\partial}{\partial v}$. The symmetry generator $\frac{\partial}{\partial t}+t\frac{\partial}{\partial t}-\frac{v}{2}\frac{\partial}{\partial v}$ provides

$$v(x,y,z,t) = \frac{p(j,k,l)}{\sqrt{1+t}},\quad j=x,\quad k=y,\quad l=z. \tag{3.35}$$

Eq. (1.2) with the variables (3.35) becomes

$$-2p\left(\lambda bpp_{jj}+\lambda bpp_{kk}+\lambda bpp_{ll}+2\lambda bp_j^2+2\lambda bp_k^2+2\lambda bp_l^2+\frac{1}{2}\right)=0. \tag{3.36}$$

For equation (3.36), the infinitesimal operators are $\mathcal{Y}_1 = \frac{\partial}{\partial j}$, $\mathcal{Y}_2 = \frac{\partial}{\partial k}$, $\mathcal{Y}_3 = \frac{\partial}{\partial l}$, $\mathcal{Y}_4 = k\frac{\partial}{\partial j} - j\frac{\partial}{\partial k}$, $\mathcal{Y}_5 = l\frac{\partial}{\partial j} - j\frac{\partial}{\partial l}$, $\mathcal{Y}_6 = l\frac{\partial}{\partial k} - k\frac{\partial}{\partial l}$, $\mathcal{Y}_7 = j\frac{\partial}{\partial j} + k\frac{\partial}{\partial k} + l\frac{\partial}{\partial l} + p\frac{\partial}{\partial p}$. $\mathcal{Y}_2 = \frac{\partial}{\partial k}$ gives $p(j,k,l) = q(\sigma,\tau)$, where $\sigma = j, \tau = l$, which ensures the reduction of (3.36) into the equation

$$-2q\left(b\lambda q q_{\sigma\sigma} + \lambda b q q_{\tau\tau} + 2q_\sigma^2 b\lambda + 2b\lambda q_\tau^2 + \frac{1}{2}\right) = 0. \tag{3.37}$$

For further reduction of (3.37), we use the symmetries stated below

$$\mathcal{Z}_1 = \frac{\partial}{\partial\sigma},\ \mathcal{Z}_2 = \frac{\partial}{\partial\tau},\ \mathcal{Z}_3 = \tau\frac{\partial}{\partial\sigma} - \sigma\frac{\partial}{\partial\tau},\ \mathcal{Z}_4 = \sigma\frac{\partial}{\partial\sigma} + \tau\frac{\partial}{\partial\tau} + q\frac{\partial}{\partial q}.$$

$\mathcal{Z}_1 + \mathcal{Z}_2 = \frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}$ yields $q(\sigma,\tau) = \theta(s)$, where $s = \tau - \sigma$. Using the invariants of $\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}$, we acquire the ODE

$$-4\theta(\theta\theta_{ss} b\lambda + 2\theta_s^2 b\lambda + \frac{1}{4}) = 0.$$

Since $\theta(s) \neq 0$, therefore $\theta\theta_{ss} b\lambda + 2\theta_s^2 b\lambda + \frac{1}{4} = 0$, and its solution is

$$\begin{aligned}\theta(s) = \text{InverseFunction}\Big[\pm \Big(\Big(2\sqrt{2}\sqrt{b}\sqrt{\lambda}\,\text{Hypergeometric2F1}\Big[\frac{1}{2}, \frac{3}{4}, \frac{7}{4},\\ e^{-16b\lambda c_1 \#1^4}\Big]\#1^3\sqrt{e^{-16b\lambda c_1 \#1^4}}\Big)\Big/\big(3\sqrt{e^{16b\lambda c_1 \#1^4}}\big)\Big)\&\Big][s + c_2].\end{aligned}$$

For (1.2), the solution takes the form

$$\begin{aligned}v(x,y,z,t) = \frac{1}{\sqrt{1+t}}\Big(\text{InverseFunction}\Big[\pm \Big(\Big(2\sqrt{2}\sqrt{b}\sqrt{\lambda}\,\text{Hypergeometric2F1}\Big[\frac{1}{2}, \frac{3}{4}, \frac{7}{4},\\ e^{-16b\lambda c_1 \#1^4}\Big]\#1^3\sqrt{e^{-16b\lambda c_1 \#1^4}}\Big)\Big/\big(3\sqrt{e^{16b\lambda c_1 \#1^4}}\big)\Big)\&\Big][z - x + c_2]\Big).\end{aligned}$$

