J. Math. Computer Sci., 41 (2026), 25–33

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

A unified analytical treatment for solving delay differential equations: exact solution

Essam R. El-Zahar[a], Abdelhalim Ebaid[b,*], Laila F. Seddek[a]

[a]Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, P.O. Box 83, Al-Kharj 11942, Saudi Arabia.

[b]Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia.

Abstract

The delay differential equations (DDEs) are widely used to explore various engineering and physical applications. An example of DDEs with proportional delays is known as the pantograph model which governs the current collection in electric trains. DDEs with constant delays also have different applications. This paper introduces a unified approach to analyze a class of first order DDEs under arbitrary history functions (HFs). The proposed approach assumes that the arbitrary HF $\phi(t)$ can be represented as Maclaurin series with coefficients ϕ_m, $m \geqslant 0$. Based on this assumption, the solution in each sub-interval of the problem's domain is obtained in explicit form in terms of the coefficients ϕ_m. Exact solutions are obtained for several examples subjected to history functions of different forms. Properties of the solution and its derivative are proved and examined theoretically. Existing results in the literature are derived from the current ones as special cases. In view of the obtained results, the exact solution of any first order linear delay differential equation can be directly determined once the coefficients ϕ_m of the given history function is inserted into the standard solution. This reflects the advantage of the proposed approach over other techniques. Moreover, the suggested analysis can be easily extended to include higher order linear delay models.

Keywords: Ordinary differential equation, delay, initial value problem.

2020 MSC: 34K06, 34K07.

1. Introduction

The delay differential equations (DDEs) are of great importance due to their applications in various fields [18, 25, 26, 35, 36, 38]. DDEs have been used to model the nutrient-autotroph-herbivore interaction with nutrient recycling [29] and the nutrient–plankton system [34]. Other kinds of DDEs with a proportional delay parameter and functional differential equations including the pantograph model have applications in railways electrification [1, 8, 12, 20] and electric trains/locomotive [3, 19, 22, 23, 31]. A well-known special case of the pantograph model is the so called Ambartsumian equation [6, 7, 16, 27, 33], which concerns with the surface brightness in the Milky Way. This paper presents a new approach to treat the basic DDE:

$$y'(t) = \alpha y(t) + \beta y\left(t - \tau\right), \quad t \geqslant 0, \tag{1.1}$$

*Corresponding author

Email address: aebaid@ut.edu.sa (Abdelhalim Ebaid)

doi: 10.22436/jmcs.041.01.03

Received: 2025-04-18 *Revised:* 2025-05-24 *Accepted:* 2025-06-21

subject to

$$y(t) = \phi(t) = \sum_{m=0}^{\infty} \phi_m t^m, \quad t \in [-\tau, 0], \tag{1.2}$$

where $\alpha, \beta \in \mathbb{R}$ and $\tau \geqslant 0$. In Eq. (1.2), the history function (HF) is considered in the form of Maclaurin expansion of a given HF $\phi(t)$ in terms of the coefficients ϕ_m.

Although the Adomian decomposition method (ADM) [2, 5, 13, 14, 28, 39], the regular perturbation method [17], the homotopy perturbation method [10, 11, 15] and its modification [21, 32], and the Laplace transform (LT) [4, 24], have been found effective to solve various models, their applications on DDEs are relatively rare. Also, there are a number of analytical approaches which have been applied to solve other scientific models in different areas [9, 30, 37]. However, the current procedure facilitates the derivation of the explicit solution for Eqs. (1.1)-(1.2) under any given HF once the coefficients ϕ_m are specified. As will be seen later, the solution is to be determined explicitly in sequential intervals in terms of ϕ_m and in closed series form utilizing the method of steps (MoS). It will also be shown that such closed series forms convert to certain entire functions once the coefficients ϕ_m of the given HF are inserted.

