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Novel fixed point theorems for orbital continuity in b-metric spaces: applications to integral equations and neural stability

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Abstract

This article establishes novel fixed point theorems for Ψ-orbitally continuous mappings in b-metric spaces, extending the foundational results. The findings are applied to demonstrate the existence and uniqueness solutions for nonlinear integral equations and analyzing the stability of neural networks.

Keywords: Orbitally continuous mappings, nonlinear integral equations, neural network, b-metric spaces.

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1. Introduction

Fixed point theory is a cornerstone of nonlinear analysis and is applicable to differential equations, optimization, and fractal theory. Recent advances have extended classical results to generalized spaces, such as b-metric spaces presented by Bakhtin [5] and Czerwik [9], which relax the triangle inequality to allow real-world complications.

Czerwik [10] constructed fixed point theory in b-metric space, demonstrating that Banach-type contractions [6] hold if the coefficient λ satisfies $\lambda s < 1$. Significant advancements in fixed-point theory have resulted from the study of b-metric spaces, presenting novel ideas and extensions of the traditional findings. Many researchers have found several fixed point results in b-metric spaces; see, e.g., ([1–4]).

A critical development in this domain involves Ψ-orbitally continuous mapping, which generalizes continuity requirements to sequences generated by iterative mappings. Orbital continuity, introduced by Rhoades [18] and Ćirić [7, 8], requires that for any starting point $\varkappa_0 \in F$, if the orbit sequence $\{\Psi^{\kappa}\varkappa_0\}$ converges to a point z, then the sequence $\{\Psi(\Psi^{\kappa}\varkappa_0)\}$ converges to Ψz. Since this is weaker than requiring Ψ to be continuous (i.e., if $\{\varkappa_{\kappa}\}$ converges to $\varkappa$, then $\{\Psi\varkappa_{\kappa}\}$ converges to $\Psi\varkappa$ for any convergent sequence $\{\varkappa_{\kappa}\}$), it enables fixed point results for mappings exhibiting irregular behavior. In mappings where standard continuity fails, the idea of Ψ-orbital continuity allows for fixed point the results by applying continuity to the orbit of Ψ.

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Nashine et al. [13] and Fallahi et al. [11] investigate fixed point results in b-metric spaces, with the first focusing on orbitally lower semicontinuous functions and the second introducing wt-distance and graph structures. Tsegaye [19] presented fixed point theorems for generalized α-Suzuki-Geraghty type contractions in f-orbitally complete b-metric spaces. Nashine and Karapınar [14] extended these concepts to partial metric spaces by introducing orbitally complete and orbitally continuous mappings. All studies aim to generalize and improve existing fixed point theorems, with applications ranging from fractal integral equations [13] to antenna energy reduction and solutions to integral equations [19]. In addition to providing new methods for the analysis of contractive mappings and their applications, these studies collectively contribute to fixed point theory in a variety of metric space generalizations; see, e.g., ([12], [15–17]).

This article establishes novel fixed point theorems for Ψ-orbitally continuous mappings in b-metric spaces and discusses their important applications in nonlinear integral equations and neural network stability analysis.

We review the fundamental concepts and our results in the following.

Definition 1.1. Let $\digamma$ be a non empty set. A function $\rho : \digamma \times \digamma \to \mathbb{R}^+$ is called a b-metric space if there exists a constant $s \geqslant 1$ such that for all $\varkappa_1, \varkappa_2, \varkappa_3 \in \digamma$, the following conditions are holds:

i. $\rho(\varkappa_1, \varkappa_2) = 0$ if and only if, $\varkappa_1 = \varkappa_2$;
ii. $\rho(\varkappa_1, \varkappa_2) = \rho(\varkappa_2, \varkappa_1)$;
iii. $\rho(\varkappa_1, \varkappa_3) \leqslant s\,[\rho(\varkappa_1, \varkappa_2) + \rho(\varkappa_2, \varkappa_3)]$.

The pair $(\digamma, \rho)$ is then called a b-metric space with parameter $s \geqslant 1$.

As shown in the following examples, b-metric spaces do not satisfy the conditions of metric spaces.

Example 1.2. Define the metric $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by the formula $\rho(\varkappa_1, \varkappa_2) = \mid \varkappa_1 - \varkappa_2 \mid^p$, $p > 1$, $\varkappa_1, \varkappa_2 \in \mathbb{R}$. Thus $(\mathbb{R}, \rho)$ is a b-metric space with $s = 2^{p-1}$.

Example 1.3. Let $p \in (0, 1)$, and $\digamma = l^p(\mathbb{R}) = \{\varkappa = \{\varkappa_\kappa\} \subset \mathbb{R} : \sum_{\kappa=1}^{\infty} |\varkappa_\kappa|^p < \infty\}$. For $\varkappa, \omega \in \digamma$, set $\rho(\varkappa, \omega) = [\sum_{\kappa=1}^{\infty} |\varkappa_\kappa - \omega_\kappa|^p]^{\frac{1}{p}}$. Then $(\digamma, \rho)$ is a b-metric space with $s = 2^{1/p}$.

Example 1.4. Let $(\digamma, J)$ be a metric space. For parameters $\beta > 1$, $\lambda \geqslant 0$, and $\mu > 0$, define $\rho(\varkappa, \omega) = \lambda J(\varkappa, \omega) + \mu J(\varkappa, \omega)^\beta$ for all $\varkappa, \omega \in \digamma$. Then ρ is not a metric on $\digamma$. However $(\digamma, \rho)$ is a b-metric space with $s = 2^{\beta-1}$. Indeed, for any $\upsilon \in \digamma$,

$$\begin{aligned}\rho(\varkappa, \omega) = \lambda J(\varkappa, \omega) + \mu J(\varkappa, \omega)^\beta &\leqslant \lambda[J(\varkappa, \upsilon) + J(\upsilon, \omega)] + \mu[J(\varkappa, \upsilon) + J(\upsilon, \omega)]^\beta \\ &\leqslant \lambda[J(\varkappa, \upsilon) + J(\upsilon, \omega)] + 2^{\beta-1}\mu[J(\varkappa, \upsilon)^\beta + J(\upsilon, \omega)^\beta] \\ &\leqslant 2^{\beta-1}[\rho(\varkappa, \upsilon) + \rho(\upsilon, \omega)].\end{aligned}$$

It is obvious that the distance function in a b-metric space does not have to be continuous, and open balls in these areas do not necessarily have to be open sets.