(iv) $\mathcal{X}_4 + \mathcal{X}_6 = \frac{\partial}{\partial z} + z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$. The symmetry generator $\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$ provides

$$v(x,y,z,t) = p(j,k,l),\quad j = t,\quad k = y,\quad l = x^2 + z^2 - 2x. \tag{3.38}$$

Eq. (1.2) with the variables (3.38) becomes

$$-4b\lambda p^2 (l+1) p_{ll} - 8b\lambda p (l+1) p_l^2 - 4b\lambda p^2 p_l - \lambda b p_{kk} p^2 - 2\lambda b p p_k^2 + p_j = 0. \tag{3.39}$$

For equation (3.39), the infinitesimal operators are $\mathcal{Y}_1 = \frac{\partial}{\partial j}$, $\mathcal{Y}_2 = \frac{\partial}{\partial k}$, $\mathcal{Y}_3 = j\frac{\partial}{\partial j} - \frac{p}{2}\frac{\partial}{\partial p}$, $\mathcal{Y}_4 = k\frac{\partial}{\partial k} + (2 + 2l)\frac{\partial}{\partial l} + p\frac{\partial}{\partial p}$. $\mathcal{Y}_1 = \frac{\partial}{\partial j}$ gives $p(j,k,l) = q(\sigma,\tau)$, where $\sigma = k, \tau = l$, which ensures the reduction of (3.39) into the equation

$$-4b\lambda q\left(q(\tau+1) q_{\tau\tau} + \frac{q q_{\sigma\sigma}}{4} + (2\tau+2) q_\tau^2 + q q_\tau + \frac{q_\sigma^2}{2}\right) = 0. \tag{3.40}$$

For further reduction of (3.40), we use the symmetries stated below

$$\mathcal{Z}_1 = \frac{\partial}{\partial\sigma},\ \mathcal{Z}_2 = q\frac{\partial}{\partial q},\ \mathcal{Z}_3 = \sigma\frac{\partial}{\partial\sigma} + (2\tau+2)\frac{\partial}{\partial\tau},\ \mathcal{Z}_4 = (\tau - \sigma^2)\frac{\partial}{\partial\sigma} - 4\sigma(\tau+1)\frac{\partial}{\partial\tau} + \frac{\sigma q}{3}\frac{\partial}{\partial q}.$$

$\mathcal{Z}_1 + \mathcal{Z}_2 = \frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}$ yields $q(\sigma,\tau) = e^\sigma\theta(s)$, where $s = \tau$. Using the invariants of $\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}$, we acquire the ODE

$$\theta(-4\theta b\lambda\theta_{ss} s - 4\theta b\lambda\theta_{ss} - 8b\lambda\theta_s^2 s - 8b\lambda\theta_s^2 - 4b\lambda\theta\theta_s - 3\theta^2 b\lambda) = 0.$$

Since $\theta(s) \neq 0$, therefore

$$-4\theta b\lambda\theta_{ss}s - 4\theta b\lambda\theta_{ss} - 8b\lambda\theta_s^2 s - 8b\lambda\theta_s^2 - 4b\lambda\theta\theta_s - 3\theta^2 b\lambda = 0,$$

and its solution is

$$\begin{aligned}\theta(s) = &-\frac{\left(-6\,\mathrm{BesselI}(0,3\sqrt{-s-1})\,\mathrm{csgn}(s+1)\,c_2 + 6\,\mathrm{BesselK}(0,3\sqrt{-s-1})\,\mathrm{csgn}(s+1)\,c_1\right)^{\frac{1}{3}}}{2}\\ &\pm \frac{\iota}{2}\sqrt{3}\left(-6\,\mathrm{BesssellI}(0,3\sqrt{-s-1})\,\mathrm{csgn}(s+1)\,c_2 + 6\,\mathrm{BesselK}(0,3\sqrt{-s-1})\,\mathrm{csgn}(s+1)\,c_1\right)^{\frac{1}{3}}.\end{aligned}$$

