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Fixed point iterative algorithm with double inertial steps for solving data classification problems

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Abstract

The aim of this paper is to propose Krasnosel'skii-Mann type iteration with double inertial steps for approximating fixed points of nonexpansive mappings in real Hilbert spaces. The weak convergence is proved under some suitable conditions of the parameters. Some applications to the problems of finding a common fixed point of a family of mappings are also given. Finally, several numerical experiments to show the efficiency and accuracy of our method in breast and cervical cancer diseases predictions are presented.

Keywords: Fixed point, nonexpansive mapping, Hilbert space, weak convergence, data classification problem.

2020 MSC: 47H09, 47H10, 47J05, 47J25.

1. Introduction

Throughout this paper, let us assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively. Let $T : H \to H$ be a mapping. Recall that a mapping $T : H \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in H.$$

Let $T : H \to H$ be a mapping. The *fixed point problem* (shortly, FPP) is to find a point $x^* \in H$ such that

$$x^* = Tx^*. \tag{1.1}$$

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From now on, we denote the fixed points set of T by $F(T) := \{x \in H : x = Tx\}$. It is well-known that numerous problems in optimization theory, such as convex minimization problems, equilibrium problems, variational inequalities, inclusion problems, and split feasibility problems, can be formulated as fixed-point problems of a nonexpansive mapping or as the common fixed point of a family of nonexpansive mappings (see, e.g., [11, 16, 45, 52–54] and references therein).

In recent years, various types of iterative method have been established for solving FPP of nonexpansive mappings in Hilbert spaces by many authors. Next, we shall mention some known methods in the literature for solving FPP (1.1), which motivate us to establish a new iterative method. The *Krasnosel'skii-Mann type iteration* [30, 36] is one of the classical method for solving FPP in a real Hilbert space H. This method is of the form:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geqslant 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$ and T is a nonexpansive mapping. It's worth noting that the Krasnosel'skii-Mann type iteration (1.2) was proved the weak convergence under various relaxed conditions on the parameter $\{\alpha_n\}$. Moreover, the Krasnosel'skii-Mann type iteration has been further modified in various ways and has been implemented to solve many types of optimization. By these reasons, the Krasnosel'skii-Mann type iteration (1.2) has attracted a lot of attention from by many authors (see, e.g., [15, 18, 28, 41, 43, 44, 53]).

In order to speed up the convergence rate of the Krasnosel'skii-Mann type iteration (1.2), Maingé [35] proposed an *Inertial Krasnosel'skii-Mann type iteration* for the fixed point problems of a nonexpansive mapping T as follows:

$$\begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, \quad \forall n \geqslant 1. \end{cases} \tag{1.3}$$

Here the term $\theta_n(x_n - x_{n-1})$ is called the *inertial term (or momentum term)*, which used to accelerate the rate of convergence of the algorithms and θ_n is called the *inertial parameter*, which controls the contribution of the inertial term (see [42]). He also proved the weak convergence theorem under the following conditions:

(C1) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$;
(C2) $\theta_n \in [0, \theta)$ for some $\theta \in [0, 1)$;
(C3) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$.

Note that the condition (C3) is extremely restrictive. This condition makes the algorithm difficult to implement in practical applications, so it would be better to simplify the condition required for its implementation. In this regard, Alvarez and Attouch [3] replaced the condition (C2) with the following condition:

(C2*) $\theta_n \in [0, \theta)$ for some $\theta \in [0, 1/3)$ and $\{\theta_n\}$ is nondecreasing.

Then $\sum_{n=1}^{\infty} \|x_n - x_{n-1}\|^2 < \infty$. In a particular case of the condition (C2*), if the sequence $\{\theta_n\}$ is chosen as a constant in $[0, 1/3)$, then the condition (C3) is automatically satisfied. However, it is observed that the inertial parameter θ_n of the condition (C2) can not be equal to 1. To overcome this limitation, Iyiola et al. [25] proposed a modification of Inertial Krasnosel'skii-Mann type iteration (1.3) for which the inertial parameter $\theta_n = 1$. This method is called *Reflected Krasnosel'skii-Mann type iteration* and it is of the form:

$$\begin{cases} x_0, x_1 \in H, \\ y_n = 2x_n - x_{n-1}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad \forall n \geqslant 1, \end{cases} \tag{1.4}$$

where $T : H \to H$ is a nonexpansive mapping with $F(T) \neq \emptyset$. The weak convergence result is proved under the following condition:

$$0 < \alpha \leqslant \alpha_n \leqslant \alpha_{n+1} \leqslant \frac{1}{2+\epsilon}, \ \epsilon \in (0, \infty).$$

Very recently, Izuchukwu and Shehu [26] combined Inertial Krasnosel'skii-Mann type iteration (1.3) and Reflected Krasnosel'skii-Mann type iteration (1.4), they proposed iterative algorithm, so-called *Inertial Reflected Krasnosel'skii-Mann type iteration*. This method is of the form:

$$\begin{cases} x_0, x_1 \in H, \\ z_n = 2x_n - x_{n-1}, \\ y_n = x_n + \theta(x_n - x_{n-1}), \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T z_n, \quad \forall n \geqslant 1, \end{cases} \tag{1.5}$$

They proved the weak convergence theorem of Inertial Reflected Krasnosel'skii-Mann type iteration (1.5) to a fixed point of T provided $\{\alpha_n\}$ and θ satisfy the following conditions:

(C1′) $0 < \alpha \leqslant \alpha_n \leqslant \alpha_{n+1} \leqslant \frac{1}{1+\epsilon}$, $\epsilon \in (2, \infty)$;

(C2′) $0 \leqslant \theta \leqslant \frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}$, $\epsilon \in (2, \infty)$.

However, the implementation of Inertial Reflected Krasnosel'skii-Mann type iteration (1.5) is limited due to the inertial parameter θ being a fixed parameter.

It was discussed in [19] (see, also [18]) that algorithms with multi-step inertial extrapolation converge significantly faster than those using a single inertial step. In recent years, numerous studies have shown that methods with multi-step inertial extrapolation can significantly accelerate the convergence rate in solving various types of optimization problems (see, e.g., [18, 40, 49, 50, 55]).

Motivated by the aforementioned research works, this paper proposes a double inertial Krasnosel'skii-Mann type iteration involving two sequences of inertial parameters for approximating fixed points of nonexpansive mappings in real Hilbert spaces. In the proposed method, one of the inertial parameters is permitted to be equal to 1, while the other can be chosen arbitrarily close to 1. The weak convergence of the proposed method to a fixed point of the mapping is established under appropriate conditions. We further utilize our main result to solve the fixed point problems of a countable family of nonexpansive mappings and nonexpansive semigroups. Also, the preconditioning algorithm with double inertial steps is obtained from the main result for solving the monotone inclusion problem for the sum of the M-cocoercive operator and monotone operator, where M is a linear bounded operator. Finally, we perform several numerical experiments to illustrate the computational efficiency of our algorithm in solving fixed point problems of a family of mappings and prediction problems of breast and cervical cancer diseases.

This paper is structured as follows. We recall some results and lemmas that will be used in the sequel in Section 2. The rationale behind the algorithm proposed is discussed in Section 3. The main result of this paper is given in Section 4. Applications of our main result are presented in Section 5 and, finally, several numerical results are presented in Section 6.

2. Preliminaries

Throughout this paper, we use I to denote the identity mapping. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. For each $x, y \in H$ and $\alpha \in \mathbb{R}$, the following equalities hold:

$$\begin{aligned} \|x-y\|^2 &= \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \\ \|x+y\|^2 &= \|x\|^2 + 2\langle y, x+y\rangle, \end{aligned} \tag{2.1}$$

and

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.2}$$

Let C be a nonempty, closed and convex subset of H. Then for each $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leqslant \|x - y\|$, for all $y \in C$. Such a P_C is called the *metric* *projection* of H onto C. It is well-known that P_C is a nonexpasive mapping. Moreover, P_C is characterized by the following properties: for each $x \in H$ and $y \in C$,

$$\langle x - P_C(x), y - P_C(x)\rangle \leqslant 0 \quad \text{and} \quad \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \leqslant \|x - y\|^2.$$

Lemma 2.1 ([2]). *Let $\{\varphi_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be three nonnegative real sequences such that*

$$\varphi_{n+1} \leqslant \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \beta_n, \ \forall n \geqslant 1,$$

with $\sum_{n=1}^{\infty} \beta_n < \infty$ and there exists a real number α such that $0 \leqslant \alpha_n \leqslant \alpha < 1$ for all $n \in \mathbb{N}$. Then the following results hold:

(i) $\sum_{n=1}^{\infty} [\varphi_n - \varphi_{n-1}]_+ < \infty$, *where* $[t]_+ := \max\{t, 0\}$;
(ii) *there exists* $\varphi^* \in [0, \infty)$ *such that* $\lim_{n\to\infty} \varphi_n = \varphi^*$.

Lemma 2.2 ([23]). *Let C be nonempty, closed and convex of H and $T : C \to H$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$ for some $x \in H$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.3 ([41]). *Let C be a nonempty subset of H. Let $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(i) $\lim_{n\to\infty} \|x_n - x\|$ *exists for each* $x \in C$;
(ii) *every weak cluster point of* $\{x_n\}$ *is in* C.

Then $\{x_n\}$ converges weakly to a point in C.