Lemma 1.5. *Let $(\digamma, \rho)$ be a b-metric space with $s \geqslant 1$, and assume that $\{\varkappa_\kappa\}$ and $\{y_\kappa\}$ are convergent to $\varkappa$ and y, respectively. Then we get*

$$\frac{1}{s^2}\rho(\varkappa, y) \leqslant \lim_{\kappa\to\infty} \inf \rho(\varkappa_\kappa, y_\kappa) \leqslant \lim_{\kappa\to\infty} \sup \rho(\varkappa_\kappa, y_\kappa) \leqslant s^2\rho(\varkappa, y).$$

And also, if $\varkappa = y$, then $\lim\limits_{\kappa\to\infty} \rho(\varkappa_\kappa, y_\kappa) = 0$. Furthermore, for every $c \in \digamma$, hence

$$\frac{1}{s^2}\rho(\varkappa, c) \leqslant \lim_{\kappa\to\infty} \inf \rho(\varkappa_\kappa, c) \leqslant \lim_{\kappa\to\infty} \sup \rho(\varkappa_\kappa, c) \leqslant s\rho(\varkappa, c).$$

Definition 1.6. Let $\{\varkappa_\kappa\} = \{\varkappa_\kappa : \kappa \in \mathbb{N}\}$ be a sequence in a b-metric space $(\digamma, \rho)$.

i. A sequence $\{\varkappa_\kappa\}$ *converges in* $\digamma$ *if there exists* $\varkappa \in \digamma$ *such that* $\lim\limits_{\kappa\to\infty} \rho(\varkappa_\kappa, \varkappa) = 0$.
ii. A sequence $\{\varkappa_\kappa\}$ *is a Cauchy sequence in* $\digamma$ *if for every* $\varepsilon > 0$, there exists $\kappa_0 \in \mathbb{N}$ *such that:*
$$\rho(\varkappa_\kappa, \varkappa_l) < \varepsilon, \ \forall \kappa, l > \kappa_0.$$
iii. The space $\digamma$ *is called complete if every Cauchy sequence in* $\digamma$ *converges in* $\digamma$.

Orbital continuity refines continuity by focusing on sequences generated by iterative mappings.

Definition 1.7. Let $\Psi : \digamma \to \digamma$ be a mapping of a b-metric space $\digamma$ into itself. Ψ is said to be orbitally continuous if $\lim\limits_{i\longrightarrow\infty} \Psi^{\kappa_i}\varkappa = \omega$ implies that, $\lim\limits_{i\longrightarrow\infty} \Psi\Psi^{\kappa_i}\varkappa = \Psi\omega$ and that $\digamma$ is said to be Ψ-orbitally complete if every Cauchy sequence of the form $\{\Psi^{\kappa_i}\varkappa\}_{i=1}^{\infty}$ converges in $\digamma$.

2. Main results

Now we are discussing our main result

Theorem 2.1. *Let* $(\digamma, \rho)$ *be a* b*-metric space with parameter* $s \geqslant 1$, *and let* $\Psi : \digamma \to \digamma$ *be a self-mapping satisfying the following: for each* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that*
$$\varepsilon \leqslant \max\left\{\rho(\varkappa_1, \varkappa_2), \rho(\varkappa_1, \Psi\varkappa_1), \rho(\varkappa_2, \Psi\varkappa_2), \frac{1}{2s^2}[\rho(\varkappa_1, \Psi\varkappa_2) + \rho(\varkappa_2, \Psi\varkappa_1)\right\} < \varepsilon + \delta(\varepsilon) \tag{2.1}$$
implies $\rho(\Psi\varkappa_1, \Psi\varkappa_2) < \varepsilon$. *Then, for each* $\varkappa \in \digamma$, *the sequence* $\{\Psi\varkappa_\kappa\}$ *is Cauchy. Furthermore, if* $\digamma$ *is* Ψ*-orbitally complete and satisfies either*
$$\rho(\Psi\varkappa_1, \Psi\varkappa_2) \leqslant \max\left\{\rho(\varkappa_1, \varkappa_2), \alpha[\rho(\varkappa_1, \Psi\varkappa_1) + \rho(\varkappa_2, \Psi\varkappa_2)], \frac{1}{2s^2}[\rho(\varkappa_1, \Psi\varkappa_2) + \rho(\varkappa_2, \Psi\varkappa_1)\right\} \tag{2.2}$$
for some $\alpha \in (0, \frac{1}{s})$, *or* Ψ *is orbitally continuous, then* Ψ *has a unique fixed point* $\omega \in \digamma$, $\lim\limits_{\kappa\longrightarrow\infty} \Psi^\kappa\varkappa = \omega$ *for all* $\varkappa \in \digamma$.

Proof. Given $\varkappa \in \digamma$, define $\varkappa_\kappa = \Psi\varkappa_{\kappa-1}$, assume that:
$$\max\left\{\rho(\varkappa_{\kappa-1}, \Psi\varkappa_{\kappa-1}), \rho(\varkappa_{\kappa-1}, \Psi\varkappa_{\kappa-1}), \rho(\Psi\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}), \frac{1}{2s^2}[\rho(\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}) + \rho(\Psi\varkappa_{\kappa-1}, \Psi\varkappa_{\kappa-1})]\right\} > 0$$
implies
$$\rho(\Psi\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}) < \max\left\{\rho(\varkappa_{\kappa-1}, \Psi\varkappa_{\kappa-1}), \rho(\Psi\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}), \frac{1}{2s^2}\rho(\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1})\right\},$$
if $\max\left\{\rho(\varkappa_{\kappa-1}, \Psi\varkappa_{\kappa-1}), \rho(\Psi\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}), \frac{1}{2s^2}\rho(\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1})\right\} = \rho(\Psi\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1})$, due to contradiction, we have $\rho(\Psi\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}) < \rho(\Psi\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1})$. It follows that
$$\rho(\Psi\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}) < \max\left\{\rho(\varkappa_{\kappa-1}, \Psi\varkappa_{\kappa-1}), \frac{1}{2s^2}\rho(\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1})\right\}.$$
Suppose that $\rho(\varkappa_{\kappa-1}, \Psi\varkappa_{\kappa-1}) < \frac{1}{2s^2}\rho(\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1})$, we get $\rho(\Psi\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}) < \frac{1}{2s^2}\rho(\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1})$,
$$\rho(\varkappa_{\kappa-1}, \Psi\varkappa_{\kappa-1}) + \rho(\Psi\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}) < \frac{1}{s^2}\rho(\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}) < \frac{1}{s}\rho(\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}),$$
$$s[\rho(\varkappa_{\kappa-1}, \Psi\varkappa_{\kappa-1}) + \rho(\Psi\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1})] < \rho(\varkappa_{\kappa-1}, \Psi^2\varkappa_{\kappa-1}),$$

which is a contradiction with the s-triangle inequality, then

$$\rho(\Psi\varkappa_{\kappa-1},\Psi^2\varkappa_{\kappa-1})<\rho(\varkappa_{\kappa-1},\Psi\varkappa_{\kappa-1}) \tag{2.3}$$