For examples, if the HF $\phi(t)$ is given as e^t or e^{-t} then ϕ_m are $1/m!$ or $(-1)^m/m!$, respectively. For polynomials HF such as t, then $\phi_1 = 1$, $\phi_m = 0$ for all $m > 1$ in addition to $m = 0$. For $\phi(t) = t^2$, we have $\phi_2 = 1$, $\phi_m = 0$ for all $m > 2$ in addition to $m = 0, 1$. In a trigonometric form such as $\phi(t) = \cos t$, we have $\phi_{2m} = (-1)^m/(2m)!$ and $\phi_{2m+1} = 0$, $\forall m \geqslant 0$ while for $\phi(t) = \sin t$, we have $\phi_{2m} = 0$ and $\phi_{2m+1} = (-1)^m/(2m+1)!$, $\forall m \geqslant 0$. Similarly, for other HFs one can specify the coefficients ϕ_m and then obtain the explicit solution as will be demonstrated latter.

Although the problem (1.1)-(1.2) has been analyzed in several text books in addition to several papers, the explicit solution was only obtained depending on specifying the HF from the beginning. This explains the difference between the present approach and those previously reported on the same problem.

2. Theoretical results

In order to apply the MoS, the domain $t \geqslant 0$ is divided to sequential sub-intervals $I_n = [(n-1)\tau, n\tau]$, $n \geqslant 1$. Let us also assume that $I_0 = [-\tau, 0]$. The MoS determines the solution $y_n(t)$ in the sub-interval I_n in terms of the solution $y_{n-1}(t)$ in the previous sub-interval I_{n-1} as explained by the following theorem.

Th 2.1. *The solution $y_n(t)$ is*

$$y_n(t) = y_{n-1}((n-1)\tau)e^{\alpha(t-(n-1)\tau)} + \beta e^{\alpha t}\int_{(n-1)\tau}^{t} e^{-\alpha t} y_{n-1}(t-\tau)dt, \; t \in I_n = [(n-1)\tau, n\tau]. \tag{2.1}$$

Proof. Let $t \in I_1 = [0, \tau]$, then $t - \tau \in I_0 = [-\tau, 0]$. Since $y_0(t) = \phi(t)$ represents the given HF in I_0 and $y_1(t)$ is the solution in I_1, then $y_1(t-\tau) = y_0(t-\tau) = \phi(t-\tau)$. Accordingly, $y_1(t)$ is governed by

$$\begin{cases} y_1'(t) = \alpha y_1(t) + \beta y_0(t-\tau), & t \in I_1 = [0, \tau], \\ y_1(0) = y_0(0) = \phi(0). \end{cases} \tag{2.2}$$

Similarly, the solution $y_2(t)$ in I_2 is governed by

$$\begin{cases} y_2'(t) = \alpha y_2(t) + \beta y_1(t-\tau), & t \in I_2 = [\tau, 2\tau], \\ y_2(\tau) = y_1(\tau). \end{cases} \tag{2.3}$$

Repeating this process n-times, one obtains the following initial value problem (IVP) for $y_n(t)$:

$$\begin{cases} y_n'(t) = \alpha y_n(t) + \beta y_{n-1}(t-\tau), & t \in I_n = [(n-1)\tau, n\tau], \\ y_n((n-1)\tau) = y_{n-1}((n-1)\tau). \end{cases} \tag{2.4}$$

The solution of the IVP (2.4) gives the result in Eq. (2.1), which completes the proof. □

Lemma 2.2. *The reduced model:*

$$\begin{cases} y'(t) = \beta y\,(t-\tau)\,, \\ y(t) = \phi(t) = \sum_{m=0}^{\infty} \phi_m t^m, \quad t \in [-\tau, 0], \end{cases}$$

has the solution:

$$y_n(t) = y_{n-1}((n-1)\tau) + \beta \int_{(n-1)\tau}^{t} y_{n-1}(t-\tau)dt,$$

in the interval $I_n = [(n-1)\tau, n\tau]$, $n \geqslant 1$.

Proof. The proof follows immediately from Theorem 2.1 by setting $\alpha = 0$ in Eq. (2.1). □

3. Properties

This section introduces some properties about the continuity of the solution $y(t)$ and its derivative $y'(t)$ at the nodes $t_n = n\tau$, $n = 0, 1, 2, \ldots$, as described in the next theorem.

Th 3.1. *The solution* $y(t)$ *of the delay model* (1.1)-(1.2) *is continuous at every point* $t_n = n\tau$, $\forall n \geqslant 0$. *Moreover,* $y'(t)$ *is continuous at* $t_0 = 0$ *if the given HF satisfies the condition:*

$$\phi'(0) = \alpha\phi(0) + \beta\phi(-\tau), \tag{3.1}$$

while $y'(t)$ *is continuous at every point* $t_n = n\tau$, $\forall n \geqslant 1$.

