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Some quantum corrected dual Euler-Simpson type inequalities for q-differentiable convex functions

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Abstract

The calculus of integrals is used to solve the majority of physics and engineering issues, which are frequently not immediately solved. This compels us to use approximation techniques, the selection of which is based on the class of functions that meet the necessary criteria as well as known points. Within the context of quantum calculus, we present an assessment of the error of the well-known corrected dual Euler-Simpson quadrature rule in this paper. As an auxiliary result, we create a new quantum identity. Using this identity, we prove several quantum-corrected dual Euler-Simpson type integral inequalities for functions with convex q-derivatives. We allow q to go towards 1^- in order to obtain the classical inequalities. We provide a few applications to wrap up this research.

Keywords: Corrected dual Euler-Simpson inequality, convex functions, q-derivative, q-integration.

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1. Introduction

A function $\mathcal{X} : I \to \mathbb{R}$ is said to be convex, where I is a real interval, if

$$\mathcal{X}(z\Delta_1 + (1-z)\Delta_2) \leqslant z\mathcal{X}(\Delta_1) + (1-z)\mathcal{X}(\Delta_2)$$

for all $\Delta_1, \Delta_2 \in I$ and all $z \in [0,1]$ (see [22]).

The development of the theory of inequalities is closely related to the concept of convexity, which is essential and fundamental in many disciplines. This concept is an important tool for studying the qualitative properties of solutions to differential and integral equations. In other words, the majority of problems are solved by calculating integrals, in which we most often adopt approximate methods, which

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brings us back to the evaluation of the error committed in the approximate calculation, see [5, 13, 14, 17, 18, 26].

One may think of quantum calculus as a link between physics and mathematics. It was initially presented by Euler and Jacobi in the traces of Newton's infinite series, which is where it all began. Often regarded as a limitless calculus, it has rapidly advanced as a result of Jackson's study [12]. Numerous scientists have recently expressed a strong interest in studying and researching quantum calculus in various domains.

Recently, Tariboon and Ntouyas [29] introduced the notions of quantum derivative and quantum integral over finite intervals and developed various q-analogues of classical integral inequalities. Inspired by this work, many authors have established estimates of the error bounds of different quadrature rules for different classes of functions. Sudsutad et al. [27] explored the trapezoid and Pachpatte type inequalities. Noor et al. [23] studied the Ostowski type inequalities. Noor et al. [20] gave the quantum analogue of the trapezoid type inequalities for convex and quasi-convex q-derivatives. Tunç et al. [30], presented the first Simpson formula. Noor et al. [21], investigated the Hermite-Hadamard, trapezoid and Ostrowski type inequalities for strongly convex functions. Butt et al. [6] developed the Simpson-Mercer 1/8 formula and the Simpson -Mercer 3/8 formula for convex functions. In [7], Butt et al. constructed a midpoint variant for twice q-differentiable convex functions. Saleh et al. [2], discussed some Simpson-type inequalities via convexity. Meftah et al. [19] explored a new closed four point inequalities for convex q-differentiable functions. Zhang et al. [32], obtained several closed inequalities involving three points for (α, m)-convex functions. For further work on quantum inequalities, we refer interested readers to [1, 3, 4, 8–10, 16].

Motivated by the above-mentioned research on this dynamic domain, we propose in this paper to study the corrected dual Euler-Simpson type integral inequalities which can be stated as follows:

$$\left|\frac{8\mathcal{X}\left(\frac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\frac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\frac{\rho+3\xi}{4}\right)}{15}-\frac{1}{\xi-\rho}\int_{\rho}^{\xi}\mathcal{X}(u)\,du\right| \leqslant \frac{17(\xi-\rho)}{120}\left\|\mathcal{X}'\right\|_{\infty},$$

where $\|\mathcal{X}'\|_{\infty}=\sup_{x\in(\rho,\xi)}|\mathcal{X}'(x)|$ (see [11]).