for each $\kappa=1,2,3,\ldots$. Then $\{\rho(\varkappa_\kappa,\varkappa_{\kappa+1})\}$ non increasing and bound below and hence has a limit $\xi\geqslant 0$. Assume that $\xi>0$. Then $\delta(\xi)>0$ and for some ι, $\rho(\Psi^{\iota-1}\varkappa,\Psi^\iota\varkappa)<\xi+\delta(\xi)$. Then by (2.1) and (2.3), it implies, $\rho(\Psi\Psi^{\iota-1}\varkappa,\Psi\Psi^\iota\varkappa)=\rho(\Psi^\iota\varkappa,\Psi^{\iota+1}\varkappa)<\xi$, which is a contradiction; therefore, $\lim\limits_{\kappa\longrightarrow\infty}\rho(\varkappa_{\kappa-1},\Psi\varkappa_{\kappa-1})=0$. Now, we are proving it $(\varkappa_\kappa)$ is a Cauchy sequence in $\digamma$. Assume that $(\varkappa_\kappa)$ is not Cauchy; then there exist $\varepsilon>0$ and the subsequences $(\varkappa_{\kappa_i})$ and $(\varkappa_{m_i})$ of $(\varkappa_\kappa)$ with $\kappa_i>m_i>i$ such that $\rho(\varkappa_{\kappa_i},\varkappa_{m_i})\geqslant\varepsilon$, and κ_i is the least number. We obtain $\rho(\varkappa_{\kappa_{i-1}},\varkappa_{m_i})<\varepsilon$. By using the s-triangular inequality,

$$\varepsilon\leqslant\rho(\varkappa_{\kappa_i},\varkappa_{m_i})\leqslant s[\rho(\varkappa_{\kappa_i},\varkappa_{m_{i+1}})+\rho(\varkappa_{m_{i+1}},\varkappa_{m_i})],\quad \frac{\varepsilon}{s}\leqslant\limsup\rho(\varkappa_{\kappa_i},\varkappa_{m_{i+1}}). \tag{2.4}$$

Also, $\limsup\rho(\varkappa_{\kappa_{i-1}},\varkappa_{m_i})\leqslant\varepsilon$, and by (2.1),

$$\max\left\{\rho(\varkappa_1,\varkappa_2),\rho(\varkappa_1,\Psi\varkappa_1),\rho(\varkappa_2,\Psi\varkappa_2),\frac{1}{2s^2}[\rho(\varkappa_1,\Psi\varkappa_2)+\rho(\varkappa_2,\Psi\varkappa_1)\right\}>0,$$

imply

$$\begin{aligned}
\rho(\Psi\varkappa_1,\Psi\varkappa_2)&<\max\left\{\rho(\varkappa_1,\varkappa_2),\rho(\varkappa_1,\Psi\varkappa_1),\rho(\varkappa_2,\Psi\varkappa_2),\frac{1}{2s^2}[\rho(\varkappa_1,\Psi\varkappa_2)+\rho(\varkappa_2,\Psi\varkappa_1)\right\},\\
\rho(\varkappa_{\kappa_i},\varkappa_{m_{i+1}})&=\rho(\Psi\varkappa_{\kappa_{i-1}},\Psi\varkappa_{m_i})\\
&<\max\left\{\rho(\varkappa_{\kappa_{i-1}},\varkappa_{m_i}),\rho(\varkappa_{\kappa_{i-1}},\Psi\varkappa_{\kappa_i-1}),\rho(\varkappa_{m_i},\Psi\varkappa_{m_i}),\frac{1}{2s^2}[\rho(\varkappa_{\kappa_{i-1}},\Psi\varkappa_{m_i})+\rho(\varkappa_{m_i},\Psi\varkappa_{\kappa_{i-1}})]\right\}\\
&<\max\left\{\rho(\varkappa_{\kappa_{i-1}},\varkappa_{m_i}),\rho(\varkappa_{\kappa_{i-1}},\varkappa_{\kappa_i}),\rho(\varkappa_{m_i},\varkappa_{m_{i+1}}),\frac{1}{2s^2}[\rho(\varkappa_{\kappa_{i-1}},\varkappa_{m_{i+1}})+\rho(\varkappa_{m_i},\varkappa_{\kappa_i})]\right\},
\end{aligned}$$

using the s-triangle inequality,

$$\rho(\varkappa_{\kappa_{i-1}},\varkappa_{m_{i+1}})\leqslant s[\rho(\varkappa_{\kappa_{i-1}},\varkappa_{\kappa_i})+\rho(\varkappa_{\kappa_i},\varkappa_{m_{i+1}})],\quad \rho(\varkappa_{m_i},\varkappa_{\kappa_i})\leqslant s[\rho(\varkappa_{m_i},\varkappa_{\kappa_{i-1}})+\rho(\varkappa_{\kappa_{i-1}},\varkappa_{\kappa_i})],$$

we get

$$\begin{aligned}
\rho(\varkappa_{\kappa_i},\varkappa_{m_{i+1}})&\leqslant\frac{1}{2s^2}\left[s[\rho(\varkappa_{\kappa_{i-1}},\varkappa_{\kappa_i})+\rho(\varkappa_{\kappa_i},\varkappa_{m_{i+1}})]+s[\rho(\varkappa_{m_i},\varkappa_{\kappa_{i-1}})+\rho(\varkappa_{\kappa_{i-1}},\varkappa_{\kappa_i})]\right]\\
&\leqslant\frac{1}{2s}[2\rho(\varkappa_{\kappa_{i-1}},\varkappa_{\kappa_i})+\rho(\varkappa_{\kappa_i},\varkappa_{m_{i+1}})+\rho(\varkappa_{m_i},\varkappa_{\kappa_{i-1}})],\\
\rho(\varkappa_{\kappa_i},\varkappa_{m_{i+1}})[1-\frac{1}{2s}]&<\frac{1}{2s}[2\rho(\varkappa_{\kappa_{i-1}},\varkappa_{\kappa_i})+\rho(\varkappa_{m_i},\varkappa_{\kappa_{i-1}})].
\end{aligned}$$

Letting the upper limit as $i\to\infty$, $[1-\frac{1}{2s}]\lim\limits_{i\to\infty}\sup\rho(\varkappa_{\kappa_i},\varkappa_{m_{i+1}})<\frac{\varepsilon}{2s}$, then $\lim\limits_{i\to\infty}\sup\rho(\varkappa_{\kappa_i},\varkappa_{m_{i+1}})<\frac{\varepsilon}{2s-1}<\frac{\varepsilon}{s}$, which is contradiction with (2.4), hence $(\varkappa_\kappa)$ is a Cauchy sequence in $\digamma$. Since $\digamma$ is Ψ orbitally complete, the Cauchy sequence $(\varkappa_\kappa)=(\Psi^\kappa\varkappa)$ converges to some point ω. Now, if Ψ is orbitally continuous, $\Psi\omega=\lim\Psi(\Psi^\kappa\varkappa)=\Psi^{\kappa+1}\varkappa=\omega$, so ω is a fixed point. If Ψ satisfies (2.2), then we need to show that ω is a fixed point for Ψ,

$$\rho(\omega,\Psi\omega)\leqslant s\left[\rho(\omega,\varkappa_{\kappa+1})+\rho(\varkappa_{\kappa+1},\Psi\omega)\right]=s\left[\rho(\omega,\varkappa_{\kappa+1})+\rho(\Psi\varkappa_\kappa,\Psi\omega)\right],$$

using condition (2.2),

$$\rho(\Psi\varkappa_\kappa,\Psi\omega)\leqslant\max\left\{\rho(\varkappa_\kappa,\omega),\alpha\left[\rho(\varkappa_\kappa,\Psi\varkappa_\kappa)+\rho(\omega,\Psi\omega)\right],\frac{1}{2s^2}\left[\rho(\varkappa_\kappa,\Psi\omega)+\rho(\omega,\Psi\varkappa_\kappa)\right]\right\}.$$