Proof. From the analysis of Theorem 2.1, we have $y_1(0) = y_0(0)$, $y_2(\tau) = y_1(\tau), \ldots,$ and $y_n((n-1)\tau) = y_{n-1}((n-1)\tau)$, which implies the continuity of $y(t)$ at $t_{n-1} = (n-1)\tau$, $\forall n \geqslant 1$. Hence, $y(t)$ is continuous at every $t_n = n\tau$, $\forall n \geqslant 0$. The right derivative $y'_+(0)$ at $t = t_0 = 0$ is given from Eq. (2.2) as

$$y'_+(0) = y'_1(0) = \alpha y_1(0) + \beta y_0(-\tau),$$

i.e.,

$$y'_+(0) = \alpha\phi(0) + \beta\phi(-\tau).$$

From the given HF, the left derivative is $y'_-(0) = \phi'(0)$. Accordingly, $y'(t)$ is continuous at $t = t_0 = 0$ if

$$\phi'(0) = \alpha\phi(0) + \beta\phi(-\tau).$$

At $t = t_1 = \tau$, we have from (2.2) that

$$y'_-(\tau) = y'_1(\tau) = \alpha y_1(\tau) + \beta y_0(0), \tag{3.2}$$

while Eq. (2.3) implies

$$y'_+(\tau) = y'_2(\tau) = \alpha y_2(\tau) + \beta y_1(0). \tag{3.3}$$

Since $y_1(0) = y_0(0)$ and $y_2(\tau) = y_1(\tau)$, then Eqs. (3.2) and (3.3) yield $y'_-(\tau) = y'_+(\tau)$, which reveals that $y'(t)$ is continuous at $t = t_1 = \tau$. By similar analysis, one can show that $y'(t)$ is continuous at every point $t = t_n = n\tau$, $\forall n \geqslant 1$. □

Remark 3.2. As indicated in Theorem 3.1, the solution is always continuous at every point $t = t_n = n\tau$, $n \geqslant 1$ while the derivative is continuous at $t = 0$ if the given HF satisfies the condition (3.1). This means that if Eq. (3.1) is satisfied then $y'(t)$ is continuous in the whole domain $t \geqslant 0$. The next section explains how to generate different classes of DDEs for which the given HF satisfies Eq. (3.1) and accordingly the derivative of the solution(for each of these classes) is continuous in the whole domain $t \geqslant 0$.

4. Examples of special classes of DDEs

For the HF $\phi(t) = e^t$, Eq. (3.1) leads to $\beta = (1-\alpha)e^{\tau}$. By this, the system (1.1)-(1.2) takes the form:

$$\begin{cases} y'(t) = \alpha y(t) + (1-\alpha)e^{\tau} y(t-\tau), & t \geqslant 0, \\ y(t) = e^t, & t \in [-\tau, 0]. \end{cases} \tag{4.1}$$

From Theorem 3.1, the solution and its derivative of the class (4.1) are continuous in the whole domain $t \geqslant 0$. Similarly for the HF $\phi(t) = e^{-t}$, Eq. (3.1) requires $\beta = -(\alpha+1)e^{-\tau}$, which leads to the class:

$$\begin{cases} y'(t) = \alpha y(t) - (\alpha+1)e^{-\tau} y(t-\tau), & t \geqslant 0, \\ y(t) = e^{-t}, & t \in [-\tau, 0]. \end{cases} \tag{4.2}$$

Suppose that $\phi(t) = A$ (real constant), Eq. (3.1) implies $\beta = -\alpha$, a simple delay model is resulted as

$$\begin{cases} y'(t) = \alpha [y(t) - y(t-\tau)], & t \geqslant 0, \\ y(t) = A, & t \in [-\tau, 0]. \end{cases} \tag{4.3}$$

The above three classes can be combined if the HF is chosen as $\phi(t) = Ae^{\gamma t}$, γ is a real constant. In this case we have $\beta = (\gamma - \alpha)e^{\gamma\tau}$, which gives the class:

$$\begin{cases} y'(t) = \alpha y(t) + (\gamma-\alpha)e^{\gamma\tau} y(t-\tau), & t \geqslant 0, \\ y(t) = Ae^{\gamma t}, & t \in [-\tau, 0]. \end{cases}$$

