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Qualitative results and Lyapunov-type inequality for a new φ-ABC fractional boundary value problem

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Abstract

This paper investigates a new category of a fractional boundary value problem (BVP) involving the generalized Atangana-Baleanu-Caputo (ABC) derivatives of order belonging to $(2,3]$. First, the Green function with its properties for a proposed fractional BVP are derived. Then, the theoretical results are established by various techniques. The existence and uniqueness of theorems of the solutions are proved by utilizing the Weissinger, the Aghajani fixed point theorems, the Meir-Keeler condensing operator, and the Kuratowski's measure of non-compactness techniques. The sufficient conditions of the existence and non-existence results of nontrivial solutions for the proposed problem are investigated by introducing the Lyapunov-type inequality ($\mathbb{LTI}$). Finally, our findings are compared with the existing results in the literature. The validity of the main outcomes is also tested by numerical examples with graphs and tables, and the strict minimum borders of eigenvalues for several fractional BVPs are estimated. This work is the first to deal with $\mathbb{LTI}$ for fractional BVP in the sense of generalized ABC derivatives.

Keywords: Fractional Atangana-Baleanu-Caputo derivative, fractional boundary value problem, Weissinger fixed point theorems, Meir-Keeler condensing operator, Lyapunov-type inequality.

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1. Introduction

The idea of fractional derivatives extends differentiation beyond integer orders. Applications of these

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derivatives can be found in numerous fields like physics, engineering, and finance; such as these papers [22, 42]. They are utilized to model systems that demonstrate non-local behavior, long-range interactions, and memory effects [4, 36, 44]. In particular, Qiu et al. [35] presented a novel design approach for fuzzy affine model-based output feedback control, while Bi et al [8] suggested an adaptive decentralized finite-time fuzzy secure control strategy for uncertain nonlinear cyber-physical systems facing deception attacks. In the fractional calculus field, there are many non-local operators with singular and non-singular kernels. The most important of these operators are Caputo, Riemann-Liouville [45], Hilfer [19], ψ-Hilfer [5], Caputo-Fabrizio [10], Atangana-Baleanu-Caputo (ABC) [6], where extended recently to higher arbitrary orders with respect to another function [2], and generalized to mixed kernel by [38]. In certain cases, nonlocal fractional operators with singular kernels are unable to properly describe the dynamics of a system, and their implementation leads to unaffordable ill-posedness. In this manner, a group of scientists with an innovative approach and also an extra tool to provide different alternatives in order to enhance the description of models of actual phenomena. It has proposed new fractional operators with non-singular kernels; see [6, 14]. Fixed point theorems play essential roles in studying the existence and uniqueness of solutions for ordinary, partial, and fractional differential equations, see [16, 28–30]. For instance, the Banach, Weissinger fixed point theorems (FPTs), and the Meir-Keeler condensing operator were employed to discuss the qualitative properties of two classes of fractional differential problems [11]. The Banach and Schaefer FPTs utilized to analyze the solutions of a ρ-ABC fractional problem [40]; for more information see [7, 9, 37] and the references cited therein.

The Lyapunov-type inequality ($\mathbb{LTI}$) for boundary value problem (BVP) is considered an essential device for investigating the existence and non-existence of nontrivial solutions for ordinary and fractional BVP. The $\mathbb{LTI}$ has employed to determinate an eigenvalue for fractional BVPs such as Riemann BVP [15], mixed BVP [18], q-fractional BVP [21], Hadamard BVP [27], impulsive BVP [23], Katugampola BVP [20, 26], Hilfer BVP [34], Katugampola-Hilfer fractional BVP [39], Riemann-Atangana-Baleanu BVP [1], and Caputo-Fabrizio ($\mathbb{CF}$) BVP [43], for more details see these surveys [31, 32]. In particular, paper [25] considered as the first work that proved that if $\mathfrak{q}(s)$ is a continuous function satisfying $\mathbb{LTI}$:

$$\int_{\mathfrak{c}_0}^{\mathfrak{c}} |\mathfrak{q}(z)|dz > \frac{4}{\mathfrak{c}-\mathfrak{c}_0}, \tag{1.1}$$

then the following second ordinary BVP has nontrivial solution $y(s) \neq 0$:

$$\begin{cases} y''(s) + \mathfrak{q}(s)y(s) = 0, & s \in (\mathfrak{c}_0, \mathfrak{c}), \\ y(\mathfrak{c}_0) = y(\mathfrak{c}) = 0. \end{cases}$$

Further, authors of [33] proved that the following third ordinary differential equation:

$$\begin{cases} y'''(s) + \mathfrak{q}(s)y(s) = 0, & s \in \Upsilon = [\mathfrak{c}_0, \mathfrak{c}], \\ y(\mathfrak{c}_0) = y(\mathfrak{c}) = 0, \quad y''(d) = 0, & d \in [\mathfrak{c}_0, \mathfrak{c}], \end{cases} \tag{1.2}$$

has nontrivial solution if the $\mathbb{LTI}$ holds:

$$\int_{\mathfrak{c}_0}^{\mathfrak{c}} |\mathfrak{q}(z)|dz > \frac{4}{(\mathfrak{c}-\mathfrak{c}_0)^2}. \tag{1.3}$$

Also, Toprakseven [41] investigated the $\mathbb{CF}$-fractional BVP

$$\begin{cases} {}^{\mathbb{CF}}\mathfrak{D}^{u}_{\mathfrak{c}_0} y(s) + \mathfrak{q}(s)y(s) = 0, & s \in \Upsilon = [\mathfrak{c}_0, \mathfrak{c}], u \in (2,3], \\ y(\mathfrak{c}_0) = y''(\mathfrak{c}_0) = y(\mathfrak{c}) = 0, \end{cases} \tag{1.4}$$

has nontrivial solution if the $\mathbb{LTI}$ holds:

$$\int_{\mathfrak{c}_0}^{\mathfrak{c}} |\mathfrak{q}(z)|dz > \frac{4}{(3-u)(\mathfrak{c}-\mathfrak{c}_0) + (u-2)(\mathfrak{c}-\mathfrak{c}_0)^2}. \tag{1.5}$$

It is important to note that there is no work-proven $\mathbb{LTI}$ for fractional BVP in the sense of φ-ABC. Thus, inspired by the above papers, in the present work, we establish sufficient conditions of the existence and uniqueness of the solution as well as the $\mathbb{LTI}$ for the next fractional BVP involving φ-ABC derivative:

$$\begin{cases} {}^{ABC}\mathfrak{D}_{\mathfrak{c}_0}^{u,\varphi}\, y(s) + \mathfrak{h}(s, y(s)) = 0, \quad s \in \Upsilon = [\mathfrak{c}_0, \mathfrak{c}], \quad u \in (2,3], \\ y(\mathfrak{c}_0) = [y(s)]''_{\varphi(s)}\big|_{s=\mathfrak{c}_0} = y(\mathfrak{c}) = 0, \end{cases} \tag{1.6}$$

where φ is a non-decreasing and non-negative function with $\varphi(s) \in \mathcal{C}^3((\mathfrak{c}_0, \mathfrak{c}), \mathbb{R}^+)$, and $\varphi'(s) \neq 0$, $\forall\, s \in \Upsilon$, where $[y(s)]''_{\varphi(s)} = \left(\frac{1}{\varphi'(s)}\frac{d}{ds}\right)^2 y(s)$. Furthermore, ${}^{ABC}\mathfrak{D}_{\mathfrak{c}_0}^{u,\varphi}$ denotes to the φ-ABC derivatives of arbitrary order $u \in (2,3]$, and $\mathfrak{h} : \Upsilon \times \mathbb{R} \to \mathbb{R}$ is continuous function such that $\mathfrak{h}(\mathfrak{c}_0, y(\mathfrak{c}_0)) = 0$.

It's worth mentioning that this work contains the following novelty and contributions.