Letting the upper limit as $\kappa \to \infty$, Max tends to $\max\left\{0, \alpha\rho(\omega, \Psi\omega), \frac{1}{2s^2}\rho(\omega, \Psi\omega)\right\}$, since $\alpha < \frac{1}{s}$ and $\frac{1}{2s^2} < \frac{1}{s}$. Thus

$$\lim_{\kappa\to\infty} \rho(\Psi\varkappa_\kappa, \Psi\omega) \leqslant q\rho(\omega, \Psi\omega),$$

$q = \max\{\alpha, \frac{1}{2s^2}\} < \frac{1}{s}$, $\rho(\omega, \Psi\omega) \leqslant \rho(\omega, \Psi\omega)$. Hence $\Psi\omega = \omega$.

To prove the uniqueness of the fixed point, suppose that there are two fixed points υ, ω. Then $\rho(\upsilon, \omega) \leqslant \rho(\Psi\upsilon, \Psi\omega)$, applying condition (2.1), we have $\varepsilon = \rho(\upsilon, \omega)$, and thus

$$\varepsilon \leqslant \max\left\{\rho(\upsilon, \omega), \rho(\upsilon, \Psi\upsilon), \rho(\omega, \Psi\omega), \frac{1}{2s^2}\left[\rho(\upsilon, \Psi\omega) + \rho(\omega, \Psi\upsilon)\right]\right\} < \varepsilon + \delta(\varepsilon) \leqslant \max\left\{\varepsilon, 0, \frac{\varepsilon}{s^2}\right\} = \varepsilon.$$

Therefore, $\rho(\Psi\upsilon, \Psi\omega) < \varepsilon$. But $\rho(\Psi\upsilon, \Psi\omega) = \rho(\upsilon, \omega) = \varepsilon$, which is a contradiction, hence ε must be 0, so $\upsilon = \omega$. If we applied condition (2.2), we obtain

$$\rho(\upsilon, \omega) = \rho(\Psi\upsilon, \Psi\omega) \leqslant \max\left\{\rho(\upsilon, \omega), \alpha.0, \frac{1}{s^2}\rho(\upsilon, \omega)\right\} = \rho(\upsilon, \omega),$$

which implies $\rho(\upsilon, \omega) = 0$. □

Example 2.2. Let $\digamma = [0, 1]$, and $\Psi\varkappa = \frac{\varkappa}{3}$, $\rho(\varkappa_1, \varkappa_2) = |\varkappa_1 - \varkappa_2|^2$, $\varkappa_1, \varkappa_2 \in \digamma$. Then Ψ have a unique fixed point in $\digamma$. Indeed, $\rho(\Psi\varkappa_1, \Psi\varkappa_2) = \left|\frac{\varkappa_1}{3} - \frac{\varkappa_2}{3}\right|^2 = \frac{1}{9}|\varkappa_1 - \varkappa_2|^2 = \frac{1}{9}\rho(\varkappa_1, \varkappa_2)$. For $\alpha = \frac{1}{3}(\alpha < \frac{1}{s} = \frac{1}{2})$, we get the condition (2.2) holds. For any $\varkappa \in \digamma$, $\Psi^\kappa(\varkappa) = (\frac{1}{3})^\kappa\varkappa$ converges to 0.

Example 2.3. Let $\digamma = \mathbb{R}$, and $\Psi\varkappa = \theta\varkappa$, where $\theta < 2^{\frac{-(p-1)}{p}}$, $p > 1$, and $\rho(\varkappa_1, \varkappa_2) = |\varkappa_1 - \varkappa_2|^p$, $\varkappa_1, \varkappa_2 \in \digamma$, be a b-metric with $s = 2^{p-1}$. Then Ψ has a unique fixed point in $\digamma$. Indeed, $\rho(\Psi\varkappa_1, \Psi\varkappa_2) = |\theta\varkappa_1 - \theta\varkappa_2|^p = \theta^p|\varkappa_1 - \varkappa_2|^p = \theta^p\rho(\varkappa_1, \varkappa_2)$. For $\theta^p < \frac{1}{s} = 2^{-(p-1)}$, if we let $p = 2$, $\theta < \frac{1}{\sqrt{2}} = 0.7$, $\Psi\varkappa = 0.7\varkappa$, satisfies

$$\rho(\Psi\varkappa_1, \Psi\varkappa_2) = 0.49\rho(\varkappa_1, \varkappa_2) \leqslant \max\left\{\rho(\varkappa_1, \varkappa_2), \alpha[\rho(\varkappa_1, \Psi\varkappa_1) + \rho(\varkappa_2, \Psi\varkappa_2)], \frac{1}{2s^2}[\rho(\varkappa_1, \Psi\varkappa_2) + \rho(\varkappa_2, \Psi\varkappa_1)\right\}.$$

For any $\varkappa \in \digamma$, $\Psi^\kappa(\varkappa) = \theta^\kappa\varkappa$ converges to 0.

Theorem 2.4. *Let $(\digamma, \rho)$ be a b-metric space with parameter $s \geqslant 1$, and let $\Psi : \digamma \to \digamma$ be a self-mapping. Suppose $\digamma$ is Ψ-orbitally complete. If Ψ satisfies the conditions:*

$$\rho(\Psi\varkappa_1, \Psi\varkappa_2) < Q\max\left\{\rho(\varkappa_1, \varkappa_2), \frac{\rho(\varkappa_1, \Psi\varkappa_1)\rho(\varkappa_2, \Psi\varkappa_2)}{\rho(\varkappa_1, \varkappa_2)}, \sigma(\varkappa_1, \varkappa_2)\rho(\varkappa_1, \Psi\varkappa_2)\rho(\varkappa_2, \Psi\varkappa_1)\right\} \tag{2.5}$$

for all $\varkappa_1 \neq \varkappa_2$, where $Q < \frac{1}{s}$ and $\sigma(\varkappa_1, \varkappa_2)$ is a non-negative real function satisfying $\sigma(\varkappa_1, \varkappa_2) \leqslant \frac{1}{\rho(\varkappa_1, \varkappa_2)}$. Then for each $\varkappa \in \digamma$, $\lim_{\kappa\longrightarrow\infty} \Psi^\kappa\varkappa = \omega \in \digamma$ with $\Psi\omega = \omega$. Moreover, the fixed point is unique.