This class reduces to the classes (4.1), (4.2), and (4.3) at $\gamma = 1$, $\gamma = -1$, and $\gamma = 0$, respectively. In the logarithmic forms $\phi(t) = B\ln(1+t)$ and $\phi(t) = B\ln(1-t)$ (B is a real constant), we have the delay models

$$\begin{cases} y'(t) = \alpha y(t) + \frac{y(t-\tau)}{\ln(1-\tau)}, & t \geqslant 0, \\ y(t) = B\ln(1+t), & t \in [-\tau, 0], \quad 0 < \tau < 1, \end{cases} \tag{4.4}$$

and

$$\begin{cases} y'(t) = \alpha y(t) - \frac{y(t-\tau)}{\ln(1+\tau)}, & t \geqslant 0, \\ y(t) = B\ln(1-t), & t \in [-\tau, 0], \end{cases} \tag{4.5}$$

respectively. Generally, the choice $\phi(t) = B\ln(1+\mu t)$ ($\mu > 0$) gives the delay model:

$$\begin{cases} y'(t) = \alpha y(t) + \frac{\mu y(t-\tau)}{\ln(1-\mu\tau)}, & t \geqslant 0, \\ y(t) = B\ln(1+\mu t), & t \in [-\tau, 0], \quad 0 < \tau < \frac{1}{\mu}, \end{cases} \tag{4.6}$$

while the choice $\phi(t) = B\ln(1-\nu t)$ ($\nu > 0$) gives the delay model:

$$\begin{cases} y'(t) = \alpha y(t) - \frac{\nu y(t-\tau)}{\ln(1+\nu\tau)}, & t \geqslant 0, \\ y(t) = B\ln(1-\nu t), & t \in [-\tau, 0]. \end{cases} \tag{4.7}$$

It can be easily noticed that the classes (4.4) and (4.5) are special cases of the classes (4.6) and (4.7) when $\mu = 1$ and $\nu = 1$, respectively. In the trigonometric forms, one can also find the following classes, where c_1 and c_2 are real constants:

$$\begin{cases} y'(t) = \alpha y(t) - \frac{y(t-\tau)}{\sin\tau}, & t \geqslant 0, \\ y(t) = c_1 \sin t, & t \in [-\tau, 0], \end{cases} \quad \text{and} \quad \begin{cases} y'(t) = \alpha y(t) - \frac{\alpha y(t-\tau)}{\cos\tau}, & t \geqslant 0, \\ y(t) = c_2 \cos t, & t \in [-\tau, 0], \end{cases}$$

which are special cases of the general class:

$$\begin{cases} y'(t) = \alpha y(t) + \frac{(\alpha c_2 - c_1) y(t-\tau)}{c_1 \sin\tau - c_2 \cos\tau}, & t \geqslant 0, \\ y(t) = c_1 \sin t + c_2 \cos t, & t \in [-\tau, 0]. \end{cases}$$

5. Explicit solution

In this section, we show how to construct the explicit solution $y_1(t)$, $y_2(t)$, and $y_3(t)$ in the first three interval $I_1 = [0, \tau]$, $I_2 = [\tau, 2\tau]$, and $I_3 = [2\tau, 3\tau]$ for the delay model (1.1)-(1.2) at $\beta = 0$. The general case in which $\beta \neq 0$ can be treated similarly. The result of Lemma 2.2 gives $y_n(t)$ in the interval I_n as

$$y_n(t) = y_{n-1}((n-1)\tau) + \beta \int_{(n-1)\tau}^{t} y_{n-1}(t-\tau)dt, \; I_n = [(n-1)\tau, n\tau], \; n \geqslant 1.$$

At $n = 1$, we have

$$y_1(t) = y_0(0) + \beta \int_0^t y_0(t-\tau)dt, \quad I_1 = [0, \tau]. \tag{5.1}$$

However

$$y_0(t-\tau) = \phi(t-\tau) = \sum_{m=0}^{\infty} \phi_m (t-\tau)^m. \tag{5.2}$$

Inserting (5.2) into (5.1) and simplifying, then

$$y_1(t) = \phi_0 + \beta \sum_{m=0}^{\infty} \frac{\phi_m}{m+1} \left[(t-\tau)^{m+1} - (-\tau)^{m+1}\right]. \tag{5.3}$$