3. Motivation from dynamical systems

Following the works in [6, 26], we consider the following implicit second-order dynamical system for the fixed point problems:

$$\ddot{x}(t) + \Big(\gamma(t) + \alpha(t)(1 - \gamma(t))\Big)\dot{x}(t) + \alpha(t)x(t) = \alpha(t)T\Big(\theta(t)\dot{x}(t) + x(t)\Big), \tag{3.1}$$

where $\alpha, \gamma, \theta : [0, \infty) \to [0, \infty)$ are Lebesgue measurable functions and T is a mapping. Now, taking step size $h_n > 0$, set $x_n := x(t_n)$ and $\alpha_n := \alpha_n(t_n)$, $\gamma_n := \gamma_n(t_n)$, $\theta_n := \theta(t_n)$, where $t_n := \sum_{i=1}^{n} h_i$. Moreover, set $\ddot{x}(t) \approx \frac{x_{n-1} - 2x_n + x_{n+1}}{h^2}$ and $\dot{x}(t) \approx \frac{x_n - x_{n-1}}{h}$ in (3.1), we have

$$\frac{x_{n-1} - 2x_n + x_{n+1}}{h_n^2} + \Big(\gamma_n + \alpha_n(1 - \gamma_n)\Big)\Big(\frac{x_n - x_{n-1}}{h_n}\Big) + \alpha_n x_n = \alpha_n T\Big(\theta_n \frac{x_n - x_{n-1}}{h_n} + x_n\Big).$$

Now, we set $h_n = 1$, then we have

$$x_{n-1} - 2x_n + x_{n+1} + \gamma_n(x_n - x_{n-1}) + \alpha_n(1 - \gamma_n)(x_n - x_{n-1}) + \alpha_n x_n = \alpha_n T(\theta_n(x_n - x_{n-1}) + x_n).$$

This implies that

$$\begin{aligned} x_{n+1} &= x_n + x_n - x_{n-1} - \gamma_n(x_n - x_{n-1}) - \alpha_n(1 - \gamma_n)(x_n - x_{n-1}) - \alpha_n x_n + \alpha_n T(x_n + \theta_n(x_n - x_{n-1})) \\ &= (1 - \alpha_n)x_n + (1 - \gamma_n)(x_n - x_{n-1}) - \alpha_n(1 - \gamma_n)(x_n - x_{n-1}) + \alpha_n T(x_n + \theta_n(x_n - x_{n-1})) \\ &= (1 - \alpha_n)(x_n + (1 - \gamma_n)(x_n - x_{n-1})) + \alpha_n T(x_n + \theta_n(x_n - x_{n-1})). \end{aligned} \tag{3.2}$$

In this case, we take $\delta_n := 1 - \gamma_n$. Then (3.2) can be reformulated as

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = x_n + \delta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n T z_n, \end{cases} \tag{3.3}$$

where $\theta_n(x_n - x_{n-1})$ and $\delta_n(x_n - x_{n-1})$ are called *inertial terms* with *inertial parameters* θ_n and δ_n, respectively. Note that such terms intended to accelerate the rate of convergence of the iterative algorithm. We call (3.3) the Krasnosel'skii-Mann type with double inertial steps for the fixed point problems.

4. Main result

In this section, we introduce Krasnosel'skii-Mann type iteration with double inertial steps for solving fixed point problem (FPP) and prove weak convergence theorem of the proposed algorithm in a real Hilbert space. In the sequel, we state the following assumptions.

(A1) H is a real Hilbert space.
(A2) $T : H \to H$ is a nonexpansive mapping.
(A3) $F(T) \neq \emptyset$.
(A4) The sequences $\{\theta_n\}$, $\{\delta_n\}$, $\{\alpha_n\}$ satisfy the following conditions:
  (i) $0 \leqslant \theta_n \leqslant \theta_{n+1} \leqslant 1$;
  (ii) $0 \leqslant \delta_n \leqslant \delta_{n+1} \leqslant \delta < \frac{\epsilon-\sqrt{2\epsilon}}{\epsilon}$, $\epsilon \in (2, \infty)$;
  (iii) $0 < \alpha \leqslant \alpha_n \leqslant \alpha_{n+1} \leqslant \frac{1}{1+\epsilon}$, $\epsilon \in (2, \infty)$;
  (iv) $\theta_n \geqslant \delta_n$ for all $n \geqslant 1$.

Example 4.1. Let $\epsilon = 100$. The sequences listed below satisfy assumption (A4):

$$\theta_n = 1 - \frac{1}{4^n + 17}, \quad \delta_n = 0.8 - \frac{1}{4n^2 + 17}, \quad \text{and} \quad \alpha_n = 0.009 - \frac{1}{1000n + 1}.$$

We now propose our algorithm as below.

Algorithm 4.2. For given $x_0, x_1 \in H$, let $\{x_n\}$ be a sequence generated by the following scheme:

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = x_n + \delta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T z_n, \quad \forall n \geqslant 1. \end{cases}$$

Remark 4.3.

(1) Note that our Algorithm 4.2 may seems like Algorithm (18) of [18] but its conditions (D1) and (D2) which are imposed on the parameter sequences $\{\theta_n\}$, $\{\delta_n\}$, and $\{\alpha_n\}$ of Algorithm (18) of [18] are very rigorous which makes this algorithm not easy to implement in practical applications. However, these conditions (D1) and (D2) are replaced by simpler assumptions in assumption (A4).
(2) If $\theta_n = 1$ and $\delta_n = \theta$, then our Algorithm 4.2 reduces to Inertial Reflected Krasnosel'skii-Mann type iteration proposed in [26, Algorithm (14)].
(3) If $\theta_n = 1$ and $\delta_n = 0$, then Algorithm 4.2 reduces to Reflected Krasnosel'skii-Mann type iteration proposed in [25, Algorithm (11)].
(4) If $\theta_n = \delta_n$, then Algorithm 4.2 reduces to Inertial Krasnosel'skii-Mann type iteration proposed in [35, Algorithm (1.2)].
(5) If $\theta_n = \delta_n = 0$, then Algorithm 4.2 reduces to Krasnosel'skii-Mann type iteration proposed in [30, 36].

Lemma 4.4. *Assume that assumptions* (A1)-(A3) *are satisfied. Let* $\{x_n\}$ *be the sequence generated by Algorithm* 4.2. *Then for each* $p \in F(T)$, *the following inequality holds for all* $n \geqslant 1$,

$$\|x_{n+1} - p\|^2 \leqslant (1 + a_n)\|x_n - p\|^2 - a_n\|x_{n-1} - p\|^2 + b_n\|x_n - x_{n-1}\|^2 - c_n\|x_{n+1} - x_n\|^2,$$

where

$$a_n := (1-\alpha_n)\delta_n + \alpha_n\theta_n,$$

$$b_n := (1-\alpha_n)\delta_n(1+\delta_n) + \alpha_n\theta_n(1+\theta_n) - \frac{(1-\alpha_n)(\delta_n^2-\delta_n)}{\alpha_n},$$

and

$$c_n := \frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n}.$$

Proof. Let $p \in F(T)$. From (2.2), we see that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)y_n + \alpha_n Tz_n - p\|^2 \\ &= \|(1-\alpha_n)(y_n-p) + \alpha_n(Tz_n-p)\|^2 \\ &= (1-\alpha_n)\|y_n-p\|^2 + \alpha_n\|Tz_n-p\|^2 - \alpha_n(1-\alpha_n)\|y_n-Tz_n\|^2 \\ &\leqslant (1-\alpha_n)\|y_n-p\|^2 + \alpha_n\|z_n-p\|^2 - \alpha_n(1-\alpha_n)\|y_n-Tz_n\|^2.\end{aligned} \tag{4.1}$$

Since $x_{n+1} = (1-\alpha_n)y_n + \alpha_n Tz_n$, we have

$$Tz_n - y_n = \frac{1}{\alpha_n}(x_{n+1}-y_n).$$

From (2.1) and the inequality $2\langle x, y\rangle \leqslant \|x\|^2 + \|y\|^2$ for all $x, y \in H$, we get

$$\begin{aligned}\|Tz_n-y_n\|^2 &= \frac{1}{\alpha_n^2}\|x_{n+1}-y_n\|^2 \\ &= \frac{1}{\alpha_n^2}\Big[\|x_{n+1}-x_n\|^2 + \delta_n^2\|x_n-x_{n-1}\|^2 - 2\delta_n\langle x_{n+1}-x_n, x_n-x_{n-1}\rangle\Big] \\ &\geqslant \frac{1}{\alpha_n^2}\Big[\|x_{n+1}-x_n\|^2 + \delta_n^2\|x_n-x_{n-1}\|^2 - \delta_n(\|x_{n+1}-x_n\|^2 + \|x_n-x_{n-1}\|^2)\Big] \\ &= \frac{1}{\alpha_n^2}\Big[(1-\delta_n)\|x_{n+1}-x_n\|^2 + (\delta_n^2-\delta_n)\|x_n-x_{n-1}\|^2\Big].\end{aligned} \tag{4.2}$$

Now, substituting (4.2) into (4.1), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leqslant (1-\alpha_n)\|y_n-p\|^2 + \alpha_n\|z_n-p\|^2 \\ &\quad - \frac{1-\alpha_n}{\alpha_n}\Big[(1-\delta_n)\|x_{n+1}-x_n\|^2 + (\delta_n^2-\delta_n)\|x_n-x_{n-1}\|^2\Big].\end{aligned} \tag{4.3}$$

From (2.2), we see that

$$\begin{aligned}\|y_n-p\|^2 &= \|x_n + \delta_n(x_n-x_{n-1}) - p\|^2 \\ &= \|(1+\delta_n)(x_n-p) - \delta_n(x_{n-1}-p)\|^2 \\ &= (1+\delta_n)\|x_n-p\|^2 - \delta_n\|x_{n-1}-p\|^2 + \delta_n(1+\delta_n)\|x_n-x_{n-1}\|^2\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\|z_n-p\|^2 &= \|x_n + \theta_n(x_n-x_{n-1}) - p\|^2 \\ &= \|(1+\theta_n)(x_n-p) - \theta_n(x_{n-1}-p)\|^2 \\ &= (1+\theta_n)\|x_n-p\|^2 - \theta_n\|x_{n-1}-p\|^2 + \theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2.\end{aligned} \tag{4.5}$$