1. The fractional derivative used in the problem (1.6), is generalized more than the ABC fractional derivatives [1, 6], and allows for dealing with the variety of a function φ.
2. The main problem (1.6) is more general of the existing in the literature. For instance, if $\varphi(s) = s$, $u = 3$, and $\mathfrak{h}(s, y(s)) = \mathfrak{q}(s)y(s)$, then problem (1.6) returns to problem (1.2) when $d = \mathfrak{c}_0$. Also, the problem (1.6) becomes (1.4) if we replace ${}^{ABC}\mathfrak{D}_{\mathfrak{c}_0}^{u}$ by ${}^{\mathbb{CF}}\mathfrak{D}_{\mathfrak{c}_0}^{u}$, and $\mathfrak{h}(s, y(s)) = \mathfrak{q}(s)y(s)$.
3. The existence and uniqueness results are established by utilizing the Weissinger, Aghajani FPTs, the Meir-Keeler condensing operator, and the Kuratowski's measure of non-compactness techniques.
4. The existence of a nontrivial solution is proved by using the $\mathbb{LTI}$.
5. The estimating of minimum borders for eigenvalues of new classes of φ-ABC fractional BVPs are provided.

The rest of this manuscript is assorted as: In Section 2 several important preliminaries are provided. Section 3 investigates the Green function with its properties. Sections 4 and 5 prove the existence and uniqueness of the solution for the proposed problem. Section 6 is introduced the $\mathbb{LTI}$ for φ-ABC-fractional BVP (1.6). Section 7 is focused on testing the effectiveness of main outcomes by some examples. Finally, the conclusion of this work is provided in Section 8.

2. Preliminaries

In this part, we recall some important definitions and lemma, which are related to fractional φ-ABC derivative and our used techniques.

Definition 2.1 ([5])**.** Let $y : [\mathfrak{c}_0, \mathfrak{c}] \to \mathbb{R}$ and $u > 0$, then the equation

$$^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{u,\varphi} y(s) = \frac{1}{\Gamma(u)} \int_{\mathfrak{c}_0}^{s} (\varphi(s) - \varphi(z))^{u-1} \varphi'(z) y(z)\, dz,$$

is u^{th} left-side φ-Riemann-Liouville fractional integral, such that Γ is Gamma function.

Definition 2.2. ([2]). The φ-ABC-fractional derivatives of $y^{(m)} \in \mathcal{H}^1(\mathfrak{c}_0, \mathfrak{c})$ with order $u \in (m, m+1]$, $\nu = u - m$, is defined as

$$\begin{aligned} ({}^{ABC}\mathfrak{D}_{\mathfrak{c}_0}^{u,\varphi} y)(s) &= ({}^{ABC}\mathfrak{D}_{\mathfrak{c}_0}^{\nu,\varphi} y_{\varphi}^{(m)})(s) \\ &= \frac{\Theta(u-m)}{(m+1-u)} \int_{\mathfrak{c}_0}^{s} \varphi'(z) \mathbb{E}_{u-m}\left(\frac{-(u-m)}{(m+1-u)} (\varphi(s) - \varphi(z))^{u-m}\right) y_{\varphi}^{(m+1)}(z)\, dz, \end{aligned}$$

where $\mathbb{E}_u$ is the Mittag-Leffler function, and $\Theta(u)$ denotes the normalization function endowed by $\Theta(0) = \Theta(1) = 1$. Also, the φ-ABC-fractional integral of a function y with order u is defined by

$$\begin{aligned} ({}^{ABC}\mathfrak{I}_{\mathfrak{c}_0}^{u,\varphi} y)(s) &= ({}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{m,\varphi}\ {}^{ABC}\mathfrak{I}_{\mathfrak{c}_0}^{\nu,\varphi}\ y)(s) \\ &= ({}^{ABC}\mathfrak{I}_{\mathfrak{c}_0}^{\nu,\varphi}\ {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{m,\varphi}\ y)(s) = \frac{m+1-u}{\Theta(u-m)}\ {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{m,\varphi} y(s) + \frac{(u-m)}{\Theta(u-m)}\ {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{u,\varphi} y(s). \end{aligned} \tag{2.1}$$

Lemma 2.3 ([2]). *For $u \in (m, m+1]$, and $y \in \mathcal{H}^1(\Upsilon, \mathbb{R})$, then,*

$$\left({}^{ABC}\mathfrak{J}^{u,\varphi}_{\mathfrak{c}_0}\,{}^{ABC}\mathfrak{D}^{u,\varphi}_{\mathfrak{c}_0}y\right)(s) = y(s) - \sum_{i=0}^{m} d_i\left(\varphi(s) - \varphi(\mathfrak{c}_0)\right)^i, \quad d_i \in \mathbb{R}.$$

Now, let us present a definition of the Kuratowski's measure of non-compactness denoted by $\mu(\cdot)$. Let $\mathcal{A}$ be a bounded subset of a Banach space X, then

$$\mu(\mathcal{A}) := \inf\{r > 0 : \ \mathcal{A} = \bigcup_{i=1}^{m}\mathcal{A}_i, \ \text{ and } \ \operatorname{diam}(\mathcal{A}_i) \leqslant r, \ i = \overline{1,m}\}.$$

Clearly, $0 \leqslant \mu(\mathcal{A}) \leqslant \operatorname{diam}(\mathcal{A}) < \infty$ ([17]).

Lemma 2.4 ([24]). *For each bounded subset $\mathcal{A}$ of a Banach space X, there is a countable subset $\mathcal{A}_0$ of $\mathcal{A}$, supplied with $\mu(\mathcal{A}) \leqslant 2\,\mu(\mathcal{A}_0)$.*

Lemma 2.5 ([17]). *Consider $\mathcal{A} \subseteq \mathcal{C}(J, X)$ to be a equicontinuous bounded set such that X is a Banach space, then $\mu(\mathcal{A}(s))$ is continuous on J, and $\mu(\mathcal{A}) = \max_{s\in J}\mu(\mathcal{A}(s))$.*

Lemma 2.6 ([17]). *Consider $\mathcal{A} = \{x_m\}_{m\in\mathbb{N}} \subseteq \mathcal{C}(J, X)$ to be a countable and bounded set such that X is a Banach space, then $\mu(\mathcal{A}(s))$ is the Lebesgue integrable on J, and*

$$\mu\left(\left\{\int_0^s x_m(z)\,dz\right\}_{m\in\mathbb{N}}\right) \leqslant 2\int_0^s \mu(\{x_m(\varpi)\}_{m\in\mathbb{N}})\,dz.$$

Definition 2.7. ([3]) Consider $\mathcal{A}$ to be a nonempty subset of a Banach space X, and μ to be a measure of noncompactness on X. Then $Y : \mathcal{A} \to \mathcal{A}$ is a Meir-Keeler condensing mapping if for any $\epsilon > 0$, there exists $\sigma > 0$ such that

$$\epsilon < \mu(\mathcal{B}) < \epsilon + \sigma \Longrightarrow \mu(Y\mathcal{B}) < \epsilon, \quad \text{for any } \ \mathcal{B} \subset \mathcal{A}.$$

Theorem 2.8 (Aghajani FPT, [3]). *Consider $\mathcal{A}$ to be a bounded, convex, closed, and nonempty subset of a Banach space X, and μ to be a measure of noncompactness on X. If $Y : \mathcal{A} \to \mathcal{A}$ is a Meir-Keeler condensing and continuous mapping. Then, a mapping Y possesses at least one fixed point, and their set in $\mathcal{A}$ is compact.*

Theorem 2.9 (Weissinger's FPT, [13]). *Consider $Y : X \to X$ to be a mapping verifying the identity:*

$$d\left(Y^m x, Y^m \bar{x}\right) \leqslant \zeta_m d(x, \bar{x}), \quad \forall m \in \mathbb{N}, x, \bar{x} \in X,$$

such that (X, d) be a complete metric space, $\zeta_m \geqslant 0, \forall m \in \mathbb{N}$, where $\sum_{m=1}^{\infty}\zeta_m$ is a convergence series. Then, a mapping Y possesses a unique fixed point x^*. Furthermore, for any $x_0 \in X$, the sequence $\{Y^m x_0\}_{m=1}^{\infty}$ tends to a fixed point x^*.