At $n = 2$, we have

$$y_2(t) = y_1(\tau) + \beta \int_\tau^t y_1(t-\tau)dt, \quad I_2 = [\tau, 2\tau]. \tag{5.4}$$

Employing (5.3) in (5.4) gives

$$\begin{aligned} y_2(t) = \phi_0 + \beta(t-\tau) - \beta \sum_{m=0}^{\infty} \frac{\phi_m}{m+1}(-\tau)^{m+1} \\ + \beta^2 \sum_{m=0}^{\infty} \frac{\phi_m}{m+1} \left[\frac{(t-2\tau)^{m+2}}{m+2} - \frac{(-\tau)^{m+2}}{m+2} - (-\tau)^{m+1}(t-\tau)\right]. \end{aligned} \tag{5.5}$$

The case $n = 3$ leads to

$$y_3(t) = y_2(2\tau) + \beta \int_{2\tau}^t y_2(t-\tau)dt, \quad I_3 = [2\tau, 3\tau],$$

and hence

$$\begin{aligned} y_3(t) = \phi_0 \sum_{k=0}^{2} \frac{\beta^k}{k!}(t-k\tau)^k - \beta \sum_{m=0}^{\infty} \frac{\phi_m}{m+1}(-\tau)^{m+1} + \beta^2 \sum_{m=0}^{\infty} \frac{\phi_m}{m+2}(-\tau)^{m+2} \\ - \beta^2 \sum_{m=0}^{\infty} \frac{\phi_m}{m+1}(-\tau)^{m+1}(t-2\tau) + \beta^3 \sum_{m=0}^{\infty} \frac{\phi_m}{(m+1)(m+2)(m+3)} \left[(t-3\tau)^{m+3} - (-\tau)^{m+3}\right] \\ - \beta^3 \sum_{m=0}^{\infty} \frac{\phi_m}{(m+1)(m+2)}(-\tau)^{m+2}(t-2\tau) - \frac{1}{2}\beta^3 \sum_{m=0}^{\infty} \frac{\phi_m}{m+1}(-\tau)^{m+1}(t-2\tau)^2. \end{aligned} \tag{5.6}$$

6. Exact solutions at different HFs

In this section, the exact solutions at different forms of the HF $\phi(t)$ are to be obtained via investing the result of the previous section.

6.1. $\phi(t) = e^t$

In this case, one find that $\phi_m = 1/m!$, $m \geqslant 0$. So, the solution $y_1(t)$ in the interval $I_1 = [0, \tau]$ can be directly derived from Eq. (5.3) as

$$\begin{aligned} y_1(t) &= \phi_0 + \beta \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \left[(t-\tau)^{m+1} - (-\tau)^{m+1}\right], \\ &= 1 + \beta \sum_{m=1}^{\infty} \frac{1}{m!} \left[(t-\tau)^m - (-\tau)^m\right], = 1 + \beta\left(e^{t-\tau} - 1\right) - \beta\left(e^{-\tau} - 1\right), = 1 + \beta\left(e^{t-\tau} - e^{-\tau}\right). \end{aligned}$$

From Eq. (5.5), we obtain

$$\begin{aligned} y_2(t) = 1 + \beta(t-\tau) &- \beta \sum_{m=0}^{\infty} \frac{(-\tau)^{m+1}}{(m+1)!} \\ &+ \beta^2 \sum_{m=0}^{\infty} \left[\frac{(t-2\tau)^{m+2}}{(m+2)!} - \frac{(-\tau)^{m+2}}{(m+2)!} - \frac{(-\tau)^{m+1}}{(m+1)!}(t-\tau)\right]. \end{aligned} \tag{6.1}$$

Implementing the relations:

$$\sum_{m=0}^{\infty} \frac{x^{m+1}}{(m+1)!} = e^x - 1, \qquad \sum_{m=0}^{\infty} \frac{x^{m+12}}{(m+2)!} = e^x - 1 - x,$$

for $x = -\tau$ and $x = t - 2\tau$, then Eq. (6.1) can be simplified to

$$y_2(t) = 1 + \beta\left(t - \tau - e^{-\tau} + 1\right) + \beta^2\left[e^{t-2\tau} - e^{-\tau}(1 + t - \tau)\right].$$