Substituting (4.4) and (4.5) into (4.3), we get

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\leqslant (1-\alpha_n)\Big[(1+\delta_n)\|x_n-p\|^2-\delta_n\|x_{n-1}-p\|^2+\delta_n(1+\delta_n)\|x_n-x_{n-1}\|^2\Big]\\
&\quad+\alpha_n\Big[(1+\theta_n)\|x_n-p\|^2-\theta_n\|x_{n-1}-p\|^2+\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2\Big]\\
&\quad-\frac{1-\alpha_n}{\alpha_n}\Big[(1-\delta_n)\|x_{n+1}-x_n\|^2+(\delta_n^2-\delta_n)\|x_n-x_{n-1}\|^2\Big]\\
&=(1-\alpha_n)(1+\delta_n)\|x_n-p\|^2-(1-\alpha_n)\delta_n\|x_{n-1}-p\|^2+(1-\alpha_n)\delta_n(1+\delta_n)\|x_n-x_{n-1}\|^2\\
&\quad+\alpha_n(1+\theta_n)\|x_n-p\|^2-\alpha_n\theta_n\|x_{n-1}-p\|^2+\alpha_n\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2 \qquad (4.6)\\
&\quad-\frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n}\|x_{n+1}-x_n\|^2-\frac{(1-\alpha_n)(\delta_n^2-\delta_n)}{\alpha_n}\|x_n-x_{n-1}\|^2\\
&=\Big[1+(1-\alpha_n)\delta_n+\alpha_n\theta_n\Big]\|x_n-p\|^2-\Big[(1-\alpha_n)\delta_n+\alpha_n\theta_n\Big]\|x_{n-1}-p\|^2\\
&\quad+\Big[(1-\alpha_n)\delta_n(1+\delta_n)+\alpha_n\theta_n(1+\theta_n)-\frac{(1-\alpha_n)(\delta_n^2-\delta_n)}{\alpha_n}\Big]\|x_n-x_{n-1}\|^2\\
&\quad-\frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n}\|x_{n+1}-x_n\|^2.
\end{aligned}$$

By the definitions of a_n, b_n, and c_n, then (4.6) can be written shortly as

$$\|x_{n+1}-p\|^2\leqslant(1+a_n)\|x_n-p\|^2-a_n\|x_{n-1}-p\|^2+b_n\|x_n-x_{n-1}\|^2-c_n\|x_{n+1}-x_n\|^2.$$

□

Theorem 4.5. *Assume that assumptions* (A1)-(A4) *are satisfied. Let* $\{x_n\}$ *be the sequence generated by Algorithm 4.2. Then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Proof. From Lemma 4.4, we deduce that

$$\|x_{n+1}-p\|^2\leqslant\|x_n-p\|^2-a_n\|x_{n-1}-p\|^2+b_n\|x_n-x_{n-1}\|^2+a_n\|x_n-p\|^2-c_n\|x_{n+1}-x_n\|^2.$$

Hence

$$\begin{aligned}
&\|x_{n+1}-p\|^2-a_{n+1}\|x_n-p\|^2+b_{n+1}\|x_{n+1}-x_n\|^2\\
&\quad\leqslant\|x_n-p\|^2-a_n\|x_{n-1}-p\|^2+b_n\|x_n-x_{n-1}\|^2+a_n\|x_n-p\|^2-c_n\|x_{n+1}-x_n\|^2\\
&\qquad-a_{n+1}\|x_{n+1}-p\|^2+b_{n+1}\|x_{n+1}-x_n\|^2.
\end{aligned}$$

Now, let

$$\Theta_n:=\|x_n-p\|^2-a_n\|x_{n-1}-p\|^2+b_n\|x_n-x_{n-1}\|^2,\quad\forall n\geqslant1.$$

Then by the definition of Θ_n, we have

$$\Theta_{n+1}\leqslant\Theta_n-(a_{n+1}-a_n)\|x_n-p\|^2-(c_n-b_{n+1})\|x_{n+1}-x_n\|^2. \qquad (4.7)$$

Since $\{\theta_n\}$, $\{\delta_n\}$, $\{\alpha_n\}$ satisfy the assumption (A4), it follows from the definition of a_n that

$$\begin{aligned}
a_{n+1}-a_n&=(1-\alpha_{n+1})\delta_{n+1}+\alpha_{n+1}\theta_{n+1}-(1-\alpha_n)\delta_n-\alpha_n\theta_n\\
&=(1-\alpha_{n+1})\delta_{n+1}-(1-\alpha_n)\delta_n+\alpha_{n+1}\theta_{n+1}-\alpha_n\theta_n\\
&\geqslant(1-\alpha_{n+1})\delta_n-(1-\alpha_n)\delta_n+\alpha_{n+1}\theta_n-\alpha_n\theta_n=(\theta_n-\delta_n)(\alpha_{n+1}-\alpha_n)\geqslant0.
\end{aligned}$$

This together with (4.7) implies that

$$\Theta_{n+1} - \Theta_n \leqslant -(c_n - b_{n+1})\|x_{n+1} - x_n\|^2. \tag{4.8}$$

By our assumption, we see that

$$\begin{aligned}
c_n - b_{n+1} &= \frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n} - (1-\alpha_{n+1})\delta_{n+1}(1+\delta_{n+1}) \\
&\quad - \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1}) + \frac{(1-\alpha_{n+1})(\delta_{n+1}^2 - \delta_{n+1})}{\alpha_{n+1}} \\
&\geqslant \frac{(1-\alpha_{n+1})(1-\delta_{n+1})}{\alpha_{n+1}} + \frac{(1-\alpha_{n+1})(\delta_{n+1}^2 - \delta_{n+1})}{\alpha_{n+1}} \\
&\quad - (1-\alpha_{n+1})\delta_{n+1}(1+\delta_{n+1}) - \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1}) \\
&\geqslant \left(\frac{1-\alpha_{n+1}}{\alpha_{n+1}}\right)(1 - 2\delta_{n+1} + \delta_{n+1}^2) - (1-\alpha_{n+1})\delta_{n+1}(1+\delta_{n+1}) - 2\alpha_{n+1} \\
&\geqslant \left(\frac{1-\alpha_{n+1}}{\alpha_{n+1}}\right)(1-\delta_{n+1})^2 - 2(1-\alpha_{n+1}) - 2\alpha_{n+1} \geqslant \epsilon(1-\delta)^2 - 2 = \epsilon\delta^2 - 2\epsilon\delta + \epsilon - 2.
\end{aligned} \tag{4.9}$$

Let $\sigma := \epsilon\delta^2 - 2\epsilon\delta + \epsilon - 2$. Then from (4.8) and (4.9), we have

$$\Theta_{n+1} - \Theta_n \leqslant -\sigma\|x_{n+1} - x_n\|^2. \tag{4.10}$$

It is easy to see that $\sigma > 0$ if $\delta < \frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}$ with $\epsilon > 2$. Consequently, $\{\Theta_n\}$ is nonincreasing. Clearly, it follows from the definition of b_n that $0 \leqslant b_n \leqslant \mu$, where $\mu := (1-\alpha)\left(2 + \frac{\delta}{\alpha}\right) + \frac{2}{1+\epsilon}$. This implies that $\{b_n\}$ is bounded. Then by the definitions of Θ_n and a_n, we have

$$\begin{aligned}
\|x_n - p\|^2 &= a_n\|x_{n-1} - p\|^2 + \Theta_n - b_n\|x_n - x_{n-1}\|^2 \\
&\leqslant a_n\|x_{n-1} - p\|^2 + \Theta_n \\
&\leqslant a_n\|x_{n-1} - p\|^2 + \Theta_1 \\
&\leqslant \tau\|x_{n-1} - p\|^2 + \Theta_1 \\
&\leqslant \tau(\tau\|x_{n-2} - p\|^2 + \Theta_1) + \Theta_1 \\
&= \tau^2\|x_{n-2} - p\|^2 + \tau\Theta_1 + \Theta_1 \\
&\;\;\vdots \\
&\leqslant \tau^n\|x_0 - p\|^2 + (1 + \tau + \tau^2 + \ldots + \tau^{n-1})\Theta_1 \\
&\leqslant \tau^n\|x_0 - p\|^2 + \frac{\Theta_1}{1-\tau},
\end{aligned} \tag{4.11}$$

where $\tau := (1-\alpha)\delta + \frac{1}{1+\epsilon} < 1$. Again, since $b_n \geqslant 0$ for all $n \geqslant 1$, we have

$$\Theta_{n+1} = \|x_{n+1} - p\|^2 - a_{n+1}\|x_n - p\|^2 + b_{n+1}\|x_{n+1} - x_n\|^2 \geqslant -a_{n+1}\|x_n - p\|^2 \geqslant -\tau\|x_n - p\|^2. \tag{4.12}$$

It then follows from (4.10), (4.11), and (4.12) that

$$\sigma\sum_{n=1}^{k}\|x_{n+1} - x_n\|^2 \leqslant \sum_{n=1}^{k}(\Theta_n - \Theta_{n+1}) = \Theta_1 - \Theta_{k+1} \leqslant \Theta_1 + \tau\|x_k - p\|^2 \leqslant \Theta_1 + \tau^{k+1}\|x_0 - p\|^2 + \frac{\tau\Theta_1}{1-\tau}.$$

Thus

$$\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2 \leqslant \frac{1}{\sigma}\lim_{k\to\infty}\left(\Theta_1+\tau^{k+1}\|x_0-p\|^2+\frac{\tau\Theta_1}{1-\tau}\right)<\infty.$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{4.13}$$

From Lemma 4.4, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leqslant (1+a_n)\|x_n-p\|^2-a_n\|x_{n-1}-p\|^2+b_n\|x_n-x_{n-1}\|^2-c_n\|x_{n+1}-x_n\|^2\\ &\leqslant \|x_n-p\|^2+a_n(\|x_n-p\|^2-\|x_{n-1}-p\|^2)+b_n\|x_n-x_{n-1}\|^2.\end{aligned}$$