As an auxiliary finding, we first develop a new quantum integral identity for this. We get certain quantum analogues of the corrected dual Euler-Simpson type inequalities for functions with convex q-derivatives depending on this identity. Finally, we present applications of our findings.

2. Preliminaries

Definition 2.1 ([15, 29])**.** The q-derivative at $u \in J=[\rho, \xi]$ of continuous function $\mathcal{X}: J \to \mathbb{R}$ is defined as:

$${}_{\rho}D_{q}\mathcal{X}(u)=\frac{\mathcal{X}(u)-\mathcal{X}(qu+(1-q)\rho)}{(1-q)(u-\rho)},\ u\neq\rho, \quad \text{and} \quad {}_{\rho}D_{q}\mathcal{X}(\rho)=\lim_{u\to\rho}{}_{\rho}D_{q}\mathcal{X}(u).$$

Definition 2.2 ([15, 29])**.** The q-integral on $J=[\rho, \xi]$ of continuous function $\mathcal{X}: J \to \mathbb{R}$ is defined as:

$$\int_{\rho}^{u}\mathcal{X}(t)\,{}_{\rho}d_{q}t=(1-q)(u-\rho)\sum_{n=0}^{\infty}q^{n}\mathcal{X}(q^{n}u+(1-q^{n})\rho),$$

for $u \in J$. Furthermore, we have

$$\int_{c}^{u}\mathcal{X}(t)\,{}_{\rho}d_{q}t=\int_{\rho}^{u}\mathcal{X}(t)\,{}_{\rho}d_{q}t-\int_{\rho}^{c}\mathcal{X}(t)\,{}_{\rho}d_{q}t,$$

where $c \in (\rho, u)$.

Theorem 2.3 ([28]). *For all continuous functions $\mathcal{X}: J \to \mathbb{R}$, we have*

- $_{\rho}D_q \int\limits_{\rho}^{k} \mathcal{X}(t)_{\rho}\, d_q t = \mathcal{X}(k)$;
- $\int\limits_{c}^{k} {}_{\cdot\rho}D_q \mathcal{X}(t)_{\rho}\, d_q t = \mathcal{X}(k) - \mathcal{X}(c)$ *for* $c \in (\rho, k)$.

Theorem 2.4 ([28]). *Let the continuous functions $\mathcal{X}, \psi : J \to \mathbb{R}$, for $\nu \in \mathbb{R}$ and $u \in J$, we have*

- $\int\limits_{\rho}^{u} [\mathcal{X}(t) + \psi(t)]\, {}_{\cdot\rho}d_q t = \int\limits_{\rho}^{u} \mathcal{X}(t)\, {}_{\cdot\rho}d_q t + \int\limits_{a}^{u} \psi(t)\, {}_{\cdot\rho}d_q t$;
- $\int\limits_{\rho}^{u} (\nu\mathcal{X})(t)\, {}_{\cdot\rho}d_q t = \nu \int\limits_{\rho}^{u} \mathcal{X}(t)\, {}_{\cdot\rho}d_q t$;
- $\int\limits_{c}^{u} \mathcal{X}(t)_{\rho} D_q \psi(t)\, {}_{\cdot\rho}d_q t = \mathcal{X}(t)\psi(t)|_{c}^{u} - \int\limits_{c}^{u} \psi(qt + (1-q)\rho)\, {}_{\cdot\rho}D_q \mathcal{X}(t)\, {}_{\cdot\rho}d_q t$ *for* $c \in (\rho, u)$.

Lemma 2.5 ([17]). *Let $\vartheta \in (0,1)$ and consider $\mathcal{X}(\varepsilon) = 1$, then we have $\int_0^1 {}_{\cdot 0}d_\vartheta t = 1$.*

Lemma 2.6 ([17]). *Let $\vartheta \in (0,1)$ and consider $\mathcal{X}(\varepsilon) = \varepsilon$, then we have $\int_0^{1} \varepsilon {}_{\cdot 0}d_\vartheta t = \frac{1}{1+\vartheta}$.*