Proof. Let $\varkappa \in \digamma$. If $\Psi\varkappa = \varkappa$, the result is trivial. Assume $\Psi\varkappa \neq \varkappa$. Applying (2.5):

$$\begin{aligned}
&\rho(\Psi\varkappa, \Psi^2\varkappa) < Q\max\left\{\rho(\varkappa, \Psi\varkappa), \frac{\rho(\varkappa, \Psi\varkappa)\rho(\Psi\varkappa, \Psi^2\varkappa)}{\rho(\varkappa, \Psi\varkappa)}, \sigma(\varkappa, \Psi\varkappa)\rho(\varkappa, \Psi^2\varkappa)\rho(\Psi\varkappa, \Psi\varkappa)\right\},\\
&\rho(\Psi\varkappa, \Psi^2\varkappa) < Q\max\left\{\rho(\varkappa, \Psi\varkappa), \rho(\Psi\varkappa, \Psi^2\varkappa), 0\right\},
\end{aligned}$$

if $\rho(\Psi\varkappa, \Psi^2\varkappa) \geqslant \rho(\varkappa, \Psi\varkappa)$, this would imply $\rho(\Psi\varkappa, \Psi^2\varkappa) < Q\rho(\Psi\varkappa, \Psi^2\varkappa)$, which is impossible since $Q < \frac{1}{s}$. Therefore $\rho(\Psi\varkappa, \Psi^2\varkappa) < Q\rho(\varkappa, \Psi\varkappa)$. By induction, $\rho(\Psi^\kappa\varkappa, \Psi^{\kappa+1}\varkappa) < Q^\kappa\rho(\varkappa, \Psi\varkappa)$. To show $\{\Psi^\kappa\varkappa\}$ is Cauchy, note that for $\iota > \kappa$,

$$\rho(\Psi^\kappa\varkappa, \Psi^\iota\varkappa) \leqslant s\sum_{i=\kappa}^{\iota-1} Q^i\rho(\varkappa, \Psi\varkappa) \leqslant s\frac{Q^\kappa}{1-Q}\rho(\varkappa, \Psi\varkappa).$$

Since $Q < \frac{1}{s}$, $Q^\kappa \to 0$, then $\{\Psi^\kappa \varkappa\}$ is Cauchy. By Ψ-orbitally completeness, $\Psi^\kappa \varkappa \to \omega \in \digamma$. Orbitally continuity of Ψ gives $\Psi\omega = \lim_{\kappa\to\infty} \Psi^{\kappa+1}\varkappa = \omega$, proving ω is a fixed point. For uniqueness, suppose υ, ω are distinct fixed points. Applying (2.5) and $\sigma(\varkappa_1, \varkappa_2) \leqslant \frac{1}{\rho(\varkappa_1,\varkappa_2)}$, then

$$\rho(\upsilon, \omega) = \rho(\Psi\upsilon, \Psi\omega) < Q \max\{\rho(\upsilon, \omega), 0, \sigma(\upsilon, \omega)\rho(\upsilon, \Psi\omega)\rho(\omega, \Psi\upsilon)\} = Q\rho(\upsilon, \omega).$$

That is a contradiction; hence, the fixed point is unique. □

3. Applications

3.1. Solving a Volterra integral equation

Theorem 3.1. *Let $\digamma = C[0, 1]$ be the space of continuous functions on $[0, 1]$, equipped with the b-metric:*

$$\rho(\upsilon, \omega) = \sup_{\varkappa\in[0,1]} |\upsilon(\varkappa) - \omega(\varkappa)|^2, \ \upsilon, \omega \in \digamma$$

with coefficient $s = 2$. The Volterra integral equation is investigated:

$$\phi(\varkappa) = \eta \int_0^{\varkappa} k(\varkappa, \tau)\phi(\tau)d\tau + c(\varkappa), \ \varkappa \in [0, 1],$$

where $k : [0, 1] \times [0, 1] \to R$ is continuous and bounded by $|k(\varkappa, \tau)| \leqslant \beta$, $c \in C([0, 1])$, $|\eta| < \frac{1}{\sqrt{2}\beta}$. Then there exists a unique continuous solution $\zeta \in C[0, 1]$.

Proof. Define the operator Ψ:

$$(\Psi\phi)(\varkappa) = \eta \int_0^{\varkappa} k(\varkappa, \tau)\phi(\tau)d\tau + c(\varkappa).$$

For $\phi, \varphi \in \digamma$, compute:

$$|\Psi\phi - \Psi\varphi| \leqslant |\eta|\, \beta \int_0^{\varkappa} |\phi(\tau) - \varphi(\tau)|\, d\tau \leqslant |\eta|\, \beta \sup_{\tau} |\phi(\tau) - \varphi(\tau)|\, .$$

Squaring both sides, we get

$$\rho(\Psi\phi, \Psi\varphi) \leqslant |\eta|^2\, \beta^2 \rho(\phi, \varphi).$$

Let $\alpha = |\eta|^2\, \beta^2$. Choose $|\eta| < \frac{1}{\sqrt{2}\beta}$, ensuring $\alpha \in (0, \frac{1}{2})$. For a given $\varepsilon > 0$, set $\delta(\varepsilon) = \frac{\varepsilon}{2}(\frac{1}{\alpha} - 1)$, if the maximum in (2.1) satisfies:

$$\varepsilon \leqslant \max\left\{\rho(\phi, \varphi), \rho(\phi, \Psi\phi), \rho(\varphi, \Psi\varphi), \frac{1}{8}[\rho(\phi, \Psi\varphi) + \rho(\varphi, \Psi\phi)]\right\} < \varepsilon + \delta(\varepsilon),$$

then

$$\rho(\Psi\phi, \Psi\varphi) \leqslant \alpha(\varepsilon + \delta(\varepsilon)) < \alpha(\varepsilon + \frac{\varepsilon}{2}(\frac{1}{\alpha} - 1)) < \varepsilon.$$