Now, we know that $0\leqslant a_n\leqslant\tau<1$ and $\sum_{n=1}^{\infty}b_n\|x_{n+1}-x_n\|^2<\infty$, then by Lemma 2.1, we obtain $\lim_{n\to\infty}\|x_n-p\|^2$ exists. Also, from (4.13), we have

$$\|z_n-x_n\|\leqslant\theta_n\|x_n-x_{n-1}\|\leqslant\|x_n-x_{n-1}\|\to 0 \tag{4.14}$$

and

$$\|y_n-x_n\|\leqslant\delta_n\|x_n-x_{n-1}\|\leqslant\delta\|x_n-x_{n-1}\|\to 0. \tag{4.15}$$

Thus we have

$$\|x_{n+1}-y_n\|\leqslant\|x_{n+1}-x_n\|+\|x_n-y_n\|\to 0.$$

From the definition of x_{n+1}, we see that

$$\|x_{n+1}-y_n\|=\alpha_n\|y_n-Tz_n\|\geqslant\alpha\|y_n-Tz_n\|.$$

Thus

$$\lim_{n\to\infty}\|y_n-Tz_n\|=0. \tag{4.16}$$

It then follows from (4.14), (4.15), and (4.16) that

$$\|z_n-Tz_n\|\leqslant\|z_n-x_n\|+\|x_n-y_n\|+\|y_n-Tz_n\|\to 0. \tag{4.17}$$

Since $\lim_{n\to\infty}\|x_n-p\|^2$ exists, we also have that $\{x_n\}$ is bounded. Then we can assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup q$ for some $q\in H$. Since $\lim_{n\to\infty}\|z_n-x_n\|=0$, we also get $z_{n_k}\rightharpoonup q$. This together with (4.17) and Lemma 2.2 yields that $q\in F(T)$. In summary, we have shown that the assumptions of Lemma 2.3 are hold. Therefore, we conclude that $\{x_n\}$ converges weakly to a point in $F(T)$. This completes the proof. □

Also, the result of Theorem 4.5 still holds when T is a quasi-nonexpansive mapping (T is said to be *quasi-nonexpansive* if $F(T)\neq\emptyset$ and $\|Tx-p\|\leqslant\|x-p\|$ for all $x\in H$ and $p\in F(T)$). Then we obtain the following result:

Theorem 4.6. *Let $T:H\to H$ be a quasi-nonexpansive mapping such that $F(T)\neq\emptyset$ and $I-T$ is demiclosed at zero. Then the sequence $\{x_n\}$ generated by Algorithm 4.2 converges weakly to a fixed point of T.*

5. Some applications

In this section, we give some applications of our main result to fixed point problems of a family of mappings and monotone inclusion problem for the sum of M-cocoercive operator and monotone operator, where M is a linear bounded operator.

5.1. Countable family of nonexpansive mappings

Let C be a subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself. In this case, we denote the common fixed point set of a sequence of such mappings by $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^{\infty}$ satisfies the *AKTT-property* [5] if $\sum_{n=1}^{\infty} \sup_{x\in B} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset B of C.

Next, we give a nice property for a family of mappings, which will be useful for proving the asymptotic regularity property of T.

Lemma 5.1 ([5])**.** *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-property. Then we can define a nonexpansive mapping $T : C \to C$ such that $Tx := \lim_{n\to\infty} T_n x$ for all $x \in C$ and $\lim_{n\to\infty} \sup_{x\in B} \|Tx - T_n x\| = 0$.*

Next, we give some examples of a countable family of nonexpansive mappings which satisfy *AKTT-property*.

Example 5.2. For each $n \geqslant 1$ and $x \in \mathbb{R}$, let $T_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_n x := \left(\frac{3}{4} + \frac{1}{n+3}\right) \sin x, \ \forall x \in \mathbb{R}.$$

It is easy to verify that $\{T_n\}_{n=1}^{\infty}$ is a nonexpansive mapping on $\mathbb{R}$ with $\bigcap_{n=1}^{\infty} F(T_n) = \{0\}$. Moreover, $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-property. Indeed, for each bounded subset C of $\mathbb{R}$ and $x \in C$, we have

$$\sum_{n=1}^{\infty} \sup_{x\in C} |T_{n+1}x - T_n x| = \sum_{n=1}^{\infty} \left(\frac{1}{n+3} - \frac{1}{n+4}\right) \sup_{x\in C} |\sin x| \leqslant \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k+3} - \frac{1}{k+4}\right) < \infty.$$

In this case, we can define a nonexpansive mapping $T : \mathbb{R} \to \mathbb{R}$ by

$$Tx := \lim_{n\to\infty} T_n x = \frac{3}{4} \sin x, \ \forall x \in \mathbb{R},$$

with $F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \{0\}$.

From Theorem 4.5, we obtain the following result for a countable family of nonexpansive mappings.

Theorem 5.3. *Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For given $x_0, x_1 \in H$, let $\{x_n\}$ be a sequence generated by the following scheme:*

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = x_n + \delta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T_n z_n, \ \ \forall n \geqslant 1, \end{cases} \tag{5.1}$$

where $\{\theta_n\}$, $\{\delta_n\}$, and $\{\alpha_n\}$ satisfy the assumption (A4). Suppose, in addition, that $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-property. Let $T : H \to H$ be a mapping defined by $Tx := \lim_{n\to\infty} T_n x$ for all $x \in H$ with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ generated by (5.1) converges weakly to a common fixed point of $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Almost all of the proof can follow from the method of proof in Theorem 4.5. Then we only prove the demiclosedness of T. From Theorem 4.5, we know that $\{z_n\}$ is bounded. Now, from (4.17), we have

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0. \tag{5.2}$$

For each $n \geqslant 1$, we see that

$$\|z_n - Tz_n\| \leqslant \|z_n - T_n z_n\| + \|T_n z_n - Tz_n\| \leqslant \|z_n - T_n z_n\| + \sup_{x\in\{z_n\}} \|T_n x - Tx\|.$$

From (5.2) and since $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-property, thus we get $\lim_{n\to\infty}\|z_n - Tz_n\| = 0$. Therefore, in the same way as in the proof of Theorem 4.5, we can now complete the rest of the proof. □

5.2. *Semigroup of nonexpansive mappings*

It is known that a semigroup of operators plays an important role in the study of continuous and discrete dynamical systems (see [9, 21, 29, 56]).

First, we give some examples of semigroup operator which is the achievement of semigroup theory, provide a powerful tool for solving differential equations. Consider the following initial value problem (or abstract Cauchy problem):

$$\begin{cases} \dot{x}(t) = \mathcal{A}x(t), & t \geqslant 0, \\ x(0) = x, \end{cases} \tag{5.3}$$

where $\mathcal{A}$ be an $n \times n$ matrix and x be an n-vector whose components are unknown function. Then (5.3) has a unique solution of the form $x(t) = e^{t\mathcal{A}}x$ for $t \geqslant 0$, where $e^{t\mathcal{A}}$ is a matrix exponential defined by

$$e^{t\mathcal{A}} := \sum_{k=0}^{\infty} \frac{t^k \mathcal{A}^k}{k!}.$$

Then a family $\{e^{t\mathcal{A}} : t \geqslant 0\}$ is a semigroup of matrices (see [8, 21]).

A one-parameter family $\Gamma = \{T_t : t \geqslant 0\}$ of mappings of C into itself is said to be *nonexpansive semigroup* if it satisfies the following conditions:

(S1) $T_0 x = x$ for all $x \in C$;
(S2) $T_{s+t} = T_s T_t$ for all $s, t \geqslant 0$;
(S3) for each $x \in C$, the mapping $t \mapsto T_t x$ is continuous;
(S4) for each $t \geqslant 0$, T_t is nonexpansive, that is, $\|T_t x - T_t y\| \leqslant \|x - y\|$ for all $x, y \in C$.

We denote the set of all common fixed points of Γ by

$$F(\Gamma) := \bigcap_{t \geqslant 0} F(T_t) = \{x \in H : x = T_t x,\ t \geqslant 0\}.$$

A continuous operator of semigroup $\{T_t : t \geqslant 0\}$ is said to be *uniformly asymptotically regular* if for all $s \geqslant 0$ and any bounded subset C of H such that $\lim_{t\to\infty}\sup_{x\in C}\|T_t x - T_s T_t x\| = 0$.

Next, we give some more examples of a semigroup of operators and matrices.

Example 5.4. This example is modified from [20, Example 3.20]. For each $t \geqslant 0$ and $x = (x_1, x_2, x_3, \ldots) \in \ell_2$, let $T_t : \ell_2 \to \ell_2$ be defined by

$$T_t x := \left(e^{-t}x_1, e^{-2t}x_2, e^{-3t}x_3, \ldots\right),\ \forall x \in \ell_2.$$

It is easy to check that $\Gamma := \{T_t : t \geqslant 0\}$ satisfies the Conditions (S1)-(S3). However, Γ is nonexpansive. Indeed, for any $x, y \in \ell_2$, and using the fact that $e^{-\alpha t} \leqslant 1$ for $\alpha > 0$, $t \geqslant 0$, we have

$$\begin{aligned} \|T_t x - T_t y\|_{\ell_2}^2 &= \left\|\left(e^{-t}(x_1 - y_1), e^{-2t}(x_2 - y_2), e^{-3t}(x_3 - y_3), \ldots\right)\right\|_{\ell_2}^2 \\ &= e^{-2t}(x_1 - y_1)^2 + e^{-4t}(x_2 - y_2)^2 + e^{-6t}(x_3 - y_3)^2 + \cdots \\ &\leqslant (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + \cdots = \|x - y\|_{\ell_2}^2, \end{aligned}$$

which implies that $\|T_t x - T_t y\|_{\ell_2} \leqslant \|x - y\|_{\ell_2}$, $\forall x, y \in \ell_2$. Then Γ is a nonexpansive semigroup on ℓ_2 with $F(\Gamma) = \{(0, 0, 0, \ldots)\}$. Moreover, we can show that Γ is uniformly asymptotically regular.