3. Green function

Firstly, we introduce the Green function of the φ-ABC-fractional BVP (1.6) as follows.

Lemma 3.1. *Let $u \in (2, 3]$, and $y^{(3)} \in \mathcal{H}^1(\mathfrak{c}_0, \mathfrak{c})$. Then, the integral equation corresponding to the φ-ABC-fractional* BVP (1.6) *is given by*

$$y(s) = \int_{\mathfrak{c}_0}^{\mathfrak{c}} \mathcal{K}(s, z)\varphi'(z)\,\mathcal{F}(z; y(z))\,dz, \tag{3.1}$$

where

$$\mathcal{K}(s, z) = \begin{cases} \dfrac{(\varphi(s) - \varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))}\left(\varphi(\mathfrak{c}) - \varphi(z)\right), & \mathfrak{c}_0 \leqslant s \leqslant z \leqslant \mathfrak{c}, \\ \dfrac{(\varphi(s) - \varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))}\left(\varphi(\mathfrak{c}) - \varphi(z)\right) - \left(\varphi(s) - \varphi(z)\right), & \mathfrak{c}_0 \leqslant z \leqslant s \leqslant \mathfrak{c}, \end{cases} \tag{3.2}$$

and

$$\mathcal{F}(s;y(s)) = \frac{(3-u)}{\Theta(u-2)}\mathfrak{h}(s,y(s)) + \frac{(u-2)}{\Theta(u-2)}\ {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{u-2,\varphi}\mathfrak{h}(s,y(s)). \tag{3.3}$$

Proof. Due to Lemma 2.3, Definition 2.1, and by applying ${}^{ABC}\mathfrak{I}_{\mathfrak{c}_0}^{u,\varphi}$ on both sides of Eq. (1.6), we obtain

$$\begin{aligned} y(s) &= d_0 + d_1\left(\varphi(s)-\varphi(\mathfrak{c}_0)\right) + d_2\left(\varphi(s)-\varphi(\mathfrak{c}_0)\right)^2 - {}^{ABC}\mathfrak{I}_{\mathfrak{c}_0}^{u,\varphi}\mathfrak{h}(s,y(s)) \\ &= d_0 + d_1\left(\varphi(s)-\varphi(\mathfrak{c}_0)\right) + d_2\left(\varphi(s)-\varphi(\mathfrak{c}_0)\right)^2 - {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{2,\varphi}\mathcal{F}(s;y(s)). \end{aligned}$$

By applying the conditions $y(\mathfrak{c}_0)=0$, we find that $d_0=0$. Thus, one obtains

$$y(s) = d_1\left(\varphi(s)-\varphi(\mathfrak{c}_0)\right) + d_2\left(\varphi(s)-\varphi(\mathfrak{c}_0)\right)^2 - {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{2,\varphi}\mathcal{F}(s;y(s)),$$

it implies that

$$[y(s)]'_{\varphi(s)} = \left(\tfrac{1}{\varphi'(s)}\tfrac{d}{ds}\right)y(s) = d_1 + 2d_2\left(\varphi(s)-\varphi(\mathfrak{c}_0)\right) - {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{1,\varphi}\mathcal{F}(s;y(s)),$$

and

$$[y(s)]''_{\varphi(s)} = \left(\tfrac{1}{\varphi'(s)}\tfrac{d}{ds}\right)^2 y(s) = 2d_2 - \mathcal{F}(s;y(s)),$$

so by the conditions $[y(s)]''_{\varphi(s)}\big|_{s=\mathfrak{c}_0} = 0$ and $\mathfrak{h}(\mathfrak{c}_0,y(\mathfrak{c}_0))=0$, one finds that $d_2=0$. Therefore, we have

$$y(s) = d_1\left(\varphi(s)-\varphi(\mathfrak{c}_0)\right) - {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{2,\varphi}\mathcal{F}(s;y(s)). \tag{3.4}$$

Then, by the boundary condition $y(\mathfrak{c})=0$, one has

$$d_1 = \frac{1}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\,{}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{2,\varphi}\mathcal{F}(\mathfrak{c};y(\mathfrak{c})).$$

Thus, by putting d_1 into Eq. (3.4), we find

$$\begin{aligned} y(s) &= \frac{(\varphi(s)-\varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\,{}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{2,\varphi}\mathcal{F}(\mathfrak{c};y(\mathfrak{c})) - {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{2,\varphi}\mathcal{F}(s;y(s)) \\ &= \frac{(\varphi(s)-\varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\int_{\mathfrak{c}_0}^{\mathfrak{c}}\varphi'(z)\left(\varphi(\mathfrak{c})-\varphi(z)\right)\mathcal{F}(z;y(z))\,dz - \int_{\mathfrak{c}_0}^{s}\varphi'(z)\left(\varphi(s)-\varphi(z)\right)\mathcal{F}(z;y(z))\,dz. \end{aligned}$$

Hence, finished the proof. □

Now, we study characterization of Green function $\mathcal{K}(s,z)$, which is given in (3.2).

Lemma 3.2. *The Green function $\mathcal{K}(s,z)$ satisfies the following properties:*

1. $\mathcal{K}(s,z)\geqslant 0,\ \forall (s,z)\in\Upsilon^2$;
2. $\max_{(s,z)\in\Upsilon^2}\mathcal{K}(s,z)\leqslant \dfrac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}{4}$, *with equality at the point* $s=z=\dfrac{(\varphi(\mathfrak{c})+\varphi(\mathfrak{c}_0))}{2}$.

Proof. Consider,

$$\begin{aligned} \mathcal{K}_1(s,z) &= \frac{(\varphi(s)-\varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\left(\varphi(\mathfrak{c})-\varphi(z)\right), \quad \mathfrak{c}_0\leqslant s\leqslant z\leqslant \mathfrak{c}, \\ \mathcal{K}_2(s,z) &= \frac{(\varphi(s)-\varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\left(\varphi(\mathfrak{c})-\varphi(z)\right) - \left(\varphi(s)-\varphi(z)\right), \quad \mathfrak{c}_0\leqslant z\leqslant s\leqslant \mathfrak{c}. \end{aligned}$$

Obviously, $\mathcal{K}_1(s,z)$ is positive and increasing with respect to s. Hence,

$$0 \leqslant \mathcal{K}_1(s,z) \leqslant \mathcal{K}_1(z,z) = \frac{(\varphi(z)-\varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\left(\varphi(\mathfrak{c})-\varphi(z)\right).$$

Next, one has

$$\begin{aligned}\mathcal{K}_2(s,z) &= \frac{(\varphi(s)-\varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\left(\varphi(\mathfrak{c})-\varphi(z)\right)-\left(\varphi(s)-\varphi(z)\right)\\ &= \frac{(\varphi(s)-\varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\left(\varphi(\mathfrak{c})-\varphi(z)\right)-\left[\varphi(s)-\varphi(\mathfrak{c})+\varphi(\mathfrak{c})-\varphi(z)\right]\\ &= \left(\varphi(\mathfrak{c})-\varphi(z)\right)\left[\left(1-\frac{(\varphi(\mathfrak{c})-\varphi(s))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\right)-\left(1-\frac{\varphi(\mathfrak{c})-\varphi(s)}{(\varphi(\mathfrak{c})-\varphi(z))}\right)\right].\end{aligned}$$

Since $(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0)) \geqslant (\varphi(\mathfrak{c})-\varphi(z))$, then

$$\frac{(\varphi(\mathfrak{c})-\varphi(s))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))} \leqslant \frac{\varphi(\mathfrak{c})-\varphi(s)}{(\varphi(\mathfrak{c})-\varphi(z))}.$$