Thus, condition (2.1) holds. $C[0, 1]$ is complete under the supremum norm. Since the b-metric topology corresponds to the uniform convergence topology, $\digamma$ is Ψ-orbitally complete. Now we verify condition (2.2), $\rho(\Psi\phi, \Psi\varphi) \leqslant \alpha\rho(\phi, \varphi)$, where $\alpha \in (0, \frac{1}{2})$. Since

$$\alpha\rho(\phi, \varphi) \leqslant \max\left\{\rho(\phi, \varphi), \alpha[\rho(\phi, \Psi\phi) + \rho(\varphi, \Psi\varphi)], \frac{1}{8}[\rho(\phi, \Psi\varphi) + \rho(\varphi, \Psi\phi)]\right\},$$

then condition (2.2) is satisfied. All conditions of Theorem 2.1 are satisfied. Therefore the sequence $\{\Psi^\kappa\phi\}$ is Cauchy for any $\phi \in \digamma$. Hence Ψ has a unique fixed point $\zeta \in \digamma$ and for all $\phi \in \digamma$, $\lim_{\kappa\to\infty} \Psi^\kappa\phi = \zeta$. The fixed point ζ satisfies:

$$\zeta(\varkappa) = \eta \int_0^{\varkappa} k(\varkappa, \tau)\zeta(\tau)d\tau + c(\varkappa),$$

which is the unique solution to the volterra equation under the condition $|\eta| < \frac{1}{\sqrt{2}\beta}$. □

Example 3.2. Solve the Volterra integral equation:

$$\phi(\varkappa) = \frac{1}{2}\int_0^{\varkappa} \tau\phi(\tau)d\tau + 1, \ \varkappa \in [0,1].$$

Solution: Let $\digamma = C[0,1]$ with $\rho(\upsilon,\omega) = \sup_{\varkappa\in[0,1]} |\upsilon(\varkappa)-\omega(\varkappa)|^2$, $\upsilon,\omega \in \digamma$ and $s = 2$. Define an operator:

$$\Psi\phi(\varkappa) = \frac{1}{2}\int_0^{\varkappa} \tau\phi(\tau)d\tau + 1.$$

Now we compute $\rho(\Psi\phi,\Psi\varphi)$:

$$|\Psi\phi-\Psi\varphi| \leqslant \frac{1}{2}\int_0^{\varkappa} \tau|\phi(\tau)-\varphi(\tau)|\,d\tau \leqslant \frac{\varkappa^2}{4}\sup_{\tau\in[0,1]}|\phi(\tau)-\varphi(\tau)| \leqslant \frac{1}{4}\sup_{\tau\in[0,1]}|\phi(\tau)-\varphi(\tau)|,$$
$$\rho(\Psi\phi,\Psi\varphi) \leqslant \frac{1}{16}\rho(\phi,\varphi).$$

Here $\alpha = \frac{1}{16} \in (0,\frac{1}{2})$, satisfying condition(2.2). For iterative solution, start with $\phi_0(\varkappa) = 1$. Compute successive approximations as follows.

Iteration 1:

$$\phi_1(\varkappa) = \Psi\phi_0(\varkappa) = \frac{1}{2}\int_0^{\varkappa} \tau d\tau + 1 = \frac{\varkappa^2}{4} + 1.$$

Iteration 2:

$$\phi_2(\varkappa) = \Psi\phi_1(\varkappa) = \frac{1}{2}\int_0^{\varkappa} \tau(\frac{\tau^2}{4}+1)d\tau + 1 = \frac{\varkappa^4}{32} + \frac{\varkappa^2}{4} + 1.$$

Iteration 3:

$$\phi_3(\varkappa) = \Psi\phi_2(\varkappa) = \frac{1}{2}\int_0^{\varkappa} \tau(\frac{\tau^4}{32}+\frac{\tau^2}{4}+1)d\tau + 1 = \frac{\varkappa^6}{384} + \frac{\varkappa^4}{32} + \frac{\varkappa^2}{4} + 1.$$

Analytical solution: if $\phi'(\varkappa) = \frac{1}{2}\varkappa\phi(\varkappa)$, then $\phi(\varkappa) = Ae^{\frac{\varkappa^2}{4}}$. Using this, $\phi(0) = 1$, we find $A = 1$. Thus $\phi(\varkappa) = e^{\frac{\varkappa^2}{4}}$. The Taylor expansion is satisfied by the iterative approximation solution of $e^{\frac{\varkappa^2}{4}}$:

$$e^{\frac{\varkappa^2}{4}} = 1 + \frac{\varkappa^2}{4} + \frac{\varkappa^4}{32} + \frac{\varkappa^6}{384} + \cdots.$$

The sequence $\{\phi_\kappa(\varkappa)\}$ converges uniformly to $\phi(\varkappa) = e^{\frac{\varkappa^2}{4}}$, verifying Theorem 2.1, which guarantees a unique fixed point.

3.2. Solving nonlinear integral equations

Theorem 3.3. *Let $\digamma = C[a,b]$ be the space of continuous functions on $[a,b]$, equipped with the b-metric:*

$$\rho(\upsilon,\omega) = \sup_{\varkappa\in[a,b]} |\upsilon(\varkappa)-\omega(\varkappa)|^p, \ \textit{for } p > 1,$$

with coefficient $s = 2^{p-1}$. Consider the nonlinear integral equation:

$$\phi(\varkappa) = \eta\int_a^b \Lambda(\varkappa,\tau)\varpi(\tau,\phi(\tau))d\tau, \ \varkappa \in [a,b],$$

where $\Lambda : [a,b]\times[a,b] \to R$ is a continuous function and $\varpi : [a,b]\times R \to R$ is continuous and Lipschitz in ϕ. Then there is a unique solution $\phi \in C[a,b]$.

Proof. Define the operator $\Psi : \mathcal{F} \to \mathcal{F}$ by

$$(\Psi\phi)(\varkappa) = \eta \int_a^b \Lambda(\varkappa,\tau)\varpi(\tau,\phi(\tau))d\tau.$$

Since ϖ is Lipschitz in ϕ, there exists a constant $L \geqslant 0 : |\varpi(\tau,\phi) - \varpi(\tau,\varphi)| \leqslant L\,|\phi - \varphi|$. Then

$$\begin{aligned}\rho(\Psi\phi,\Psi\varphi) &= \sup_{\varkappa} \left|\eta \int_a^b \Lambda(\varkappa,\tau)[\varpi(\tau,\phi(\tau)) - \varpi(\tau,\varphi(\tau))]d\tau\right|^p \\ &\leqslant |\eta|^p \sup_{\varkappa} \left[\int_a^b |\Lambda(\varkappa,\tau)|\, L\,|\phi(\tau) - \varphi(\tau)|\, d\tau\right]^p \leqslant |\eta|^p\, \|\Lambda\|^p\, L^p (b-a)^p \rho(\phi,\varphi),\end{aligned}$$

where $\|\Lambda\| = \sup_{\varkappa} |\Lambda(\varkappa,\tau)|$. To satisfy (2.5) with $Q < \frac{1}{s}$, choose η such that $|\eta| < \left(\frac{1}{s\|\Lambda\|^p L^p (b-a)^p}\right)^{\frac{1}{p}}$. We get

$$\rho(\Psi\phi,\Psi\varphi) < Q \max\left\{\rho(\phi,\varphi), \frac{\rho(\phi,\Psi\phi)\rho(\varphi,\Psi\varphi)}{\rho(\phi,\varphi)}, \sigma(\phi,\varphi)\rho(\phi,\Psi\varphi)\rho(\varphi,\Psi\phi)\right\},$$