For (1.2), the solution takes the form

$$\begin{aligned}v(x,y,z,t) = &-\frac{e^y}{2}\left((6c_1 - c_2)\,\mathrm{BesselI}(0,3\sqrt{-(x^2+z^2-2x)-1})\,\mathrm{csgn}\big((x^2+z^2-2x)+1\big)\right)^{\frac{1}{3}}\\ &\pm \frac{\iota}{2}\sqrt{3}e^y\left((6c_1 - c_2)\,\mathrm{BesselI}(0,3\sqrt{-(x^2+z^2-2x)-1})\,\mathrm{csgn}\big((x^2+z^2-2x)+1\big)\right)^{\frac{1}{3}}.\end{aligned} \tag{3.41}$$

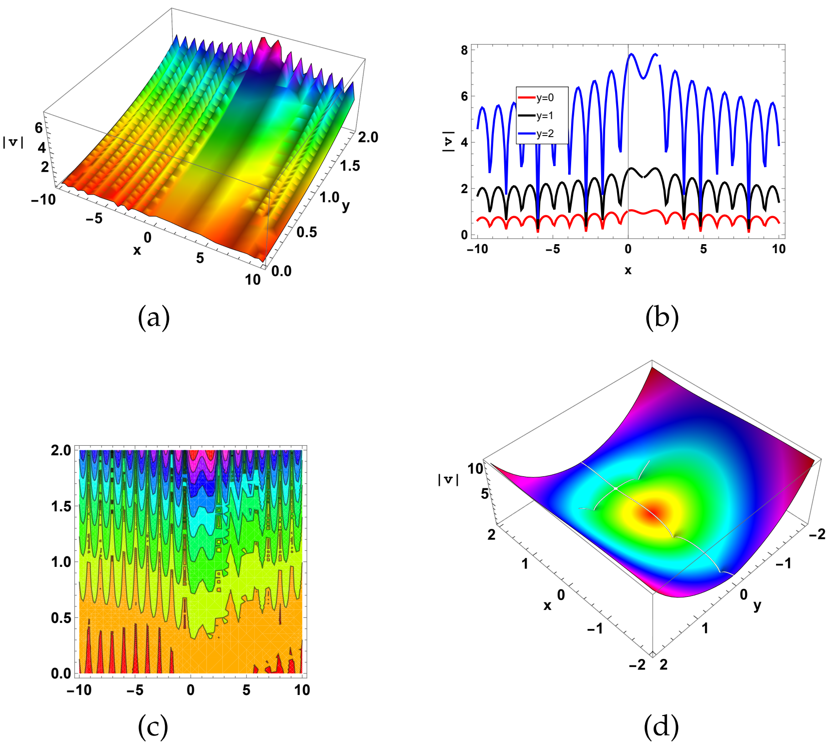


Figure 5: Visual analysis of the solution (3.41): (a) three-dimensional dynamics; (b) two-dimensional dynamics for the values $y = 1.0, y = 2.0, y = 3.0$; (c) contour dynamics; and (d) complex dynamics.

(v) $\mathcal{X}_3 + \mathcal{X}_7 = \frac{\partial}{\partial y} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$. The symmetry generator $\frac{\partial}{\partial y} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$ provides

$$v(x,y,z,t) = p(j,k,l), \quad j = t, k = x, l = -\frac{1}{2}z^2 - \frac{1}{2}y^2 - z. \tag{3.42}$$

Eq. (1.2) with the variables (3.42) becomes

$$2\lambda p^2 b\left(l - \frac{1}{2}\right)p_{ll} + 4\lambda p b\left(l - \frac{1}{2}\right)p_l^2 + 2p_l p^2 b\lambda - \lambda b p^2 p_{kk} - 2\lambda b p p_k^2 + p_j = 0. \tag{3.43}$$

For equation (3.43), the infinitesimal operators are $\mathcal{Y}_1 = \frac{\partial}{\partial j}$, $\mathcal{Y}_2 = \frac{\partial}{\partial k}$, $\mathcal{Y}_3 = j\frac{\partial}{\partial j} - \frac{p}{2}\frac{\partial}{\partial p}$, $\mathcal{Y}_4 = k\frac{\partial}{\partial k} + (2l-1)\frac{\partial}{\partial l} + p\frac{\partial}{\partial p}$. $\mathcal{Y}_1 + \mathcal{Y}_2 = \frac{\partial}{\partial j} + \frac{\partial}{\partial k}$ gives $p(j,k,l) = q(\sigma,\tau)$, where $\sigma = l, \tau = k - j$, which ensures the reduction of (3.43) into the equation

$$2q^2\lambda\left(\sigma-\frac{1}{2}\right)bq_{\sigma\sigma}-\lambda bq^2q_{\tau\tau}+4q\lambda\left(\sigma-\frac{1}{2}\right)bq_{\sigma}^2+2\lambda bq^2q_{\sigma}-2\lambda bqq_{\tau}^2-q_{\tau}=0. \tag{3.44}$$