From Eq. (5.6), we have

$$\begin{aligned} y_3(t) = \sum_{k=0}^{2} \frac{\beta^k}{k!}(t-k\tau)^k &- \beta \sum_{m=0}^{\infty} \frac{1}{(m+1)!}(-\tau)^{m+1} + \beta^2 \sum_{m=0}^{\infty} \frac{1}{m!(m+2)}(-\tau)^{m+2} \\ &- \beta^2(t-2\tau) \sum_{m=0}^{\infty} \frac{1}{(m+1)!}(-\tau)^{m+1} + \beta^3 \sum_{m=0}^{\infty} \frac{1}{(m+3)!}\left[(t-3\tau)^{m+3} - (-\tau)^{m+3}\right] \\ &- \beta^3(t-2\tau) \sum_{m=0}^{\infty} \frac{1}{(m+2)!}(-\tau)^{m+2} - \frac{1}{2}\beta^3(t-2\tau)^2 \sum_{m=0}^{\infty} \frac{1}{(m+1)!}(-\tau)^{m+1}. \end{aligned} \tag{6.2}$$

Using the equality

$$\sum_{m=0}^{\infty} \frac{(-\tau)^{m+2}}{m!(m+2)} = \sum_{m=0}^{\infty} \left[\frac{(-\tau)^{m+2}}{(m+1)!} - \frac{(-\tau)^{m+2}}{(m+2)!}\right],$$

into Eq. (6.2) and simplifying, yields

$$\begin{aligned} y_3(t) = 1 - \beta\left(e^{-\tau} - 1\right) + \beta^2\left(-\tau e^{-\tau} - e^{-\tau} + 1\right) + \beta(t-\tau) + \frac{1}{2}(t-2\tau)^2\left[\beta^2 - \beta^3(e^{-\tau} - 1)\right] \\ - (t-2\tau)\left[\beta^2(e^{-\tau} - 1) + \beta^3(e^{-\tau} - 1 + \tau)\right] + \beta^3\left[e^{t-3\tau} - t - \frac{1}{2}(t-3\tau)^2 + 2\tau - e^{-\tau} + \frac{1}{2}\tau^2\right]. \end{aligned}$$

Remark 6.1. It may be important to refer to that the solutions in the first three intervals satisfy the continuity condition at the points $t = 0, \tau, 2\tau$, where $y_0(0) = y_1(0) = 1$, $y_1(\tau) = y_2(\tau) = 1 + \beta(1 - e^{-\tau})$, and $y_2(2\tau) = y_3(2\tau) = 1 + \beta(1 + \tau - e^{-\tau}) + \beta^2[1 - e^{-\tau}(1 + \tau)]$. However, the derivative is not continuous at $t = 0$, where $y'_-(0) = \phi(0) = 1$ and $y'_+(0) = y_1(0) = \beta e^{-\tau}$. If $y'_-(0) = y'_+(0)$, i.e., $1 = \beta e^{-\tau}$ or $\beta = e^{\tau}$, then $y'(t)$ will be continuous at $t = 0$. This conclusion confirms the validity of the condition (3.1) provided by Theorem 2.1. This condition at $\alpha = 0$ becomes $\phi'(0) = \beta\phi(-\tau)$, which implies $\beta e^{-\tau} = 1$ or $\beta = e^{\tau}$, which agrees with above conclusion.

6.2. $\phi(t) = e^{-t}$

This case implies $\phi_m = (-1)^m/m!$, $m \geqslant 0$. Eq. (5.3) gives $y_1(t)$ as

$$\begin{aligned} y_1(t) &= \phi_0 + \beta \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \left[(t-\tau)^{m+1} - (-\tau)^{m+1}\right], \\ &= 1 - \beta \sum_{m=1}^{\infty} \frac{1}{m!} \left[(-t+\tau)^m - (\tau)^m\right], = 1 - \beta\left(e^{-t+\tau} - 1\right) + \beta\left(e^{\tau} - 1\right), = 1 - \beta\left(e^{-t+\tau} - e^{\tau}\right). \end{aligned}$$