Hence, we infer that $\mathcal{K}_2(s,z) \geqslant 0$, for $\mathfrak{c}_0 \leqslant z \leqslant s \leqslant \mathfrak{c}$, and $\mathcal{K}_2(\mathfrak{c},z) = 0$. Therefore, the first property holds. Now, let us take the first derivative for $\mathcal{K}_1(z,z)$ with respect to z, we have

$$\partial_z\mathcal{K}_1(z,z) = \frac{\varphi'(z)}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\left(\varphi(\mathfrak{c})-\varphi(z)\right)-\frac{(\varphi(z)-\varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\varphi'(z),$$

which yields that $\mathcal{K}_1(z,z)$ has zero when $\varphi(r^*) = \frac{\varphi(\mathfrak{c})+\varphi(\mathfrak{c}_0)}{2}$, that is $r^* = \varphi^{-1}\left(\frac{\varphi(\mathfrak{c})+\varphi(\mathfrak{c}_0)}{2}\right)$, and $\partial_z\mathcal{K}_1(z,z) > 0$ on $(\mathfrak{c}_0, r^*)$ and $\partial_z\mathcal{K}_1(z,z) < 0$ on $(r^*,\mathfrak{c})$. Then,

$$\max_{(s,z)\in\Upsilon^2}\mathcal{K}_1(s,z) \leqslant \max_{z\in\Upsilon}\mathcal{K}_1(z,z) = \frac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}{4}. \tag{3.5}$$

Also, by taking the first derivative for $\mathcal{K}_2(s,z)$ with respect to z, we get

$$\partial_z\mathcal{K}_2(s,z) = -\varphi'(z)\frac{(\varphi(s)-\varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}+\varphi'(z).$$

Its easy to check that $\partial_z\mathcal{K}_2(s,z) \geqslant 0$ for all $(s,z) \in \Upsilon^2$. Thus, $\mathcal{K}_2(s,z)$ is increasing with respect to z, so we have

$$\mathcal{K}_2(s,z) \leqslant \mathcal{K}_2(s,s) = \frac{(\varphi(s)-\varphi(\mathfrak{c}_0))}{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}\left(\varphi(\mathfrak{c})-\varphi(s)\right).$$

Similar to the processes with finding $\max_{z\in\Upsilon}\mathcal{K}_1(z,z)$, we can investigate that $\mathcal{K}_2(s,s)$ has a unique zero at $r^* = \varphi^{-1}\left(\frac{\varphi(\mathfrak{c})+\varphi(\mathfrak{c}_0)}{2}\right)$. So, we deduce that

$$\max_{(s,z)\in\Upsilon^2}\mathcal{K}_2(s,z) \leqslant \max_{s\in\Upsilon}\mathcal{K}_2(s,s) = \frac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))}{4}. \tag{3.6}$$

Therefore, by (3.5) and (3.6), the second property follows. □

4. Uniqueness result

In this part, we define the Banach space of continuous functions by $\Pi := \mathcal{C}(\Upsilon = [\mathfrak{c}_0, \mathfrak{c}], \mathbb{R})$, and gifted with the maximum norm $\|y\| = \max_{s\in[\mathfrak{c}_0,\mathfrak{c}]} |y(s)|$.

In what follows, we establish the uniqueness result by using the Weissinger's FPT.

Theorem 4.1. *Let a function* $\mathfrak{h}$ *is continuous, and there is* $\rho > 0$ *such that* $|\mathfrak{h}(s, y(s)) - \mathfrak{h}(s, \bar{y}(s))| \leqslant \rho\, |y(s) - \bar{y}(s)|$, *for each* $y, \bar{y} \in \Pi$ *and* $s \in [\mathfrak{c}_0, \mathfrak{c}]$. *If*

$$\chi := \frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^2}{4}\left(\frac{(3-u)}{|\Theta(u-2)|} + \frac{(u-2)}{|\Theta(u-2)|}\frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right) < \frac{1}{\rho}, \tag{4.1}$$

then there is one solution for the φ*-ABC fractional* BVP (1.6).

Proof. According to Lemma 3.1, we define the mapping $\Omega : \Pi \to \Pi$ as follows:

$$(\Omega y)(s) = \int_{\mathfrak{c}_0}^{\mathfrak{c}} \mathcal{K}(s,z)\varphi'(z)\mathcal{F}(z; y(z))\, dz, \tag{4.2}$$

where $\mathcal{K}$ and $\mathcal{F}$ are defined in (3.2) and (3.3), respectively. Now, for each $y, \bar{y} \in \Pi$ and $s \in [\mathfrak{c}_0, \mathfrak{c}]$, we have

$$\begin{aligned} |\mathcal{F}(s; y(s)) - \mathcal{F}(s; \bar{y}(s))| &\leqslant \frac{(3-u)}{|\Theta(u-2)|}|\mathfrak{h}(s, y(s)) - \mathfrak{h}(s, \bar{y}(s))| \\ &\quad + \frac{(u-2)}{|\Theta(u-2)|}\, {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{u-2,\varphi}|\mathfrak{h}(s, y(s)) - \mathfrak{h}(s, \bar{y}(s))| \\ &\leqslant \frac{\rho\,(3-u)}{|\Theta(u-2)|}|y(s) - \bar{y}(s)| + \frac{\rho\,(u-2)}{|\Theta(u-2)|}\, {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{u-2,\varphi}|y(s) - \bar{y}(s)| \\ &\leqslant \left(\frac{\rho\,(3-u)}{|\Theta(u-2)|} + \frac{\rho\,(u-2)}{|\Theta(u-2)|}\frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right)\|y - \bar{y}\|. \end{aligned} \tag{4.3}$$

Next, by Lemma 3.2, and Eq. (4.3), for every $m \in \mathbb{N}$, we obtain

$$\begin{aligned} &|(\Omega^m y)(s) - (\Omega^m \bar{y})(s)| \\ &\quad = |\Omega(\Omega^{m-1}\, y(s)) - \Omega(\Omega^{m-1}\, \bar{y}(s))| \\ &\quad \leqslant \int_{\mathfrak{c}_0}^{\mathfrak{c}} \mathcal{K}(s,z)\varphi'(z)\left|\mathcal{F}\left(z; \Omega^{m-1}\, y(z)\right) - \mathcal{F}\left(z; \Omega^{m-1}\, \bar{y}(z)\right)\right| dz \\ &\quad \leqslant \frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))}{4}\int_{\mathfrak{c}_0}^{\mathfrak{c}} \varphi'(z)\left(\frac{\rho\,(3-u)}{|\Theta(u-2)|} + \frac{\rho\,(u-2)}{|\Theta(u-2)|}\frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right)\|\Omega^{m-1}\, y - \Omega^{m-1}\, \bar{y}\| dz \\ &\quad \leqslant \left[\frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))}{4}\right]^2\left(\frac{\rho\,(3-u)}{|\Theta(u-2)|} + \frac{\rho\,(u-2)}{|\Theta(u-2)|}\frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right)^2 \\ &\qquad \times \int_{\mathfrak{c}_0}^{\mathfrak{c}}\int_{\mathfrak{c}_0}^{\mathfrak{c}} [\varphi'(z)]^2\, \|\Omega^{m-2}\, y - \Omega^{m-2}\, \bar{y}\| dz\, dz \\ &\quad \vdots \\ &\quad \leqslant \left[\frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))}{4}\right]^m\left(\frac{\rho\,(3-u)}{|\Theta(u-2)|} + \frac{\rho\,(u-2)}{|\Theta(u-2)|}\frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right)^m \\ &\qquad \times \underbrace{\int_{\mathfrak{c}_0}^{\mathfrak{c}} \cdots \int_{\mathfrak{c}_0}^{\mathfrak{c}}}_{m \text{ times}} [\varphi'(z)]^m\, \|y - \bar{y}\|\, \underbrace{dz \ldots dz}_{m \text{ times}} \end{aligned}$$

$$\leqslant \left[\frac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))^2}{4}\right]^m \left(\frac{\rho\,(3-u)}{|\Theta(u-2)|}+\frac{\rho\,(u-2)}{|\Theta(u-2)|}\frac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right)^m \|y-\bar{y}\|.$$

Hence,

$$\|\Omega^m y-\Omega^m \bar{y}\| \leqslant \left[\frac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))^2}{4}\right]^m \left(\frac{\rho\,(3-u)}{|\Theta(u-2)|}+\frac{\rho\,(u-2)}{|\Theta(u-2)|}\frac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right)^m \|y-\bar{y}\|.$$

Since $\rho\chi<1$, then $\sum_{m=1}^{\infty}(\rho\chi)^m$ is geometric convergence series. Thus, in view of the Weissinger's FPT 2.9, the mapping Ω has a unique fixed point. So, it follows that the φ-ABC fractional BVP (1.6) owns one solution on $[\mathfrak{c}_0,\mathfrak{c}]$. □

5. Existence result

This part establishes the existence result by using the Meir-Keeler condensing mappings.