Lemma 2.7 ([17]). *Let $\vartheta \in (0,1)$ and consider $\mathcal{X}(\varepsilon) = \varepsilon^2$, then we have*

$$\int_0^1 \varepsilon^2 {}_{\cdot 0}d_\vartheta t = \frac{1}{1+\vartheta+\vartheta^2}.$$

Lemma 2.8 ([29]). *For $p \in \mathbb{R}\backslash\{-1\}$, the following formula holds*

$$\int\limits_{\rho}^{x} (t-\rho)^p\, {}_{\cdot\rho}d_q t = \frac{1-q}{1-q^{p+1}} (x-\rho)^{p+1}.$$

Lemma 2.9 ([25]). *For $0 < \iota \leqslant 1$, the following equivalence is valid if $\mathcal{X}$ and ψ are continuous functions defined on the interval $[\rho, \xi]$:*

$$\int\limits_{0}^{\iota} \psi(t)_{\rho} D_q \mathcal{X}((1-t)\rho + t\xi)\, {}_{\cdot 0}d_q t = \left.\frac{\psi(t)\mathcal{X}((1-t)\rho+t\xi)}{\xi-\rho}\right|_0^{\iota} - \frac{1}{\xi-\rho} \int\limits_{0}^{\iota} \mathcal{X}((1-qt)\rho + qt\xi)_{\rho} D_q \psi(t)\, {}_{\cdot 0}d_q t.$$

3. Main results

We require the following lemma to support our findings.

Lemma 3.1. *Let $0 < q < 1$, and let $\mathcal{X} : I = [\rho, \xi] \to \mathbb{R}$ be a q-differentiable function on I, such that $_{\rho}D_q\mathcal{X}$ is q-integrable on I, then the following equality holds*

$$\begin{aligned} &\frac{1}{15}\left(8\mathcal{X}\left(\frac{3\rho+\xi}{4}\right) - \mathcal{X}\left(\frac{\rho+\xi}{2}\right) + 8\mathcal{X}\left(\frac{\rho+3\xi}{4}\right)\right) - \frac{1}{\xi-\rho}\int\limits_{\rho}^{\xi} \mathcal{X}(u)\, {}_{\cdot\rho}d_q u \\ &= (\xi-\rho)\left(\int\limits_{0}^{\frac{1}{4}} qt {}_{\cdot\rho}D_q\mathcal{X}((1-t)\rho+t\xi)\, {}_{\cdot 0}d_q t + \int\limits_{\frac{1}{4}}^{\frac{1}{2}} \left(qt - \tfrac{8}{15}\right) {}_{\cdot\rho}D_q\mathcal{X}((1-t)\rho+t\xi)\, {}_{\cdot 0}d_q t\right. \\ &\quad \left. + \int\limits_{\frac{1}{2}}^{\frac{3}{4}} \left(qt - \tfrac{7}{15}\right) {}_{\cdot\rho}D_q\mathcal{X}((1-t)\rho+t\xi)\, {}_{\cdot 0}d_q t + \int\limits_{\frac{3}{4}}^{1} (qt-1)\, {}_{\cdot\rho}D_q\mathcal{X}((1-t)\rho+t\xi)\, {}_{\cdot 0}d_q t\right). \end{aligned} \tag{3.1}$$

Proof. Let

$$I_1=\int_0^{\frac{1}{4}} qt.{}_{\rho}D_q\mathcal{X}\left((1-t)\rho+t\xi\right)_0 dt,\qquad I_2=\int_{\frac{1}{4}}^{\frac{1}{2}}\left(qt-\tfrac{8}{15}\right).{}_{\rho}D_q\mathcal{X}\left((1-t)\rho+t\xi\right)_0 dt,$$

$$I_3=\int_{\frac{1}{2}}^{\frac{3}{4}}\left(qt-\tfrac{7}{15}\right).{}_{\rho}D_q\mathcal{X}\left((1-t)\rho+t\xi\right)_0 dt,\qquad I_4=\int_{\frac{3}{4}}^{1}\left(qt-1\right).{}_{\rho}D_q\mathcal{X}\left((1-t)\rho+t\xi\right)_0 dt.$$