Substituting $\phi_m = (-1)^m/m!$ into Eqs. (5.5) and (5.6), then simplifying, we obtain

$$y_2(t) = 1 + \beta\left(t - \tau + e^{\tau} - 1\right) + \beta^2\left[e^{-t+2\tau} + e^{\tau}(t-\tau-1)\right],$$

and

$$\begin{aligned} y_3(t) &= 1 + \beta\left(e^{\tau} - 1\right) + \beta^2\left(\tau e^{\tau} - e^{\tau} + 1\right) + \beta(t-\tau) + \frac{1}{2}(t-2\tau)^2\left[\beta^2 + \beta^3(e^{\tau} - 1)\right] \\ &\quad + (t-2\tau)\left[\beta^2(e^{\tau} - 1) - \beta^3(e^{\tau} - 1 - \tau)\right] + \beta^3\left[-e^{-t+3\tau} - t + \frac{1}{2}(t-3\tau)^2 + 2\tau + e^{\tau} - \frac{1}{2}\tau^2\right]. \end{aligned}$$

6.3. $\phi(t) = -t$

This polynomial case implies $\phi_0 = 0$, $\phi_1 = -1$, and $\phi_m = 0$, $m \geqslant 2$. In other word, $\phi_m = 0$ for all $m \geqslant 0$ except at $m = 1$. Therefore, Eq. (5.3) reduces to

$$y_1(t) = -\frac{\beta}{2}\left[(t-\tau)^2 - \tau^2\right].$$

Also, Eqs. (5.5) and (5.6) give

$$y_2(t) = \frac{1}{2}\beta\tau^2 - \frac{1}{6}\beta^2\left[(t-2\tau)^3 + \tau^3 - 3\tau^2(t-\tau)\right],$$

and

$$y_3(t) = \frac{1}{2}\beta\tau^2 + \frac{1}{3}\beta^2\tau^3 + \frac{1}{6}\left(3\beta^2\tau^2 - \beta^3\tau^3\right)(t-2\tau) + \frac{1}{4}\beta^3\tau^2(t-2\tau)^2 - \frac{1}{24}\beta^3\left[(t-3\tau)^4 - \tau^4\right].$$

6.4. $\phi(t) = 1$

This HF is given as a constant, consequently, $\phi_0 = 1$ while $\phi_m = 0$ for all $m \geqslant 1$. We also consider $\beta = -1$, $\tau = 1$, and then use Eqs. (5.3), (5.5), and (5.6) to find that

$$y_1(t) = 1 - t,\ y_2(t) = 1 - t + \frac{1}{2}(1-t)^2,\ y_3(t) = -\frac{5}{3} + \frac{1}{2}t - \frac{1}{6}(t-3)^3 = 1 + \left[-t + \frac{1}{2}(t-1)^2 - \frac{1}{6}(t-2)^3\right].$$

In such a case, the general solution $y_n(t)$ in the interval $I_n = [n-1, n]$ can be written in the form:

$$y_n(t) = 1 + \sum_{i=0}^{n} \frac{(-1)^i}{i!}\left[t - (i-1)\tau\right]^i, \quad t \in [n-1, n], \quad n \geqslant 1,$$

which agrees with the corresponding solution in Ref. [26] for the delay model:

$$\begin{cases} y'(t) = -y(t-1), & t \geqslant 0, \\ y(t) = 1, & t \in [-1, 0]. \end{cases}$$

7. Conclusion

A new approach was introduced in this paper to resolve a class of DDEs under arbitrary HFs. The idea was based on expressing the arbitrary HF $\phi(t)$ in the form of the Maclaurin series with coefficients ϕ_m, $m \geqslant 0$. By the aide of the MoS, an explicit form was obtained for the solution in each sub-interval of the problem's domain in terms of the coefficients ϕ_m. Selected examples were illustrated to derive the solution under HFs in different forms. The paper also addressed the characteristics of the solution and its derivative, where the condition to obtain continuous derivative in the whole domain was proved theoretically. Existing results in the literature were derived as special cases of the present ones.

The obtained results can be invested to directly determine the exact solution of a first order linear DDE by just inserting the coefficients ϕ_m of the given HF into the standard solution. This explains the effectiveness of the suggested analysis and gives the opportunity to achieve further generalization for higher order linear DDEs. Regarding this point, the method may be applicable to nonlinear delay equations subjected to initial data described by HFs.

Acknowledgment

"The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2024/01/30885)".

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