Theorem 5.1. *Let a function $\mathfrak{h}$ is continuous, and there is $\sigma_1,\sigma_2>0$ such that $|\mathfrak{h}(s,y(s))|\leqslant \sigma_1+\sigma_2|y(s)|$, for each $y\in\Pi$ and $s\in[\mathfrak{c}_0,\mathfrak{c}]$. Also, let $\mathbb{A}$ be a bounded subset of $\mathbb{R}$ satisfying the following identity:*

$$\mu\left(\mathfrak{h}(s,\mathbb{A})\right)\leqslant \delta(s)\mu(\mathbb{A}),\ \forall\mathbb{A}.$$

Then, there are at least one solution for the φ-ABC fractional BVP (1.6)*, provided that $\chi<\frac{1}{\sigma_2}$, where χ is given in inequality* (4.1).

Proof. Consider the mapping $\Omega:\Pi\to\Pi$ to be defined as in (4.2), and let a closed bounded convex set $\mathbf{B}_\varepsilon=\{y\in\Pi:\ \|y\|\leqslant\varepsilon\}$, where $\varepsilon\geqslant\dfrac{\chi\sigma_1}{1-\chi\sigma_2}$, with $\chi\sigma_2<1$.

Now, we prove that Ω is itself mapping on $\mathbf{B}_\varepsilon$. For each $y\in\mathbf{B}_\varepsilon$ and $s\in[\mathfrak{c}_0,\mathfrak{c}]$, we get

$$\begin{aligned}|\mathcal{F}(s;y(s))| &\leqslant \frac{(3-u)}{|\Theta(u-2)|}|\mathfrak{h}(s,y(s))|+\frac{(u-2)}{|\Theta(u-2)|}\,{}^{\mathcal{R}}\mathfrak{J}_{\mathfrak{c}_0}^{u-2,\varphi}|\mathfrak{h}(s,y(s))|\\ &\leqslant \frac{\rho\,(3-u)}{|\Theta(u-2)|}\left(\sigma_1+\sigma_2|y(s)|\right)+\frac{\rho\,(u-2)}{|\Theta(u-2)|}\,{}^{\mathcal{R}}\mathfrak{J}_{\mathfrak{c}_0}^{u-2,\varphi}\left(\sigma_1+\sigma_2|y(s)|\right)\\ &\leqslant \left(\frac{(3-u)}{|\Theta(u-2)|}+\frac{(u-2)}{|\Theta(u-2)|}\frac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right)\left(\sigma_1+\sigma_2\|y\|\right).\end{aligned} \tag{5.1}$$

Next, by Lemma 3.2, and Eq. (5.1), one obtains

$$\begin{aligned}|(\Omega y)(s)| &\leqslant \int_{\mathfrak{c}_0}^{\mathfrak{c}}\mathcal{K}(s,z)\varphi'(z)|\mathcal{F}(z;y(z))|dz\\ &\leqslant \frac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))^2}{4}\left(\frac{(3-u)}{|\Theta(u-2)|}+\frac{(u-2)}{|\Theta(u-2)|}\frac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right)\left(\sigma_1+\sigma_2\|y\|\right)\leqslant\chi\left(\sigma_1+\sigma_2\varepsilon\right)\leqslant\varepsilon.\end{aligned}$$

Thus, $\|(\Omega y)\|\leqslant\varepsilon$, that means $\Omega:\mathbf{B}_\varepsilon\to\mathbf{B}_\varepsilon$. Next, we show that Ω is continuous mapping in $\mathbf{B}_\varepsilon$. Let $\{y_m\}_{m\in\mathbb{N}}$ be a convergent sequence that tends to y in $\mathbf{B}_\varepsilon$. By continuity of $\mathfrak{h}$, we find

$$\mathcal{F}(z;y_m(z))\to\mathcal{F}(z;y(z)),\ \text{as } m\to\infty,$$

and so

$$\lim_{m\to\infty}(\Omega y_m)(s)=\int_{\mathfrak{c}_0}^{\mathfrak{c}}\mathcal{K}(s,z)\varphi'(z)\lim_{m\to\infty}\mathcal{F}(z;y_m(z))\,dz=(\Omega y)(s),$$

which means that Ω is a continuous mapping in $\mathbf{B}_\varepsilon$.

In what follows, we investigate that Ω is equicontinuous mapping in $\mathbf{B}_\varepsilon$. For each $y \in \mathbf{B}_\varepsilon$ and $s_1, s_2 \in [\mathfrak{c}_0, \mathfrak{c}]$, and $s_1 < s_2$, we have

$$\begin{aligned}|(\Omega y)(s_2) - (\Omega y)(s_1)| \leqslant & \frac{|(\varphi(s_2) - \varphi(\mathfrak{c}_0)) - (\varphi(s_1) - \varphi(\mathfrak{c}_0))|}{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))} \int_{\mathfrak{c}_0}^{\mathfrak{c}} \varphi'(z) (\varphi(\mathfrak{c}) - \varphi(z)) |\mathcal{F}(z; y(z))| \, dz \\ & + \int_{\mathfrak{c}_0}^{s_1} |(\varphi(s_2) - \varphi(z)) - (\varphi(s_1) - \varphi(z))| \varphi'(z) |\mathcal{F}(z; y(z))| \, dz \\ & + \int_{s_1}^{s_2} |(\varphi(s_2) - \varphi(z))| \varphi'(z) |\mathcal{F}(z; y(z))| \, dz,\end{aligned}$$

which implies the right side tends to zero whenever $s_2 \to s_1$ independently of $y \in \mathbf{B}_\varepsilon$. Therefor, we infer that Ω is and equicontinuous mapping in $\mathbf{B}_\varepsilon$. Finally, we show that $\Omega : \mathbf{B}_\varepsilon \to \mathbf{B}_\varepsilon$ be a Meir-Keeler condensing mapping. To achieve this result, let $\epsilon > 0$, and our aim is to show that there exists $\sigma > 0$ such that

$$\epsilon < \mu(\mathbb{A}) < \epsilon + \sigma \Longrightarrow \mu(\Omega\mathbb{A}) < \epsilon, \quad \text{for any} \quad \mathbb{A} \subset \mathbf{B}_\varepsilon.$$

For any bounded, and equicontinuous subset $\mathbb{A} \subset \mathbf{B}_\varepsilon$, there is a sequence $\{y_m\}_{m\in\mathbb{N}} \subset \mathbb{A}$, by taking $\delta^* = \max_{s\in\Upsilon}\{\delta(s)\}$ and using properties of μ and Lemmas 2.4 and 2.6, one has

$$\begin{aligned}&\mu(\Omega(\mathbb{A})(s)) \\ &\leqslant 2\mu\left\{\int_{\mathfrak{c}_0}^{\mathfrak{c}} \mathcal{K}(s,z)\varphi'(z)\mathcal{F}(z; \{y_m(z)\}_{m=1}^{\infty}) \, dz\right\} \\ &\leqslant 4\int_{\mathfrak{c}_0}^{\mathfrak{c}} \mathcal{K}(s,z)\varphi'(z)\left(\frac{(3-u)}{\Theta(u-2)}\mu(\mathfrak{h}(z, \{y_m(z)\}_{m=1}^{\infty})) + \frac{2(u-2)}{\Theta(u-2)} \, {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{u-2,\varphi}\mu(\mathfrak{h}(z, \{y_m(z)\}_{m=1}^{\infty}))\right) dz \\ &\leqslant \delta^* (\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0)) \int_{\mathfrak{c}_0}^{\mathfrak{c}} \varphi'(z)\left(\frac{(3-u)}{\Theta(u-2)}\mu(\{y_m(z)\}_{m=1}^{\infty}) + \frac{2(u-2)}{\Theta(u-2)} \, {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{u-2,\varphi}\mu(\{y_m(z)\}_{m=1}^{\infty})\right) dz \\ &\leqslant \delta^*\mu(\mathbb{A}) (\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^2 \left(\frac{(3-u)}{\Theta(u-2)} + \frac{2\,(u-2)}{\Theta(u-2)} \frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right).\end{aligned}$$