From Lemma 2.9, I_1 gives

$$I_1=\int_0^{\frac{1}{4}} qt.{}_{\rho}D_q\mathcal{X}\left((1-t)\rho+t\xi\right)_0 dt=\tfrac{q}{4(\xi-\rho)}\mathcal{X}\left(\tfrac{3\rho+\xi}{4}\right)-\tfrac{q}{\xi-\rho}\int_0^{\frac{1}{4}}\mathcal{X}\left((1-qt)\rho+qt\xi\right).{}_0d_qt. \tag{3.2}$$

Similarly, we get

$$\begin{aligned}
I_2&=\int_{\frac{1}{4}}^{\frac{1}{2}}\left(qt-\tfrac{8}{15}\right).{}_{\rho}D_q\mathcal{X}\left((1-t)\rho+t\xi\right)_0 dt\\
&=\int_0^{\frac{1}{2}}\left(qt-\tfrac{8}{15}\right).{}_{\rho}D_q\mathcal{X}\left((1-t)\rho+t\xi\right)_0 dt-\int_0^{\frac{1}{4}}\left(qt-\tfrac{8}{15}\right).{}_{\rho}D_q\mathcal{X}\left((1-t)\rho+t\xi\right)_0 dt\\
&=\tfrac{15q-16}{30(\xi-\rho)}\mathcal{X}\left(\tfrac{\rho+\xi}{2}\right)+\tfrac{8}{15(\xi-\rho)}\mathcal{X}(\rho)-\tfrac{q}{\xi-\rho}\int_0^{\frac{1}{2}}\mathcal{X}\left((1-qt)\rho+qt\xi\right).{}_0d_qt\\
&\quad-\tfrac{15q-32}{60(\xi-\rho)}\mathcal{X}\left(\tfrac{3\rho+\xi}{4}\right)-\tfrac{8}{15(\xi-\rho)}\mathcal{X}(\rho)+\tfrac{q}{\xi-\rho}\int_0^{\frac{1}{4}}\mathcal{X}\left((1-qt)\rho+qt\xi\right).{}_0d_qt,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
I_3&=\int_{\frac{1}{2}}^{\frac{3}{4}}\left(qt-\tfrac{7}{15}\right).{}_{\rho}D_q\mathcal{X}\left((1-t)\rho+t\xi\right)_0 dt\\
&=\int_0^{\frac{3}{4}}\left(qt-\tfrac{7}{15}\right).{}_{\rho}D_q\mathcal{X}\left((1-t)\rho+t\xi\right)_0 dt-\int_0^{\frac{1}{2}}\left(qt-\tfrac{7}{15}\right).{}_{\rho}D_q\mathcal{X}\left((1-t)\rho+t\xi\right)_0 dt\\
&=\tfrac{45q-28}{60(\xi-\rho)}\mathcal{X}\left(\tfrac{\rho+3\xi}{4}\right)+\tfrac{7}{15(\xi-\rho)}\mathcal{X}(\rho)-\tfrac{q}{\xi-\rho}\int_0^{\frac{3}{4}}\mathcal{X}\left((1-qt)\rho+qt\xi\right).{}_0d_qt\\
&\quad-\tfrac{15q-14}{15(\xi-\rho)}\mathcal{X}\left(\tfrac{\rho+\xi}{2}\right)-\tfrac{7}{15(\xi-\rho)}\mathcal{X}(\rho)+\tfrac{q}{\xi-\rho}\int_0^{\frac{1}{2}}\mathcal{X}\left((1-qt)\rho+qt\xi\right).{}_0d_qt,
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
I_4 &= \int_{\frac{3}{4}}^{1} (qt-1)\,{}_{\cdot\rho}D_q \mathcal{X}((1-t)\rho + t\xi)_0 \, dt \\
&= \int_{0}^{1} (qt-1)\,{}_{\cdot\rho}D_q \mathcal{X}((1-t)\rho + t\xi)_0 \, dt - \int_{0}^{\frac{3}{4}} (qt-1)\,{}_{\cdot\rho}D_q \mathcal{X}((1-t)\rho + t\xi)_0 \, dt \\
&= \tfrac{q-1}{\xi-\rho}\mathcal{X}(\xi) + \tfrac{1}{\xi-\rho}\mathcal{X}(\rho) - \tfrac{q}{\xi-\rho}\int_{0}^{1} \mathcal{X}((1-qt)\rho + qt\xi)\,{}_{\cdot 0}d_q t \\
&\quad - \tfrac{3q-4}{4(\xi-\rho)}\mathcal{X}\left(\tfrac{\rho+3\xi}{4}\right) - \tfrac{1}{\xi-\rho}.\mathcal{X}(\rho) + \tfrac{q}{\xi-\rho}\int_{0}^{\frac{3}{4}} \mathcal{X}((1-qt)\rho + qt\xi)\,{}_{\cdot 0}d_q t.
\end{aligned} \tag{3.5}$$