For further reduction of (3.44), we use the symmetries stated below

$$\mathcal{Z}_1=\frac{\partial}{\partial\tau},\ \mathcal{Z}_2=(2\sigma-1)\frac{\partial}{\partial\sigma}+\tau\frac{\partial}{\partial\tau}+\frac{q}{2}\frac{\partial}{\partial q}.$$

$\mathcal{Z}_1+\mathcal{Z}_2=\frac{\partial}{\partial\tau}+(2\sigma-1)\frac{\partial}{\partial\sigma}+\tau\frac{\partial}{\partial\tau}+\frac{q}{2}\frac{\partial}{\partial q}$ yields $q(\sigma,\tau)=\theta(s)$, where $s=\sigma$. Using the invariants of $\frac{\partial}{\partial\tau}+(2\sigma-1)\frac{\partial}{\partial\sigma}+\tau\frac{\partial}{\partial\tau}+\frac{q}{2}\frac{\partial}{\partial q}$, we acquire the ODE

$$2\lambda\theta b\left(\theta\left(s-\frac{1}{2}\right)\theta_{ss}+(2s-1)\theta_s^2+\theta_s\theta\right)=0.$$

Since $\theta(s)\neq 0$, therefore

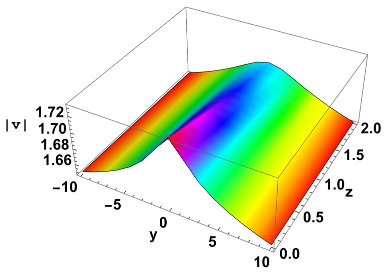
$$\theta\left(s-\frac{1}{2}\right)\theta_{ss}+(2s-1)\theta_s^2+\theta_s\theta=0,$$

and its solution is

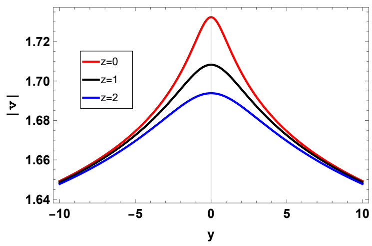
$$\theta(s)=-\frac{\left(-12c_1\ln(2s-1)+8c_2\right)^{\frac{1}{3}}}{4}\pm\frac{\iota\sqrt{3}\,\left(-12c_1\ln(2s-1)+8c_2\right)^{\frac{1}{3}}}{4}.$$

For (1.2), the solution takes the form

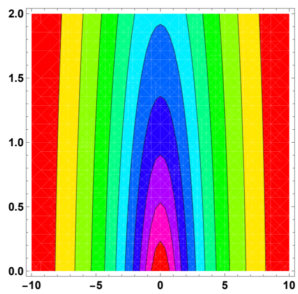
$$\begin{aligned}v(x,y,z,t)=&-\frac{\left(-12c_1\ln\left(2(-\frac{1}{2}z^2-\frac{1}{2}y^2-z)-1\right)+8c_2\right)^{\frac{1}{3}}}{4}\\&\pm\frac{\iota\sqrt{3}\,\left(-12c_1\ln\left(2(-\frac{1}{2}z^2-\frac{1}{2}y^2-z)-1\right)+8c_2\right)^{\frac{1}{3}}}{4}.\end{aligned} \tag{3.45}$$



(a)



(b)



(c)

Figure 6: Visual analysis of the solution (3.45): (a) three-dimensional dynamics, (b) two-dimensional dynamics for the values $z=1.0$, $z=2.0$, $z=3.0$; and (c) contour dynamics.