Ψ is orbitally continuous because Λ and ϖ are also continuous. By Theorem 2.4, Ψ has a unique fixed point. □

Example 3.4. Solve the nonlinear integral equation:

$$\phi(\varkappa) = 0.3 \int_0^1 (\varkappa\tau + 1)\cos(\phi(\tau))d\tau, \quad \varkappa \in [0,1].$$

Solution: Let $\mathcal{F} = C[0,1]$ with $\rho(\upsilon,\omega) = \sup_{\varkappa\in[0,1]} |\upsilon(\varkappa) - \omega(\varkappa)|^2$. Define

$$\begin{aligned}(\Psi\phi)(\varkappa) &= 0.3 \int_0^1 (\varkappa\tau + 1)\cos(\phi(\tau))d\tau, \\ \rho(\Psi\phi,\Psi\varphi) &= \sup_{\varkappa} \left|0.3 \int_0^1 (\varkappa\tau + 1)[\cos\phi(\tau) - \cos\varphi(\tau)]d\tau\right|^2 \\ &\leqslant (0.3)^2 \sup_{\varkappa}(\int_0^1 (\varkappa\tau + 1)\,|\phi(\tau) - \varphi(\tau)|\, d\tau)^2 \\ &= 0.09\rho(\phi,\varphi) \sup_{\varkappa}(\frac{\varkappa}{2} + 1)^2 \\ &= (0.09)(1.5)^2\rho(\phi,\varphi) \\ &= 0.2025\rho(\phi,\varphi) = Q \max\left\{\rho(\phi,\varphi), \frac{\rho(\phi,\Psi\phi)\rho(\varphi,\Psi\varphi)}{\rho(\phi,\varphi)}, \sigma(\phi,\varphi)\rho(\phi,\Psi\varphi)\rho(\varphi,\Psi\phi)\right\},\end{aligned}$$

$Q = 0.2025 < \frac{1}{s}$, then Ψ satisfies (2.5). A numerical solution can be obtained using the fixed point iteration approach by choosing $\phi_0(\varkappa) = 1$, as the initial function Picard iteration:

$$\phi_{n+1}(\varkappa) = 0.3 \int_0^1 (\varkappa\tau + 1)\cos(\phi_n(\tau))d\tau.$$

Split $[0,1]$ into $N = 10$ points: $\varkappa_\kappa = \tau_\kappa = \frac{\kappa}{9}$ for $\kappa = 0,1,2,\dots,9$. Use the trapezoidal rule for integration. For iteration results after 10 see Table 1.

Table 1

N	$\varkappa$	$\phi(\varkappa)$
0	0.0	0.462
1	0.111	0.465
2	0.222	0.469
3	0.333	0.473
4	0.444	0.477
5	0.556	0.481
6	0.667	0.485
7	0.778	0.489
8	0.889	0.493
9	1.0	0.497

For constant solution $\phi\varkappa = c$:

$$c = 0.3\int_0^1 (\varkappa\tau + 1)\cos(c)\ d\tau = 0.3\cos(c)(\frac{\varkappa}{2} + 1),$$

this cannot hold for all $\varkappa$ unless $c = 0.48$ (corresponding numerical results at $\varkappa = 1$). Unique solution exists by the fixed point Theorem 2.4 and numerical iteration converges to a near-linear function.

3.3. Stability analysis of neural networks

Theorem 3.5. *Let $\digamma = R^\kappa$ be state space with b-metric $\rho(\varkappa, \upsilon) = \|\varkappa - \upsilon\|_p^2$, for $\varkappa, \upsilon \in R^\kappa, 1 \leqslant p \leqslant 2$, $s = 2^{\frac{2}{p}}$. Consider a discrete-time recurrent neural network (RNN) model where the state vector $\varkappa_{i+1}$ updates as:*

$$\varkappa_{i+1} = \Psi(\varkappa_i) = \Lambda(W\varkappa_i + b),$$

where $\Lambda : R^\kappa \to R^\kappa$ is a Lipschitz-continuous activation function with constant L_Λ, $W \in R^\kappa \times R^\kappa$ is the weight matrix satisfied $\|W\|_p < \frac{1}{sL_\Lambda}$, and $b \in R^\kappa$ is the bias vector. Then there is a unique equilibrium state $\varkappa^* = \Psi(\varkappa^*)$, ensures network stability.

Proof. To show that $\rho(\varkappa, \upsilon) = \|\varkappa - \upsilon\|_p^2$, for $\varkappa, \upsilon \in R^\kappa, 1 \leqslant p \leqslant 2$, $s = 2^{\frac{2}{p}}$ is a b-metric on R^κ, where

$$\|\varkappa - \upsilon\|_p^2 = \left(\sum_{i=1}^{\kappa} |\varkappa_i - \upsilon_i|^p\right)^{\frac{2}{p}}.$$

It is easy to verify the two axioms of a b-metric. To show the third axiom, by the triangle inequality,

$$\|\varkappa - \upsilon\|_p \leqslant \|\varkappa - z\|_p + \|z - \upsilon\|_p, \text{ for all } \varkappa, \upsilon, z \in R^\kappa.$$

Now, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\rho(\varkappa, \upsilon) = \|\varkappa - \upsilon\|_p^2 &\leqslant \|\varkappa - z\|_p^2 + 2\|\varkappa - z\|_p^2 \|z - \upsilon\|_p^2 + \|z - \upsilon\|_p^2 \\ &\leqslant 2\left[\|\varkappa - z\|_p^2 + \|z - \upsilon\|_p^2\right] \\ &\leqslant 2^{\frac{2}{p}}\left[\|\varkappa - z\|_p^2 + \|z - \upsilon\|_p^2\right] = s[\rho(\varkappa, z) + \rho(z, \upsilon)], 1 \leqslant p \leqslant 2.\end{aligned}$$

Then $(\digamma, \rho)$ b-metric space with $s = 2^{\frac{2}{p}}, 1 \leqslant p \leqslant 2$. Define $\Psi : \digamma \to \digamma$ by $\Psi(\varkappa) = \Lambda(W\varkappa + b)$. Then

$$\begin{aligned}
\rho(\Psi\varkappa, \Psi\upsilon) &= \|\Lambda(W\varkappa + b) - \Lambda(W\upsilon + b)\|_p^2 \\
&\leqslant L_\Lambda^2 \|W(\varkappa - \upsilon)\|_p^2 \\
&\leqslant L_\Lambda^2 \|W\|_p^2 \|\varkappa - \upsilon\|_p^2 \\
&= Q\rho(\varkappa, \upsilon) < Q \max\left\{\rho(\varkappa, \upsilon), \frac{\rho(\varkappa, \Psi\varkappa)\rho(\upsilon, \Psi\upsilon)}{\rho(\varkappa, \upsilon)}, \sigma(\varkappa, \upsilon)\rho(\varkappa, \Psi\upsilon)\rho(\upsilon, \Psi\varkappa)\right\},
\end{aligned} \tag{3.1}$$

where $Q = L_\Lambda^2 \|W\|_p^2 < \frac{1}{s}$, and R^κ with the b-metric is complete, Ψ-orbitally complete, Ψ is continuous due to the continuity of Λ, ensuring orbital continuity. By Theorem 2.4, the RNN has a unique equilibrium state $\varkappa^* = \Psi(\varkappa^*)$. Iterations $\varkappa_{i+1} = \Psi(\varkappa_i)$ converge globally to $\varkappa^*$, guaranteeing stable network dynamics regardless of initial conditions. □