Form the above inequality, and by Lemma 2.5, we have $\mu(\Omega(\mathbb{A})) \leqslant 8\delta^*\chi\mu(\mathbb{A})$. Note that from the above inequality,

$$\mu(\Omega(\mathbb{A})) \leqslant 8\delta^*\chi\mu(\mathbb{A}) < \epsilon \Longrightarrow \mu(\mathbb{A}) < \frac{\epsilon}{8\delta^*\chi}.$$

By choosing $\sigma = \dfrac{(1 - 8\delta^*\chi)\epsilon}{8\delta^*\chi}$, we find $\epsilon < \mu(\mathbb{A}) < \epsilon + \sigma$, which implies that $\Omega : \mathbf{B}_\varepsilon \to \mathbf{B}_\varepsilon$ is Meir-Keeler condensing mapping. According to Theorem 2.8, the mapping Ω possesses at least one fixed point which represents a solution of the φ-ABC fractional BVP (1.6). □

6. Lyapunov-type inequality

In this section, we study the sufficient conditions of the existence of nontrivial solution by introducing the $\mathbb{LTI}$ for φ-ABC fractional BVP (1.6).

Theorem 6.1. *Let $|\mathfrak{h}(s, y(s))| \leqslant |\mathfrak{q}(s)| \cdot |y(s)|$, where $y, \mathfrak{q} : \Upsilon \to \mathbb{R}$ are continuous functions, and the φ-ABC fractional* BVP (1.6) *owns a nonzero solution. Then, a real continuous mapping $\mathfrak{q}$ verifying the next identity:*

$$\frac{4}{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))} \leqslant \int_{\mathfrak{c}_0}^{\mathfrak{c}} \mathfrak{p}(z) \, dz,$$

where $\mathfrak{p}(z) = \dfrac{(3-u)}{|\Theta(u-2)|}|\mathfrak{q}(z)| + \dfrac{(u-2)}{|\Theta(u-2)|} \, {}^{\mathcal{R}}\mathfrak{I}_{\mathfrak{c}_0}^{u-2,\varphi}|\mathfrak{q}(z)|.$

Proof. Firstly, for $s \in \Upsilon$, by (3.3), we have

$$|\mathcal{F}(s; y(s))| \leqslant \left[\frac{(3-u)}{|\Theta(u-2)|}|q(s)| + \frac{(u-2)}{|\Theta(u-2)|}\, {}^{\mathcal{R}}\mathfrak{I}^{u-2,\varphi}_{c_0}|q(s)|\right] \|y\| \leqslant \mathfrak{p}(s)\|y\|. \tag{6.1}$$

Next, suppose that $y \in \Pi$ is a nontrivial solution of the φ-ABC fractional BVP (1.6). Then, in view of Lemma 3.1, we have

$$y(s) = \int_{c_0}^{c} \mathcal{K}(s, z)\varphi'(z)\mathcal{F}(z; y(z))\, dz,$$

and by Lemma 3.2, and (6.1), one has

$$|y(s)| \leqslant \int_{c_0}^{c} \mathcal{K}(s, z)\ \varphi'(z)\ |\mathcal{F}(z; y(z))|dz \leqslant \frac{(\varphi(c) - \varphi(c_0))}{4} \int_{c_0}^{c} \mathfrak{p}(z)\ dz\|y\|.$$

Thus, by taking maximum norm, and since y is nontrivial, i.e., $\|y\| \neq 0$, it yields that

$$\frac{4}{(\varphi(c) - \varphi(c_0))} \leqslant \int_{c_0}^{c} \mathfrak{p}(z)\ dz.$$

Hence, the required result is proved. □

In the following, we introduce the requirement of no exists a nontrivial solution for a problem (1.6).

Corollary 6.2. *If*

$$\frac{4}{(\varphi(c) - \varphi(c_0))} > \int_{c_0}^{c} \mathfrak{p}(z)\ dz, \tag{6.2}$$

then, the φ-ABC fractional BVP (1.6)*, hasn't a nontrivial solution.*

Proof. For this end, we use the strategy of contradiction, which means we assume that the φ-ABC fractional BVP (1.6) has a nontrivial solution y. Thus, according to Theorem 6.1, we obtain

$$\frac{4}{(\varphi(c) - \varphi(c_0))} \leqslant \int_{c_0}^{c} \mathfrak{p}(z)\ dz,$$

which contradicts condition (6.2). Hence, the result is proven. □

Note that, if $\varphi(s) = s$, then the φ-ABC fractional BVP (1.6) reduces to fractional derivative in the ABC sense.

Corollary 6.3. *Let the fractional* BVP

$$\begin{cases} {}^{ABC}\mathfrak{D}^{u}_{c_0} y(s) + q(s)y(s) = 0, & s \in \Upsilon = [c_0, c], \quad u \in (2, 3], \\ y(c_0) = y''(c_0) = y(c) = 0, \end{cases}$$

possesses a nontrivial solution, then the following inequality holds:

$$\frac{4}{(\varphi(c) - \varphi(c_0))} \leqslant \int_{c_0}^{c} \tilde{\mathfrak{p}}(z)\ dz,$$

where $\tilde{\mathfrak{p}}(z) = \dfrac{(3-u)}{|\Theta(u-2)|}|q(z)| + \dfrac{(u-2)}{|\Theta(u-2)|}\, {}^{\mathcal{R}}\mathfrak{I}^{u-2}_{c_0}|q(z)|.$

Proof. The proof follows directly from Theorem 6.1. □

Remark 6.4. It's important to indicate the following.

1. If $\varphi(s) = s$ and $u = 3$, then the φ-ABC fractional BVP (1.6) reduces to third ordinary differential problem and possesses a nontrivial solution if the following inequality fulfilled:

$$\int_{\mathfrak{c}_0}^{\mathfrak{c}} |\mathfrak{q}(z)|dz \geqslant \frac{4}{(\mathfrak{c}-\mathfrak{c}_0)^2},$$

which coincides with the inequalities (1.3) and (1.5), when $u = 3$, and [12, Theorem 1.3] when $\alpha \to 3$.
2. If $\varphi(s) = s$ and $u \to 2$, then the φ-ABC fractional BVP (1.6) reduces to second ordinary differential problem and possesses a nontrivial solution if the following inequality fulfilled:

$$\int_{\mathfrak{c}_0}^{\mathfrak{c}} |\mathfrak{q}(z)|dz \geqslant \frac{4}{(\mathfrak{c}-\mathfrak{c}_0)},$$

which coincides with the inequality (1.1).

7. Applications

In this position, we are ready to present some examples as illustrative applications for essential results.