4. Local conserved flows

The conserved flows for the heat equation (1.2) acquired using the multiplier approach [1, 2] are presented in this section. The determining equations for the multiplier function $\Lambda(x, y, z, t, v)$ is given by

$$\frac{\delta}{\delta v}\left(v_t - \lambda(g'(v){v_x}^2 + g(v)v_{xx} + g'(v){v_y}^2 + g(v)v_{yy} + g'(v){v_z}^2 + g(v)v_{zz})\right) = 0, \tag{4.1}$$

where

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} - D_y\frac{\partial}{\partial v_y} - D_z\frac{\partial}{\partial v_z} + \cdots .$$

From (4.1), we obtain the following set of equations for multipliers

$$\Lambda_{xx} = -\Lambda_{yy} - \Lambda_{zz}, \quad \Lambda_t = 0, \quad \Lambda_v = 0,$$

and its solution is stated as

$$\Lambda(x, y, z, t, v) = c_1 + c_2 z + c_3 y + c_4 x. \tag{4.2}$$

The multipliers given in (4.2) fulfills the requirement

$$\begin{aligned} &D_t T^t + D_x T^x + D_y T^y + D_z T^z \\ &\quad = \Lambda(v_t - \lambda(g'(v){v_x}^2 + g(v)v_{xx} + g'(v){v_y}^2 + g(v)v_{yy} + g'(v){v_z}^2 + g(v)v_{zz})). \end{aligned} \tag{4.3}$$

Incorporating the multipliers $\Lambda_1 = 1$, $\Lambda_2 = z$, $\Lambda_3 = y$, and $\Lambda_4 = x$ into (4.3) yields the four conserved flows for (1.2) listed below, respectively,

$$T_1 = \begin{cases} {T_1}^t = v, \\ {T_1}^x = -\lambda g(v)v_x, \\ {T_1}^y = -\lambda g(v)v_y, \\ {T_1}^z = -\lambda g(v)v_z, \end{cases} \qquad T_2 = \begin{cases} {T_2}^t = zv, \\ {T_2}^x = -\lambda z g(v)v_x, \\ {T_2}^y = \lambda g(v)(-zv_y + yv_z), \\ {T_2}^z = -\lambda g(v)(yv_y + zv_z), \end{cases}$$

$$T_3 = \begin{cases} {T_3}^t = yv, \\ {T_3}^x = \lambda g(v)(-yv_x + xv_y), \\ {T_3}^y = -\lambda g(v)(xv_x + yv_y), \\ {T_3}^z = -\lambda y g(v)v_z, \end{cases} \qquad T_4 = \begin{cases} {T_4}^t = xv, \\ {T_4}^x = \lambda\left(-xg(v)v_x + \int g(v)dv\right), \\ {T_4}^y = -\lambda x g(v)v_y, \\ {T_4}^z = -\lambda x g(v)v_z. \end{cases}$$

5. Discussion and the conclusions

This study explored innovative applications of the Lie symmetry analysis method to solve heat-type equations. This approach successfully yields novel and exact invariant solutions for this equation. A diverse range of solutions exist, including trigonometric and hyperbolic solitons, Lambert functions, polynomials, exponential functions, inverse functions, hypergeometric forms, Bessel functions, logarithmic forms, rational functions, and solitary waves. These solutions are novel and have not been addressed in previous research on this topic. They provide insights into the dynamics of various soliton wave structures and serve as valuable tools for verifying the accuracy, enabling comparative analysis, and supporting numerical studies in the field. A key advantage of the results presented in this study is the practical utility for further investigation. In addition, the spatial temperature distribution on a conductive surface is depicted through different solution types, as illustrated in Figures 1–6. The dynamics of these distributions were captured through two- and three-dimensional graphs as well as contour plots. Furthermore, the intricate spatial temperature dynamics corresponding to the solution (3.41) are highlighted in Figure 5. By leveraging the advanced capabilities of these techniques, exact solutions can be obtained for a wide range of real-world problems in science and engineering. This provides a strong motivation for researchers to delve deeper into this promising field of study. Furthermore, the approach utilized in this study can be applied to solve other challenges in mathematics and physics. Ongoing research is required to develop new and more efficient analytical methods for solving nonlinear PDEs.

Data availability

All data generated or analyzed during this study are included in this published article.

Authors' contribution

Writing original draft: Akhtar Hussain and M. Usman; writing review and editing: Akhtar Hussain and M. Usman; methodology: Akhtar Hussain and M. Usman; software: Akhtar Hussain and M. Usman; supervision: Ahmed M. Zidan; project administration: Jorge Herrera and Ahmed M. Zidan; visualization: Jorge Herrera, M. Usman, Akhtar Hussain, Waleed M. Osman and Faizah D Alanazi; conceptualization: Akhtar Hussain and M. Usman; formal analysis: Ahmed M. Zidan, Jorge Herrera, M. Usman, and Akhtar Hussain.

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