Adding (3.2)-(3.5), then multiplying the resulting equality by $(\xi-\rho)$, we get

$$\begin{aligned}
(\nu-\mu)\sum_{i=1}^{4} I_i &= \tfrac{8}{15}\mathcal{X}\left(\tfrac{3\rho+\xi}{4}\right) - \tfrac{1}{15}\mathcal{X}\left(\tfrac{\rho+\xi}{2}\right) + \tfrac{8}{15}\mathcal{X}\left(\tfrac{\rho+3\xi}{4}\right) \\
&\quad + (q-1)\mathcal{X}(\xi) - q\int_{0}^{1} \mathcal{X}((1-qt)\rho + qt\xi)\,{}_{\cdot 0}d_q t.
\end{aligned} \tag{3.6}$$

On the other hand from Definition 2.1, we have

$$\mathcal{X}((1-t)\rho + t\xi) - (1-q)\,t\,(\xi-\rho)\,{}_{\cdot\rho}D_q\mathcal{X}((1-t)\rho + t\xi) = \mathcal{X}((1-qt)\rho + qt\xi). \tag{3.7}$$

Integrating both sides of (3.7) with respect to t over $[0,1]$, we get

$$\begin{aligned}
\int_{0}^{1} \mathcal{X}((1-qt)\rho + qt\xi)\,{}_{\cdot 0}d_q t &= \int_{0}^{1} \mathcal{X}((1-t)\rho + t\xi)\,{}_{\cdot 0}d_q t - (\xi-\rho)(1-q)\int_{0}^{1} t.{}_{\rho}D_q\mathcal{X}((1-t)\rho + t\xi)\,{}_{\cdot 0}d_q t \\
&= \int_{0}^{1} \mathcal{X}((1-t)\rho + t\xi)\,{}_{\cdot 0}d_q t - (1-q)\mathcal{X}(\xi) + (1-q)\int_{0}^{1} \mathcal{X}((1-qt)\rho + qt\xi)\,{}_{\cdot 0}d_q t \\
&= -(1-q)\mathcal{X}(\xi) + \int_{0}^{1} \mathcal{X}((1-t)\rho + t\xi)\,{}_{\cdot 0}d_q t + (1-q)\int_{0}^{1} \mathcal{X}((1-qt)\rho + qt\xi)\,{}_{\cdot 0}d_q t,
\end{aligned}$$

which implies

$$q\int_{0}^{1} \mathcal{X}((1-qt)\rho + qt\xi)\,{}_{\cdot 0}d_q t = -(1-q)\mathcal{X}(\xi) + \int_{0}^{1} \mathcal{X}((1-t)\rho + t\xi)\,{}_{\cdot 0}d_q t. \tag{3.8}$$