Example 5.5. For each $t \geqslant 0$ and $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$, let $T'_t : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T'_t x := \frac{1}{\beta} e^{-\beta t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma t) & \sin(\gamma t) \\ 0 & -\sin(\gamma t) & \cos(\gamma t) \end{pmatrix} x,$$

where $\beta \geqslant \sqrt{3}$ and $\gamma \in \mathbb{R}$. It can be checked that $\Gamma' := \{T'_t : t \geqslant 0\}$ satisfies the conditions (S1)-(S3) and Γ' is nonexpansive. For any $x, y \in \mathbb{R}^3$ and $t \geqslant 0$, we have

$$\|T'_t x - T'_t y\| \leqslant \frac{\sqrt{3}}{\beta} e^{-\beta t} \|x - y\| \leqslant \|x - y\|.$$

Then Γ' is a nonexpansive semigroup on $\mathbb{R}^3$ with $F(\Gamma') = \{(x_1, 0, 0)^\top : x_1 \in \mathbb{R}\}$. Moreover, we can show that Γ' is uniformly asymptotically regular.

From Theorem 4.5, we obtain the following result for a nonexpasive semigroup.

Theorem 5.6. *Let $\mathcal{S} := \{T_t : t \geqslant 0\}$ be a nonexpaisve semigorup such that $F(\mathcal{S}) \neq \emptyset$. For given $x_0, x_1 \in H$, let $\{x_n\}$ be a sequence generated by the following scheme:*

$$\begin{cases} z_n = x_n + \theta_n (x_n - x_{n-1}), \\ y_n = x_n + \delta_n (x_n - x_{n-1}), \\ x_{n+1} = (1 - \alpha_n) y_n + \alpha_n T_{t_n} z_n, \quad \forall n \geqslant 1, \end{cases} \tag{5.4}$$

where $\{\theta_n\}$, $\{\delta_n\}$, and $\{\alpha_n\}$ satisfy the assumption (A4), and $\{t_n\}$ is a real sequence in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$. Suppose, in addition, that $\mathcal{S}$ is uniformly asymptotically regular. Then the sequence $\{x_n\}$ generated by (5.4) converges weakly to a common fixed point of $\mathcal{S}$.

Proof. We only prove the demiclosedness of T_t for all $t \geqslant 0$. From Theorem 4.5, we note that $\{z_n\}$ is bounded. From (4.17), we have

$$\lim_{n\to\infty} \|z_n - T_{t_n} z_n\| = 0. \tag{5.5}$$

Then for each $t \geqslant 0$, we have

$$\begin{aligned} \|z_n - T_t z_n\| &\leqslant \|z_n - T_{t_n} z_n\| + \|T_{t_n} z_n - T_t T_{t_n} z_n\| + \|T_t T_{t_n} z_n - T_t z_n\| \\ &\leqslant 2\|z_n - T_{t_n} z_n\| + \sup_{x \in \{z_n\}} \|T_{t_n} x - T_t T_{t_n} x\|. \end{aligned}$$

From (5.5) and since $\mathcal{S}$ is uniformly asymptotically regular, we obtain

$$\lim_{n\to\infty} \|z_n - T_t z_n\| = 0, \ \forall t \geqslant 0.$$

This completes the proof. □

Remark 5.7. As shown in [46, Lemma 2] that if S_τ is a nonexpansive semigorup, then the mapping $\frac{1}{t}\int_0^t S_\tau x d\tau$ for all $x \in H$ and $t > 0$, is a uniformly asymptotically regular nonexpansive semigorup. Therefore the result of Theorem 5.6 holds when $T_{t_n} x := \frac{1}{t_n}\int_0^{t_n} S_\tau x d\tau$ for all $x \in H$.

5.3. *Monotone inclusion problems*

Consider the following *monotone inclusion problem*: find $z \in H$ such that

$$0 \in (A + B)z, \tag{5.6}$$

where $A : H \to H$ and $B : H \to 2^H$ are single and set-valued operators, respectively. We denote the solution set of (5.6) by $(A+B)^{-1}0$. The monotone inclusion problem has many applications in a wide range of science and engineering including image recovery, signal recovery, statistical regression and machine learning (see, e.g., [10, 14, 31] and references therein). Moreover, this problem includes many mathematical problems such as convex minimization problems, variational inequality problems, equilibrium problem, split feasibility problem, linear inverse problems and saddle point problems as special cases (see [1, 13, 48]).

It is known that the classical method to solve the monotone inclusion problem (5.6), when B is $1/L$-cocoercive, is the *forward-backward splitting method* [33] which is defined by the following iterative method:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (I+\lambda_n A)^{-1}(x_n - \lambda_n B x_n), \quad \forall n \geqslant 1, \end{cases} \tag{5.7}$$

where λ_n is a step size parameter. It was proved that the sequence generated by (5.7) converges weakly to an element of $(A+B)^{-1}0$ provided $\lambda_n \in (0, 2/L)$.

Using the concept of inertial technique in [42], Moudafi and Oliny [39] proposed an accelerated iterative method so-called *inertial forward-backward splitting method*. This method is a combination of the inertial method and forward-backward splitting method (5.7). The scheme is as follows:

$$\begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I+\lambda_n A)^{-1}(y_n - \lambda_n B x_n), \quad \forall n \geqslant 1. \end{cases} \tag{5.8}$$

The weak convergence of the sequence generated by (5.8) is proved under $\theta_n \in [0,1)$ is chosen such that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$ and $\lambda_n < 2/L$, where L is the Lipschitz constant of B.

In optimization theory, preconditioning techniques are often used to improve the convergence rate and efficiency of optimization methods. In recent years, Lorenz and Pock [34] proposed a modification of the inertial forward-backward splitting method so-called the *preconditioning inertial forward-backward algorithm* to solve monotone inclusion problem (5.6). The scheme is as follows:

$$\begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I+\lambda_n M^{-1} A)^{-1}(y_n - \lambda_n M^{-1} B y_n), \quad \forall n \geqslant 1, \end{cases} \tag{5.9}$$

where M is linear bounded self-adjoint and positive definite operator. They also studied the weak convergence analysis of the sequence generated by (5.9) under some certain conditions on the parameters. It is noting that the preconditioning inertial forward-backward algorithm (5.9) is reduced to forward-backward algorithm (5.7) when $\theta_n = 0$ and $M = I$.

A bounded linear operator $M : H \to H$ is said to be *self-adjoint* if $M^* = M$, where M^* is the adjoint of operator M. A self-adjoint operator is said to be *positive-definite* if $\langle Mx, x\rangle > 0$ for all $x \in H$ with $x \neq 0$. Let M be a self adjoint, positive and bounded linear operator, then we can define M-*inner product* by

$$\langle x, y\rangle_M := \langle x, My\rangle, \quad \forall x, y \in H.$$

Also, we define the *corresponding* M-*norm* induced from the M-inner product by

$$\|x\|_M^2 := \langle x, Mx\rangle, \quad \forall x \in H.$$

Let C be a nonempty subset of H, $T : C \to H$ be an operator and $M : H \to H$ be a positive-definite operator. Then T is said to be *nonexpansive with respect to* M-*norm* if $\|Tx - Ty\|_M \leqslant \|x - y\|_M$ for all

$x, y \in H$ and it is said to be M-*cocoercive* if $\|Tx - Ty\|^2_{M^{-1}} \leqslant \langle Tx - Ty, x - y\rangle$ for all $x, y \in H$.

Let $M : H \to H$ be a bounded, linear self-adjoint and positive-definite operator, $A : H \to 2^H$ be a maximal monotone operator, $B : H \to H$ be an M-cocoercive operator. If $\lambda \in (0, 1]$, then $(I + \lambda M^{-1}A)^{-1}$ and $I - \lambda M^{-1}B$ are nonexpansive operators with respect to M-norm (see [17]). Let $\lambda \in (0, 1]$, we can define a nonexpansive operator with respect to M-norm by

$$T^{A,B}_{\lambda,M} := (I + \lambda M^{-1}A)^{-1}(I - \lambda M^{-1}B)$$

with $F(T^{A,B}_{\lambda,M}) = (A + B)^{-1}0$. From Theorem 4.5, we obtain the following the convergent result for preconditioning algorithm with double inertial steps.

Theorem 5.8. *Let $M : H \to H$ be a bounded linear self-adjoint and positive-definite operator, $A : H \to 2^H$ be a maximal monotone operator, $B : H \to H$ be an M-cocoercive operator such that $(A + B)^{-1}0 \neq \emptyset$. For given $x_0, x_1 \in H$, let $\{x_n\}$ be a sequence generated by the following scheme:*

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = x_n + \delta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n T^{A,B}_{\lambda,M} z_n, \quad \forall n \geqslant 1, \end{cases} \tag{5.10}$$

where $\{\theta_n\}$, $\{\delta_n\}$, and $\{\alpha_n\}$ satisfy the assumption (A4), *and $\lambda \in (0, 1]$. Then the sequence $\{x_n\}$ generated by* (5.10) *converges weakly to a point of $(A + B)^{-1}0$.*

6. Numerical experiments

In this section, we present several numerical experiments to show the performance and advantage of our Algorithm 4.2 (Double Inertial Krasnosel'skii-Mann type iteration) (namely, DIKM) and compare the performance of it with inertial reflected Krasnosel'skii-Mann type iteration proposed in [26, Algorithm (14)] (namely, IRKM), Reflected Krasnosel'skii-Mann type iteration proposed in [25, Algorithm (11)] (namely, RKM) and Inertial Krasnosel'skii-Mann type iteration proposed in [35, Algorithm (1.2)] (namely, IKM).