Example 7.1. Consider the next φ-ABC fractional BVP:

$$\begin{cases} {}^{ABC}\mathfrak{D}_1^{u,s^2+1}y(s) + \dfrac{0.5s^2}{14+s}\sin(y(s)) = 0, \quad s \in \Upsilon = [1,2], \quad u \in (2,3], \\ y(1) = [y(s)]''_{s^2+1}\big|_{s=1} = y(2) = 0, \end{cases} \tag{7.1}$$

where $\mathfrak{c}_0 = 1, \mathfrak{c} = 2$, $\varphi(s) = s^2+1$, and $\mathfrak{h}(s, y(s)) = \dfrac{0.5s^2}{14+s}\sin(y(s))$, such that $\mathfrak{h}(1, y(1) = 0$. Thus,

$$|\mathfrak{h}(s, y(s)) - \mathfrak{h}(s, \bar{y}(s))| \leqslant \frac{2}{15}|y(s) - \bar{y}(s)|,$$

this means $\rho = \frac{2}{15}$, and for $u = 2.5$, $\Theta(\cdot) = 1$, it follows that

$$\begin{aligned} \chi := {} & \frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^2}{4}\left(\frac{(3-u)}{|\Theta(u-2)|} + \frac{(u-2)}{|\Theta(u-2)|}\frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right) \\ = {} & \frac{9}{4}\left(0.5 + \frac{0.5 \times \sqrt{3}}{\Gamma(1.5)}\right) \approx 3.32371 < \frac{1}{\rho} := 7.5. \end{aligned}$$

Figure 1 and Table 1, display the curves and values of χ is less than $1/\rho$ for the φ-ABC fractional BVP (7.1) for every $u \in (2,3]$. Hence, all conditions of Theorem 4.1 are satisfied, then the φ-ABC fractional BVP (7.1) owns one solution.

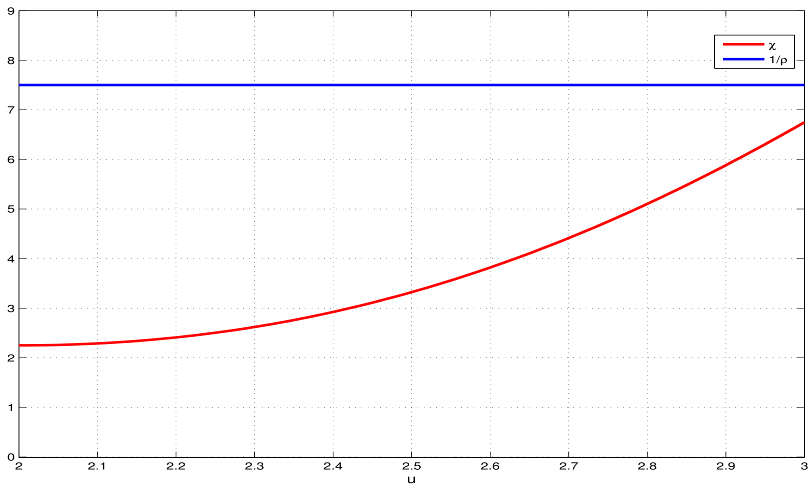


Figure 1: The curves of χ and $1/\rho$ for the φ-ABC fractional BVP (7.1), when $u \in (2,3]$.

Table 1: Numerical values of χ at $u \in (2,3]$ for the φ-ABC fractional BVP (7.1).

u	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3
χ	2.2890	2.4105	2.6207	2.9241	3.3237	3.8208	4.4150	5.1042	5.8843	6.7500

Example 7.2. Consider the next φ-ABC fractional BVP:

$$\begin{cases} {}^{ABC}\mathfrak{D}_1^{u,2\ln(s)}y(s) + \dfrac{\ln(s^2)}{16+e^s}\left(0.5+\tan^{-1}(y(s))\right) = 0, \quad s \in \Upsilon = [1,e], \quad u \in (2,3], \\ y(1) = [y(s)]''_{2\ln(s)}\big|_{s=1} = y(e) = 0, \end{cases} \tag{7.2}$$

where $\mathfrak{c}_0 = 1, \mathfrak{c} = e$, $\varphi(s) = 2\ln(s)$, and $\mathfrak{h}(s,y(s)) = \dfrac{\ln(s^2)}{16+e^s}\left(0.5+\tan^{-1}(y(s))\right)$, such that $\mathfrak{h}(1,y(1)) = 0$. Thus,

$$|\mathfrak{h}(s,y(s))| \leqslant \frac{1}{16+e} + \frac{2}{16+e}|y(s)|,$$

and for any bounded subset $\mathbb{A} \subset \mathbb{R}$, one has

$$\mu\left(\mathfrak{h}(s,\mathbb{A})\right) \leqslant \frac{\ln(s^2)}{16+e^s}\,\mu(\mathbb{A}),$$

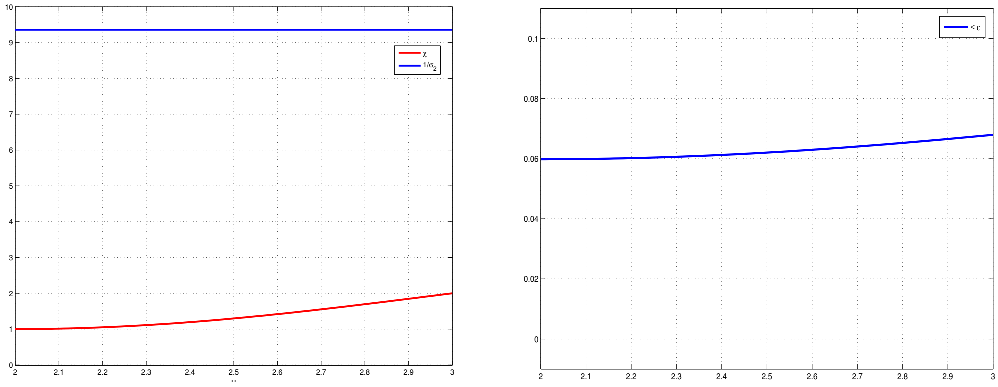
this means $\sigma_1 = \dfrac{1}{16+e}$, $\sigma_2 = \dfrac{2}{16+e}$, and $\delta(s) = \dfrac{\ln(s^2)}{16+e^s}$. Now, let $u = 2.75, \Theta(\cdot) = 1$, it follows that

$$\begin{aligned} \chi :&= \frac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))^2}{4}\left(\frac{(3-u)}{|\Theta(u-2)|} + \frac{(u-2)}{|\Theta(u-2)|}\frac{(\varphi(\mathfrak{c})-\varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)}\right) \\ &= \frac{4}{4}\left(0.25 + \frac{0.75\times 2^{0.75}}{\Gamma(1.75)}\right) \approx 1.62243 < \frac{1}{\sigma_2} := 9.35914, \end{aligned}$$

and

$$\varepsilon \geqslant \frac{\chi\sigma_1}{1-\chi\sigma_2} \approx \frac{0.086676}{1-0.173352} \approx 0.104852,$$

then we can token $\varepsilon = 0.11$. Figure 2 (a), and Table 2, display the graphics and values of χ is less than $1/\sigma_2$ for the φ-ABC fractional BVP (7.2) for each $u \in (2,3]$. Further, Figure 2 (b), and Table 2, show the graphics and values of minimum border of ε for the φ-ABC fractional BVP (7.2), when $u \in (2,3]$. Hence, all conditions of Theorem 5.1 are satisfied, then the φ-ABC fractional BVP (7.2) owns at least one solution.



(a) $\chi < 1/\sigma_2$ **(b)** ε

Figure 2: The graphics of χ, $1/\sigma_2$, and ε for the φ-ABC fractional BVP (7.2), when $u \in (2,3]$.

Table 2: Numerical values of χ and ε at $u \in (2,3]$ for the φ-ABC fractional BVP (7.2).

u	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3
χ	1.0127	1.0502	1.1115	1.1949	1.2979	1.4178	1.5515	1.6955	1.8462	2.0000
$\varepsilon \geqslant$	0.0599	0.0602	0.0606	0.0612	0.0620	0.0630	0.0640	0.0652	0.0666	0.0679

Example 7.3. Consider the next φ-ABC fractional BVP:

$$\begin{cases} {}^{ABC}\mathfrak{D}_{\mathfrak{c}_0}^{u,\varphi} y(s) + \lambda y(s) = 0, \quad s \in \Upsilon = [\mathfrak{c}_0, \mathfrak{c}], \quad u \in (2,3], \\ y(\mathfrak{c}_0) = [y(s)]''_{\varphi(s)}\big|_{s=\mathfrak{c}_0} = y(\mathfrak{c}) = 0, \end{cases} \tag{7.3}$$

where λ is an eigenvalue, $\mathfrak{q}(s) = \lambda$. So, due to Theorem 6.1, we get

$$|\lambda| \geqslant \frac{4}{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^2} \left[\frac{(3-u)}{|\Theta(u-2)|} + \frac{(u-2)}{|\Theta(u-2)|} \frac{(\varphi(\mathfrak{c}) - \varphi(\mathfrak{c}_0))^{u-2}}{\Gamma(u-1)} \right]^{-1}. \tag{7.4}$$

Now, we will study our example for four cases as follows.