Example 3.6. Let $\digamma = R^2$(state space) with b-metric $\rho(\varkappa, \upsilon) = \|\varkappa - \upsilon\|_2^2$ with coefficient $s = 2$. Define $\Psi : \digamma \to \digamma$ by $\Psi(\varkappa) = \Lambda(W\varkappa + b) = \tanh(W\varkappa + b)$, satisfying Lipschitz-continuous with $L_\Lambda = 1, W = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}$, spectral norm $\|W\|_2 = 0.4 < \frac{1}{s}$, and bias vector $b = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$, $\varkappa_0 = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$. Indeed,

$$\begin{aligned}
&\text{for } \kappa = 1,\ \varkappa_1 = \tanh(W\varkappa_0 + b) = \tanh \begin{bmatrix} 0.3 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0.2913 \\ -0.0997 \end{bmatrix}; \\
&\text{for } \kappa = 2,\ \varkappa_2 = \tanh(W\varkappa_1 + b) = \tanh \begin{bmatrix} 0.2165 \\ 0.0601 \end{bmatrix} = \begin{bmatrix} 0.2133 \\ 0.0600 \end{bmatrix}; \\
&\text{for } \kappa = 3,\ \varkappa_3 = \tanh(W\varkappa_2 + b) = \tanh \begin{bmatrix} 0.1853 \\ 0.1240 \end{bmatrix} = \begin{bmatrix} 0.1833 \\ 0.1234 \end{bmatrix}; \\
&\text{for } \kappa = 4,\ \varkappa_4 = \tanh(W\varkappa_3 + b) = \tanh \begin{bmatrix} 0.1733 \\ 0.1494 \end{bmatrix} = \begin{bmatrix} 0.1717 \\ 0.1485 \end{bmatrix}; \\
&\text{for } \kappa = 5,\ \varkappa_5 = \tanh(W\varkappa_4 + b) = \tanh \begin{bmatrix} 0.1687 \\ 0.1594 \end{bmatrix} = \begin{bmatrix} 0.1673 \\ 0.1583 \end{bmatrix}.
\end{aligned}$$

The right-hand side of (3.1) becomes

$$\max\{\rho(\varkappa_\kappa, \varkappa_{\kappa-1}), \frac{\rho(\varkappa_\kappa, \varkappa_{\kappa+1})\rho(\varkappa_{\kappa-1}, \varkappa_\kappa)}{\rho(\varkappa_\kappa, \varkappa_{\kappa-1})}, \alpha(\varkappa_\kappa, \varkappa_{\kappa-1})\rho(\varkappa_\kappa, \varkappa_\kappa)\rho(\varkappa_{\kappa-1}, \varkappa_\kappa)\} = \rho(\varkappa_\kappa, \varkappa_{\kappa-1}),$$
$$0.16\rho(\varkappa_\kappa, \varkappa_{\kappa-1}) > \rho(\varkappa_{\kappa+1}, \varkappa_\kappa),\ \kappa = 1, 2, 3, \ldots,$$

$Q = 0.16 < \frac{1}{s}$. After 10 iterations, $\varkappa_{10} = \begin{bmatrix} 0.1612 \\ 0.1612 \end{bmatrix}$, and converges to unique fixed point (equilibrium state) $\varkappa = \begin{bmatrix} 0.161 \\ 0.161 \end{bmatrix}$.

Example 3.7. Let $\digamma = [0, 1]^3$(state space) with b-metric $\rho(\varkappa, \upsilon) = \|\varkappa - \upsilon\|_2^2$ with coefficient $s = 2$, and $\Psi(\varkappa) = \Lambda(W\varkappa + b) = \tanh(W\varkappa + b)$, satisfying Lipschitz-continuous with $L_\Lambda = 1, W = \begin{bmatrix} 0.2 & 0.1 & 0.0 \\ 0.1 & 0.2 & 0.1 \\ 0.0 & 0.1 & 0.2 \end{bmatrix}$, and bias vector $b = \begin{bmatrix} 0.05 \\ -0.05 \\ 0.1 \end{bmatrix}$. Indeed, we verify spectral norm:

$$W^T W = \begin{bmatrix} 0.05 & 0.04 & 0.01 \\ 0.04 & 0.06 & 0.04 \\ 0.01 & 0.04 & 0.05 \end{bmatrix},$$

with eigen values $\lambda = 0.116, 0.05, 0.004$, and spectral norm $\|W\|_2 = \sqrt{\max\lambda} = \sqrt{0.116} = 0.34$. Thus $Q = \|W\|_2^2 = 0.116 < \frac{1}{s}$. We verify condition (2.5):

$$\begin{aligned}\rho(\Psi\varkappa, \Psi\upsilon) &= \|\tanh(W\varkappa + b) - \tanh(W\upsilon + b)\|_2^2 \\ &\leqslant \|W(\varkappa - \upsilon)\|_2^2 \\ &\leqslant \|W\|_2^2 \, \|\varkappa - \upsilon\|_2^2 \\ &= 0.116\rho(\varkappa, \upsilon) < Q \max\left\{\rho(\varkappa, \upsilon), \frac{\rho(\varkappa, \Psi\varkappa)\rho(\upsilon, \Psi\upsilon)}{\rho(\varkappa, \upsilon)}, \sigma(\varkappa, \upsilon)\rho(\varkappa, \Psi\upsilon)\rho(\upsilon, \Psi\varkappa)\right\},\end{aligned}$$

condition (2.5) holds for all $\varkappa \neq \upsilon$ in $\digamma$, satisfying Theorem 2.4. This guarantees global convergence to a unique fixed point $\omega \in \digamma$.

4. Conclusion

This article established novel fixed point theorems for Ψ-orbitally continuous mappings in b-metric spaces, extending the foundational results. The theorems were applied to prove the existence and uniqueness solutions for nonlinear integral equations and to analyze the stability of neural networks, demonstrating their adaptability to solving complicated computational and mathematical problems. These theorems can be extended to cone b-metric spaces, and neuro-inspired iterative algorithms can be developed as avenues for future research.

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