6.1. Numerical experiments for fixed point problems

Example 6.1. Let $H = \mathbb{R}$ with absolute value norm. For each $n \geqslant 1$ and $x \in \mathbb{R}$, let $T_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_n x := \left(\frac{3}{4} + \frac{1}{n+3}\right)\sin x, \ \forall x \in \mathbb{R}.$$

As shown in Example 5.2, that a family of mappings $\{T_n\}_{n=1}^{\infty}$ is nonexpansive on $\mathbb{R}$ and it satisfies the AKTT-property with $F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \{0\}$. In this experiment, we use $\epsilon = 4$ for our algorithm, RKM, and IRKM. The parameters of each algorithm are chosen as seen in Table 1.

Table 1: Chosen parameters of each algorithm for Example 6.1.

Algorithms	θ	θ_n	δ_n	α_n
IKM	-	0.25	-	$0.2 - \frac{1}{100n+1}$
RKM	-	-	-	$0.15 - \frac{1}{100n+1}$
IRKM	0.25	-	-	$0.2 - \frac{1}{100n+1}$
DIKM	-	$0.5 - \frac{1}{4^n+17}$	$0.29 - \frac{1}{4n^2+17}$	$0.2 - \frac{1}{100n+1}$

The initial points x_0, x_1 are generated randomly in $\mathbb{R}$ and we use $E_n := |x_n - 0| \leqslant \bar{\epsilon}$ to measure the iteration error of all the algorithms, where $\bar{\epsilon} \in \{10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}\}$. The numerical results for each $\bar{\epsilon}$ are shown in Table 2 and Figure 1.

Table 2: Numerical results of all algorithms for Example 6.1.

Algorithms	$\bar{\epsilon} = 10^{-4}$		$\bar{\epsilon} = 10^{-5}$		$\bar{\epsilon} = 10^{-6}$		$\bar{\epsilon} = 10^{-7}$	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
IKM	143	0.0509	164	0.0734	210	0.0627	249	0.0970
RKM	212	0.0545	337	0.1130	420	0.1388	531	0.2160
IRKM	121	0.0701	151	0.0683	179	0.0907	206	0.1184
DIKM	103	0.0493	112	0.0571	144	0.0659	166	0.0712

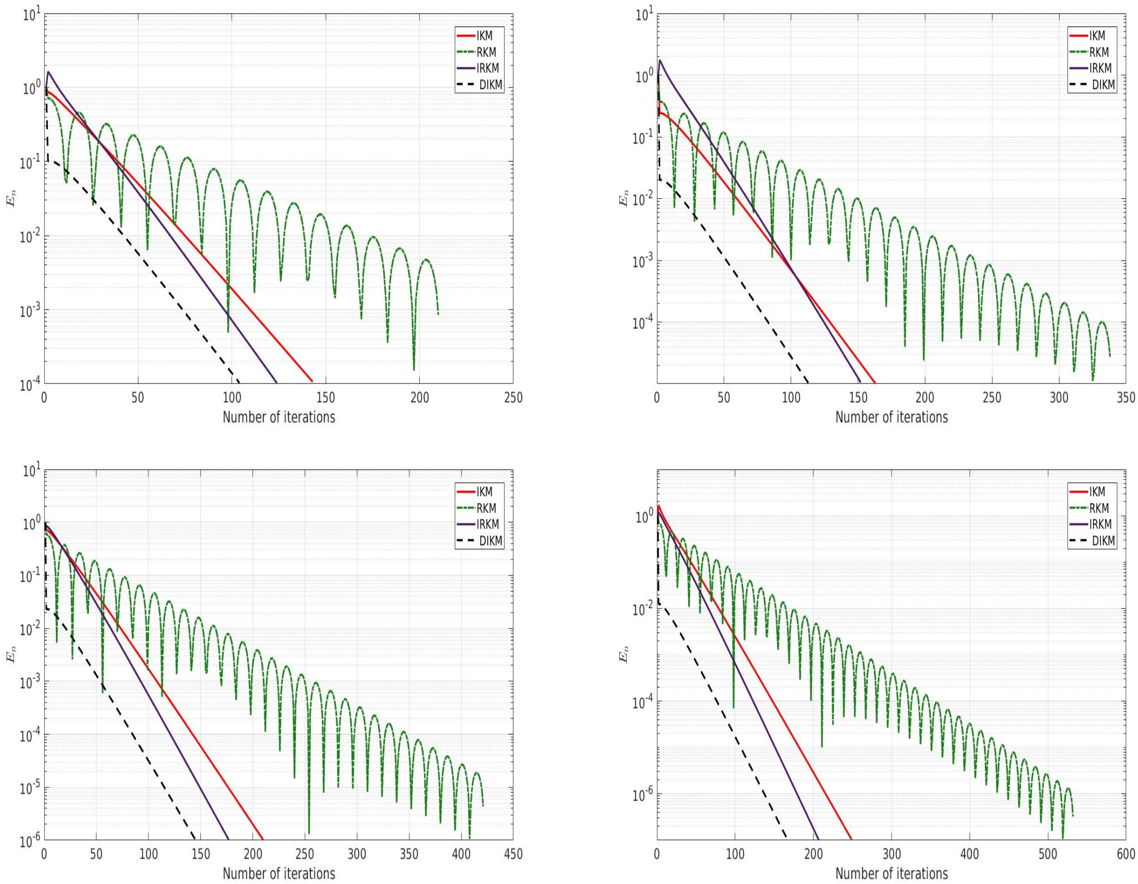

Figure 1: Numerical results for Example 6.1. Top Left: $\bar{\epsilon} = 10^{-4}$; Top Right: $\bar{\epsilon} = 10^{-5}$; Bottom Left: $\bar{\epsilon} = 10^{-5}$; Bottom Right: $\bar{\epsilon} = 10^{-7}$.

Example 6.2. Let $H = \ell_2$ with norm $\|x\|_{\ell_2} = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$. For each $t \geqslant 0$ and $x \in \ell_2$, let $T_t : \ell_2 \to \ell_2$ be defined by

$$T_t x := \left(e^{-t}x_1, e^{-2t}x_2, e^{-3t}x_3, \dots\right), \ \forall x \in \ell_2.$$

As shown in Example 5.4, a family of mappings $\Omega := \{T_t : t \geqslant 0\}$ is nonexpansive on ℓ_2 with $F(\Omega) = \{0^* = (0,0,0,\dots)\}$. All the parameters are chosen as the same as in Table 1. The initial points of this Example are generated randomly in ℓ_2. We use $E_n := \|x_n - 0^*\|_{\ell_2} \leqslant 10^{-8}$ to measure the iteration error of all the algorithms. The four cases of t_n are considered for numerical performing as follows:

Case I: $t_n = \sqrt{n}$.
Case I: $t_n = 2n$.
Case I: $t_n = n^2 + 1$.
Case I: $t_n = n!$.

The numerical results for each case are shown in Table 3 and Figure 2.

Table 3: Numerical results of all algorithms for Example 6.2.

Algorithms	Case I		Case II		Case III		Case IV	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
IKM	97	0.0132	71	0.0210	68	0.0222	95	0.0365
RKM	246	0.0426	187	0.0656	172	0.0297	229	0.0558
IRKM	88	0.0221	67	0.0307	62	0.0116	85	0.0293
DIKM	77	0.0206	57	0.0111	55	0.0178	75	0.0173

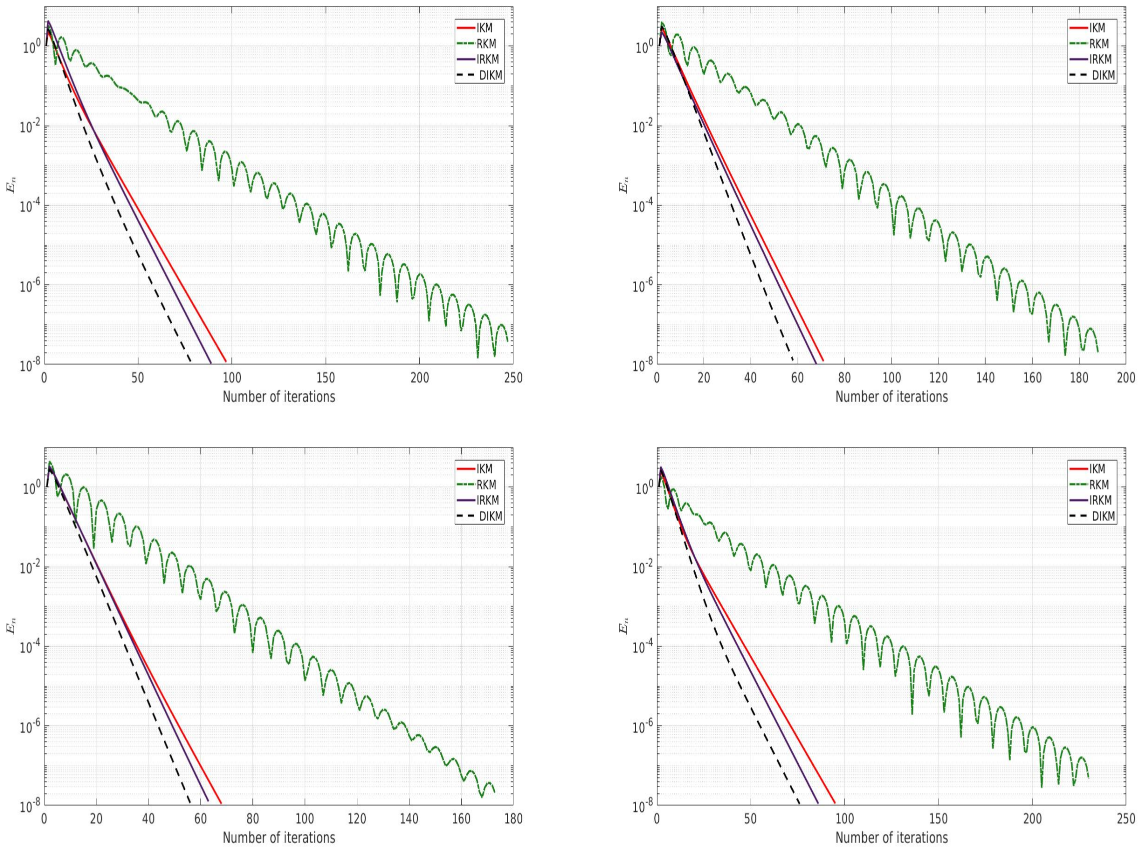

Figure 2: Numerical results for Example 6.1. Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

Remark 6.3. From the numerical results of both Examples 6.1 and 6.2, we see that our algorithm has a high superiority and efficiency than IKM, RKM and IRKM for solving the fixed point problems of a family of mappings in the sense that it requires a fewer number of iterations per time step.