Case 1: If $\varphi(s) = e^s$ and $\Upsilon = [0,2]$. Figure 3 (a) and Table 3, Case 1, display the minimum border for eigenvalues λ of the φ-ABC fractional BVP (7.3), when $\varphi(s) = e^s$, $\Upsilon = [0,2]$, $\Theta(\cdot) = 1$, and $u \in (2,3]$.

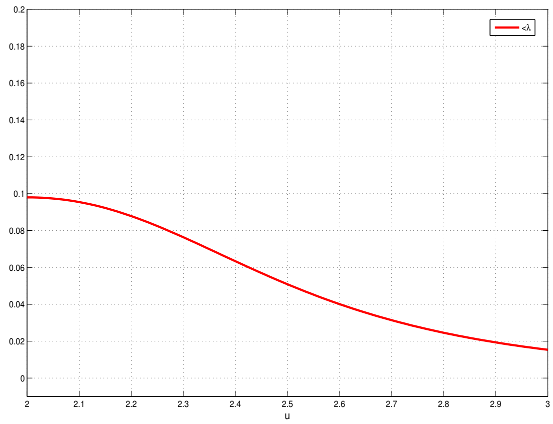
Case 2: If $\varphi(s) = \ln(s)$ and $\Upsilon = [1, e^2]$. Figure 3 (b) and Table 3, Case 2, display the minimum border for eigenvalues λ of the φ-ABC fractional BVP (7.3), when $\varphi(s) = \ln(s)$, $\Upsilon = [1, e^2]$, $\Theta(\cdot) = 1$, and $u \in (2,3]$.

Case 3: If $\varphi(s) = s^\beta, \beta > 0$ and $\Upsilon = [0,3]$. Figure 3 (c) and Table 3, Case 3, display the minimum border for eigenvalues λ of the φ-ABC fractional BVP (7.3), when $\varphi(s) = s^\beta$, $\beta = 0.25, 0.5, 1$, $\Upsilon = [0,3]$, $\Theta(\cdot) = 1$, and $u \in (2,3]$.

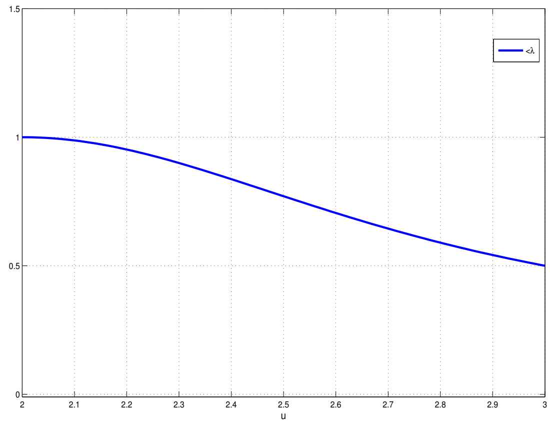
Case 4: If $\varphi(s) = s$ and $\Upsilon = [0,1]$. Figure 3 (d) and Table 3, Case 4, display the minimum border for eigenvalues λ of the φ-ABC fractional BVP (7.3), when $\varphi(s) = s$, $\Upsilon = [0,1]$, $\Theta(\cdot) = 1$, and $u \in (2,3]$, and hence a nontrivial solution of the problem (7.3) exists for all values of u. Moreover, if $u \to 2$, or $u = 3$, then inequality (7.4) becomes $\lambda \geqslant 4$, which represents the minimum border of the second or third differential equations, respectively.

Table 3: Minimum values of eigenvalues λ for φ-ABC fractional BVP (7.3), when $u \in (2,3]$.

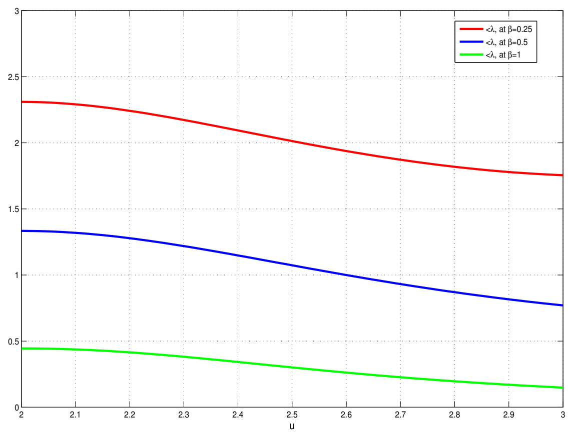
u	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3
Case 1	0.0955	0.0878	0.0764	0.0634	0.0509	0.0401	0.0314	0.0246	0.0193	0.0153
Case 2	0.9875	0.9522	0.8997	0.8369	0.7705	0.7053	0.6445	0.5898	0.5416	0.5000
Case 3, at $\beta = 0.25$	2.2910	2.2419	2.1726	2.0934	2.0130	1.9377	1.8719	1.8184	1.7789	1.7548
Case 3, at $\beta = 0.5$	1.3188	1.2782	1.2186	1.1478	1.0731	0.9998	0.9313	0.8698	0.8159	0.7698
Case 3, at $\beta = 1$	0.4369	0.4148	0.3816	0.3420	0.3009	0.2617	0.2265	0.1959	0.1699	0.1481
Case 4	3.9796	3.9299	3.8675	3.8065	3.7587	3.7331	3.7370	3.7774	3.8618	4.0000



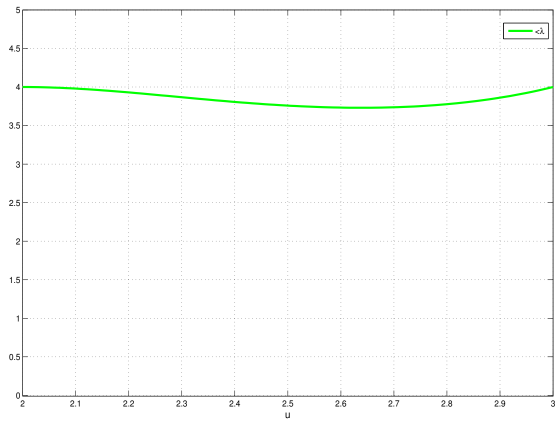
(a) Case 1



(b) Case 2



(c) Case 3



(d) Case 4

Figure 3: The minimum borders for eigenvalues λ of φ-ABC fractional BVP (7.3), at $u \in (2,3]$.

8. Conclusions

This current work was focused for the first time on investigating the existence and uniqueness of the solutions as well as the $\mathbb{LTI}$ for new φ-ABC fractional BVP (1.6). The main results of existence and uniqueness were proved by utilizing the Weissinger, Aghajani FPTs, Meir-Keeler condensing operator, and Kuratowski's measure of non-compactness approach. The sufficient conditions of the existence and non-existence results of nontrivial solutions for the proposed φ-ABC fractional BVP (1.6) were investigated by introducing the $\mathbb{LTI}$. Furthermore, numerical examples with graphics and tables were given to check the validity of the main findings. Moreover, our results were employed to approximate and sketch the minimum borders of eigenvalues for some fractional BVPs with various special cases of a function φ.

Finally, it's worth pointing out that our results are the first to be studied under the φ-ABC fractional derivative, and they are more general than existing in the literature, such as

1. if $\varphi(s) = s$, then our results coincide with the inequalities (1.3) and (1.5) when $u = 3$, and [12, Theorem 1.3] when $\alpha \to 3$;
2. if $\varphi(s) = s$ and $u \to 2$, then our results coincide with the inequality (1.1).

In the future, we will concentrate on studying the main problem under integral and multi-point boundary conditions.

Author's contributions

Every author contributed equally to each part of the paper.

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