6.2. Numerical experiments for data classifications

The Extreme Learning Machine (ELM) [24] is one of the efficient techniques for solving the data classification problems in machine learning. We first present a basic concept of ELM for such a problem and use apply ELM with our fixed pint algorithm to solve the problem of predictions of some cancers through numerical experiments.

Let $P := \{(x_n, t_n) : x_n \in \mathbb{R}^p, t_n \in \mathbb{R}^q, n = 1, 2, ..., N, p, q \in \mathbb{N}\}$ be a training set of N distinct samples, where x_n is an input training data and t_n is a target. The output function of ELM for single-hidden layer feed forward neural networks (SLFNs) with $\mathcal{S}$ hidden nodes defined by

$$O_j = \sum_{i=1}^{\mathcal{S}} \omega_i \mathcal{A}(\langle w_i, x_j \rangle + b_i),$$

where $\mathcal{A}$ is an activation function, w_i is a weight and b_i is a bias. To find the optimal output weight w_i at the i-th hidden node, we define the hidden layer output matrix F by

$$F = \begin{bmatrix} \mathcal{A}(\langle w_1, x_1\rangle + b_1) & \cdots & \mathcal{A}(\langle w_S, x_1\rangle + b_S) \\ \vdots & \ddots & \vdots \\ \mathcal{A}(\langle w_1, x_N\rangle + b_1) & \ldots & \mathcal{A}(\langle w_S, x_N\rangle + b_S) \end{bmatrix}.$$

The main goal of ELM is to find optimal output weight $\omega = [\omega_1, \omega_2, ..., \omega_S]^T$ such that

$$F\omega = G, \tag{6.1}$$

where $G = [t_1, t_2, ..., t_N]^T$ is the training target data. In general cases, finding $\omega = F^{\ddagger}G$, where $F^{\ddagger}$ denotes the Moore-Penrose generalized inverse of F may be difficult when the matrix F does not exist. It is known that (6.1) can be formulated as the following convex minimization problem:

$$\min_{\omega}\{f(\omega) + g(\omega)\}, \tag{6.2}$$

where $f(\omega) = \frac{1}{2}\|F\omega - G\|_2^2$ and $g(\omega) = \lambda\|\omega\|_1$, $\lambda > 0$. We denote the set of solutions of (6.2) by P. Then from [14, Proposition 3.1 (iii)(b)], we know that

$$\omega \in P \Leftrightarrow 0 \in \nabla f(\omega) + \partial g(\omega) \Leftrightarrow \omega = \mathrm{Prox}_{\gamma\partial g}(\omega - \gamma\nabla f(\omega)),\ \gamma > 0,$$

where $\mathrm{Prox}_{\gamma\partial g}$ is the resolvent of ∂g defined by $\mathrm{Prox}_{\gamma\partial g} := (I + \gamma\partial g)^{-1}$. We know that $\mathrm{Prox}_{\gamma\partial g}(I - \gamma\nabla f)$ is a nonexpansive mapping if $\gamma \in (0, 2/L)$, where L is the Lipschitz constant of ∇f. In this case, $L = \|F\|^2$. Therefore the proposed algorithm can be applied to solve the problem of predictions when we set

$$Tx := \mathrm{Prox}_{\gamma\partial g}(I - \gamma\nabla f)x.$$

To evaluate the quality of the algorithms in terms of prediction performance, the following metrics are used [7, 38]:

$$\text{Accuracy} = \frac{T_P + T_N}{T_P + F_P + T_N + F_N} \times 100\%, \qquad \text{Precision} = \frac{T_P}{T_P + F_P} \times 100\%,$$
$$\text{Recall} = \frac{T_P}{T_N + F_N} \times 100\%, \qquad \text{F1-score} = \frac{2 \times \text{Precision} \times \text{Recall}}{\text{Precision} + \text{Recall}},$$

where T_P represents true positive, T_N represents true negative, F_P represents false positive, and F_N represents false negative. Binary cross-entropy is a mathematical concept and a loss function used in various machine learning and deep learning applications, especially in binary classification problems. It measures the dissimilarity, often called 'cross-entropy,' between the predicted probabilities and the actual binary labels (0 or 1). The following formula defines as follows:

$$\text{Loss} = -\frac{1}{N}\sum_{i=1}^{N}\Big(y_i \cdot \log(p_i) + (1 - y_i) \cdot \log(1 - p_i)\Big),$$

where N is the number of scalar values in the model output, y_i is the corresponding target value, p_i is the i-th scalar value in the model output.

Example 6.4. In this experiment, our aim is to predict breast cancer disease using a dataset available at [51]. Breast cancer occurs when cells in the breast undergo mutations, transforming into cancerous cells that proliferate and develop tumors. While breast cancer primarily impacts women and individuals who are aged 50 and above, it can also affect men and younger women [61]. Breast cancer stands as the predominant form of cancer globally, resulting in 685,000 deaths in 2020 [63]. The survival rates for breast cancer differ depending on the stage of the cancer and can be impacted by various factors including age, race, and treatment choices. For women diagnosed with stage 1 breast cancer, the 5-year overall survival rate was reported to be 90%. This indicates that 90% of women diagnosed with stage 1 breast cancer survive for at least 5 years following diagnosis [58]. The advantage of breast cancer prediction systems lies in their capacity to employ machine learning techniques, amalgamating numerous risk factors to facilitate early disease detection. Such systems aid in formulating essential care strategies and enhancing disease management [60]. Additionally, clinical decision support systems and prognostic models offer valuable insights into breast cancer prognosis and treatment efficacy, thereby fostering better patient outcomes and informed decision-making.

The dataset used in this experiment contains 11 attributes, with the first being the ID, which we will remove since it is not a feature we want to include in our classification. The breast cancer dataset comprises 699 instances, with 458 classified as benign and 241 as malignant. Any missing instances have been removed to improve the system's accuracy, resulting in a dataset of 683 instances used for feature selection. A detailed description of the breast cancer dataset is provided in Table 4.

Table 4: Overview of breast cancer data.

Description of attributes	Domain	Missing values	Mean	Median	Standard deviation
Sample code number	ID number	0	-	-	-
			Input		
Clump thickness	1-10	0	4.4422	4	2.8208
Uniformity of cell size	1-10	0	3.1508	1	3.0651
Uniformity of cell shape	1-10	0	3.2152	1	2.9886
Marginal adhesion	1-10	0	2.8302	1	2.8646
Single epithelial cell size	1-10	16	3.2343	2	2.2231
Bare Nuclei	1-10	0	3.5447	1	3.6439
Bland Chromatin	1-10	0	3.4451	3	2.4497
Normal Nucleoli	1-10	0	2.8697	1	3.0527
Mitoses	1-10	0	1.6032	1	1.7327
			Output		
Class	2 := benign 4 := malignant	0	1.3499	1	0.4773

We partition the dataset into 70% for training and 30% for testing. The activation function is sigmoid, regularization parameter $\lambda = 10^{-5}$, $\gamma = \frac{0.99}{\|F\|^2}$, $\epsilon = 4$, and $\mathcal{S} = 270$. The parameters of each algorithm are chosen as seen in Table 5.

Table 5: Chosen parameters of each algorithm for breast cancer data.

Algorithms	θ	θ_n	δ_n	α_n
IKM	-	0.18	-	$0.19 - \frac{1}{100n+1}$
RKM	-	-	-	$0.15 - \frac{1}{100n+1}$
IRKM	0.18	-	-	$0.2 - \frac{1}{100n+1}$
DIKM	-	$0.9 - \frac{1}{4n+1}$	$0.25 - \frac{1}{4n^2+1}$	$0.2 - \frac{1}{100n+1}$

The breast cancer prediction results are presented in Table 6.

Table 6: Comparison of the performance with each algorithm for breast cancer data.

Algorithm	Iteration No.	Training time	Precision	Recall	F1-score	Accuracy
IKM	971	0.0300	95.83	95.83	95.83	97.07
RKM	961	0.0196	95.83	95.83	95.83	97.07
IRKM	911	0.0204	95.83	95.83	95.83	97.07
DIKM	**807**	**0.0115**	**95.89**	**97.22**	**96.55**	**97.56**

Remark 6.5. From Table 6, we see that DIKM has a precision, recall, F1-score and accuracy more than IKM, RKM and IRKM. Moreover, it also has the lowest number of iterations and training time. This mean that our algorithm has the highest probability of correctly classifying breast cancer compared to other existing algorithms in terms of the overall performance measurement.

Next, we compare our algorithm with some machine learning algorithms in terms of accuracy using the same set of information. The results are presented in Table 7.

Table 7: Comparison of the performance with some machine learning algorithms.

References	Algorithms	Validation	Accuracy
	Decision Tree (DT)		96.10
	Random Forest (RF)		96.10
Li et al. [32]	Support Vector Machine (SVM)	train (70%), test (30%)	95.10
	Neural Network (NN)		95.60
	Logistic Regression (LR)		93.70
Amrane et al. [4]	k-Nearest Neighbors (kNN)	k-fold cross validation (k = 5)	97.51
	Naive Bayes (NB)		96.19
Proposed algorithm	**DIKM**	**train** (70%)**, test** (30%)	**97.56**

Remark 6.6. As can be seen from Table 7 that our algorithm has the highest efficiency in accuracy, establishing it as the most accurate predictor of breast cancer.

Moreover, we also present the graphs of accuracy and loss for both the training and validation data to assess the potential overfitting of our algorithm.

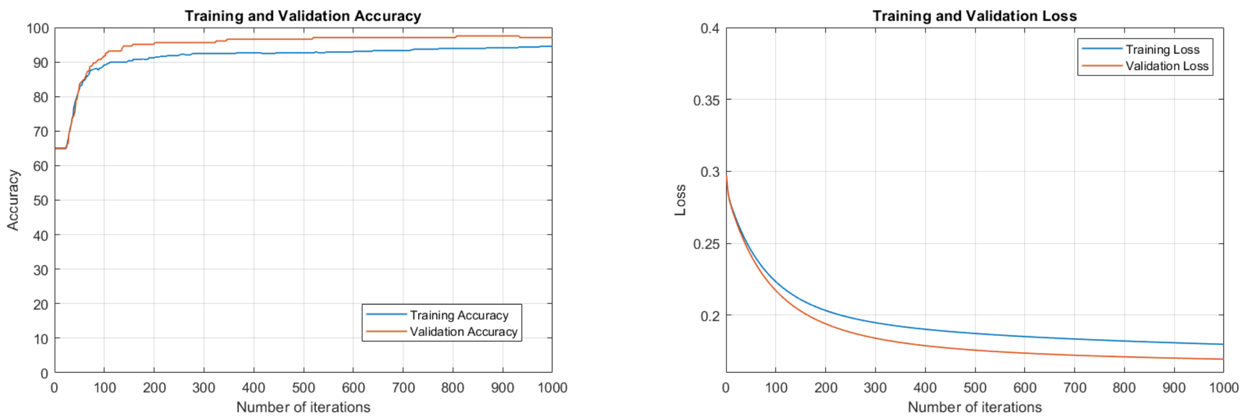

Figure 3: Accuracy and Loss plots of proposed algorithm for breast cancer data.

Remark 6.7. From Figure 3, it is evident that training loss and validation loss values tend to decrease until a certain point, after which they stabilize. In contrast, upon analyzing the accuracy graph, it becomes apparent that training and validation accuracy show an upward trend, with validation accuracy consistently surpassing training accuracy.

Example 6.8. In this experiment, we aim to predict cases of cervical cancer disease using a particular dataset available through [12]. Cervical cancer is a serious disease that starts in the cells of the cervix, the lower part of the uterus that connects to the vagina. It's important to note that most cases of cervical cancer are caused by the human papillomavirus (HPV), which is a common sexually transmitted infection. This means that anyone who has ever been sexually active can potentially be at risk for developing cervical cancer [37]. Cervical cancer remains a significant concern worldwide and continues to be a leading cause of deaths among women globally. An estimated 341,000 women lose their lives to cervical cancer annually. Furthermore, there were approximately 604,000 new cases reported in 2020 [57], highlighting the urgent need for increased awareness, prevention, and access to screening and treatment options. When cervical cancer is detected early and remains confined to the cervix, the chances of successful treatment and a positive outlook are generally much higher. In fact, for cases where the cancer is localized, the 5-year survival rate is often quite promising, ranging from 92% to 93% [59]. This means that most individuals diagnosed with early-stage cervical cancer have a good chance of surviving for at least five years after diagnosis. As cervical cancer progresses to more advanced stages, the 5-year survival rate decreases significantly, typically falling between 15% to 65% [59]. This emphasizes the importance of early detection through regular screenings, as it greatly improves the chances of successful treatment and a positive outcome for individuals diagnosed with cervical cancer. Prediction systems for cervical cancer enable personalized approaches to prevention, screening, and treatment, serving as invaluable tools for healthcare providers to make informed decisions about patient care [62]. These systems assist healthcare professionals in recommending appropriate screening intervals, follow-up testing, and treatment options based on the individual's risk profile. This enhances the quality of care and empowers patients to take proactive steps towards managing their cervical health.

The data set used in this experiment is relatively small, consisting of 72 instances or records with 19 attributes, one of which is the class attribute 'ca_cervix.' of the 72 samples, 21 were categorized as positive (has cervical cancer), while the remaining 51 were negative (no cervical cancer). Notably, the dataset contained no missing values, and all attributes, including the class variable, were in int64 format. Therefore, no encoding was necessary for their utilization.

A detailed description of the cervical cancer behavior risk dataset is provided in Table 8.

Table 8: Overview of cervical cancer behavior risk data.

Description of attributes	Max	Min	Mean	Median	Standard deviation
			Input		
behavior_sexualRisk	10	2	9.6667	10	1.1868
behavior_eating	15	3	12.7917	13	2.3613
behavior_personalHygine	15	3	11.0833	11	3.0338
intention_aggregation	10	2	7.9028	10	2.7381
intention_commitment	15	6	13.3472	15	2.3745
attitude_consistency	10	2	7.1806	7	1.5228
attitude_spontaneity	10	4	8.6111	9	1.5157
norm_significantPerson	5	1	3.1250	3	1.8457
norm_fulfillment	15	3	8.4861	7	4.9076
perception_vulnerability	15	3	8.5139	8	4.2757
perception_severity	10	2	5.3889	4	3.4007
motivation_strength	15	3	12.6528	14	3.2072
motivation_willingness	15	3	9.6944	11	4.1304
socialSupport_emotionality	15	3	8.0972	2	4.2432
socialSupport_appreciation	10	2	6.1667	6.5	2.8973
socialSupport_instrumental	15	3	10.3750	12	4.3165
empowerment_knowledge	15	3	10.5417	12	4.3668
empowerment_abilities	15	3	9.3194	10	4.1819
empowerment_desires	15	3	10.2778	11	4.4823
			Output		
ca_cervix	1	0	0.2917	0	0.4577

We partition the dataset into 80% for training and 20% for testing. All the parameters are chosen as the same as in Table 5. The cervical cancer prediction results are presented in Table 9.

Table 9: Comparison of the performance with each algorithm for cervical cancer behavior risk data.

Algorithms	Iteration No.	Training time	Precision	Recall	F1-score	Accuracy
IKM	304	0.0041	83.33	100.00	90.91	85.71
RKM	311	0.0042	83.33	100.00	90.91	85.71
IRKM	241	0.0040	83.33	100.00	90.91	85.71
DIKM	**177**	**0.0038**	**83.33**	**100.00**	**90.91**	**85.71**

Remark 6.9. From Table 9, we see that IKM, RKM, IRKM, and DIKM are equal of precision, recall, F1-score and accuracy. However, DIKM has the lowest number of iterations and training time. This mean that our algorithm still outperforms other existing algorithms.

Next, we compare our algorithm with some machine learning algorithms in terms of accuracy using the same set of information. The results are presented in Table 10.

Table 10: Comparison of the performance with some machine learning algorithms.

References	Algorithms	Validation	Accuracy
Tarakci et al. [47]	k-Nearest Neighbors (kNN)	k-fold cross validation $(k = 10)$	84.70
	Decision trees J48		70.83
Ghanem et al. [22]	RadSVM	k-fold cross validation $(k = 5)$	70.83
	Statistical implicative analysis (SIA)		80.56
Proposed algorithm	**DIKM**	**train** (80%), **test** (20%)	**85.71**

Remark 6.10. From Table 10, we see that our algorithm has the highest efficiency in accuracy, establishing it as the most accurate predictor of cervical cancer behavior risk.

Next, we also present the graphs of accuracy and loss for both the training and validation data to assess the potential overfitting of our algorithm.

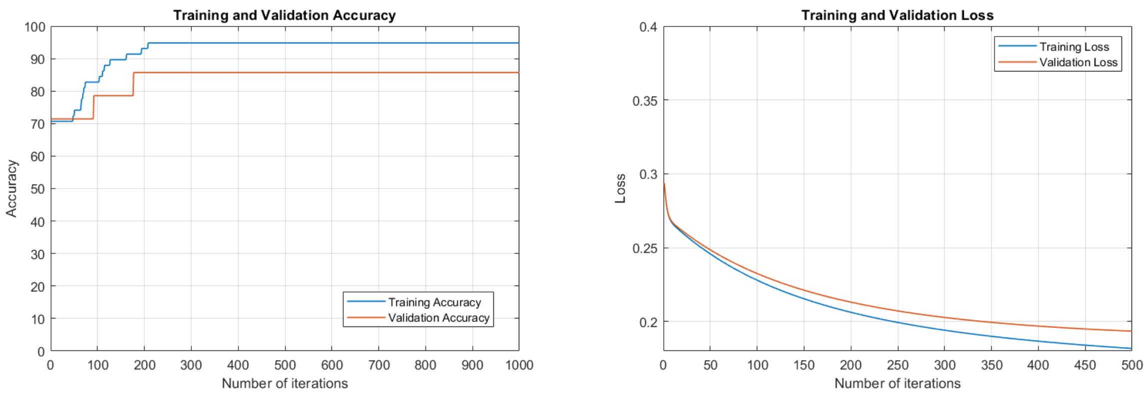

Figure 4: Accuracy and Loss plots of proposed algorithm for cervical cancer behavior risk data.

Remark 6.11. As can be seen from Figure 4, it is clear that the training loss and validation loss values tend to decrease up to a certain point, after which they remain constant. Conversely, when analyzing the accuracy graph, it is evident that both training and validation accuracies show an upward trend, with training accuracy surpassing validation accuracy.

7. Conclusions

In this paper, we proposed a Krasnosel'skii-Mann-type iteration with double inertial steps for finding fixed points of nonexpansive mappings in the framework of Hilbert spaces. The weak convergence of the proposed method was established under suitable conditions imposed on the inertial parameters.

Furthermore, we demonstrated applications of the method to fixed point problems of a countable family of nonexpansive mappings, semigroups of nonexpansive mappings, and the monotone inclusion problem. Finally, several numerical experiments were conducted to illustrate that the proposed method is effective and competitive compared to some existing algorithms presented in [25, 26, 35].

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