Using (3.8) in (3.6), we get

$$(\nu-\mu)\sum_{i=1}^{4} I_i = \tfrac{8}{15}\mathcal{X}\left(\tfrac{3\rho+\xi}{4}\right) - \tfrac{1}{15}\mathcal{X}\left(\tfrac{\rho+\xi}{2}\right) + \tfrac{8}{15}\mathcal{X}\left(\tfrac{\rho+3\xi}{4}\right) - \int_{0}^{1} \mathcal{X}((1-t)\rho + t\xi)\,{}_{\cdot 0}d_q t.$$

By changing the variable $(1-t)\rho+t\xi=u$, the aforementioned equivalence yields the intended outcome. $\square$

Theorem 3.2. *Let $0<q<1$, and let $\mathcal{X}: I=[\rho,\xi]\to\mathbb{R}$ be a q-differentiable function on I, such that ${}_{\rho}D_q\mathcal{X}$ is q-integrable on I. If $|{}_{\rho}D_q\mathcal{X}|$ is convex, then we have*

$$\begin{aligned}&\left|\frac{1}{15}\left(8\mathcal{X}\left(\frac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\frac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\frac{\rho+3\xi}{4}\right)\right)-\frac{1}{\xi-\rho}\int_{\rho}^{\xi}\mathcal{X}(u)\,{}_{\rho}d_qu\right|\\&\leqslant(\xi-\rho)\left(\left(\Omega_1+\frac{64+205q+115q^2+36q^3-90q^4}{960(1+q)(1+q+q^2)}\right)|{}_{\rho}D_q\mathcal{X}(\rho)|\right.\\&\quad\left.+\left(\Omega_2+\frac{304+231q+81q^2-208q^3-150q^4}{960(1+q)(1+q+q^2)}\right)|{}_{\rho}D_q\mathcal{X}(\xi)|\right),\end{aligned}$$

where Ω_1 and Ω_2 are defined by (3.13) and (3.17), respectively.

Proof. Applying the absolute value to both sides of (3.1), and using some of its properties, we get

$$\begin{aligned}&\left|\frac{1}{15}\left(8\mathcal{X}\left(\frac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\frac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\frac{\rho+3\xi}{4}\right)\right)-\frac{1}{\xi-\rho}\int_{\rho}^{\xi}\mathcal{X}(u)\,{}_{\rho}d_qu\right|\\&\leqslant(\xi-\rho)\left(\int_0^{\frac14}qt.|{}_{\rho}D_q\mathcal{X}((1-t)\rho+t\xi)|\,{}_0d_qt+\int_{\frac14}^{\frac12}\left|qt-\tfrac{8}{15}\right|.|{}_{\rho}D_q\mathcal{X}((1-t)\rho+t\xi)|\,{}_0d_qt\right.\\&\quad\left.+\int_{\frac12}^{\frac34}\left|qt-\tfrac{7}{15}\right||{}_{\rho}D_q\mathcal{X}((1-t)\rho+t\xi)|\,{}_0d_qt+\int_{\frac34}^{1}(1-qt)|{}_{\rho}D_q\mathcal{X}((1-t)\rho+t\xi)|\,{}_0d_qt\right).\end{aligned}\tag{3.9}$$

Given (3.9) and the convexity of $|{}_{\rho}D_q\mathcal{X}|$, we derive

$$\begin{aligned}&\left|\frac{1}{15}\left(8\mathcal{X}\left(\frac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\frac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\frac{\rho+3\xi}{4}\right)\right)-\frac{1}{\xi-\rho}\int_{\rho}^{\xi}\mathcal{X}(u)\,{}_{\rho}d_qu\right|\\&\leqslant(\xi-\rho)\left(\int_0^{\frac14}qt\left((1-t)|{}_{\rho}D_q\mathcal{X}(\rho)|+t|{}_{\rho}D_q\mathcal{X}(\xi)|\right){}_0d_qt\right.\\&\quad+\int_{\frac14}^{\frac12}\left(\tfrac{8}{15}-qt\right).\left((1-t)|{}_{\rho}D_q\mathcal{X}(\rho)|+t|{}_{\rho}D_q\mathcal{X}(\xi)|\right){}_0d_qt\\&\quad+\int_{\frac12}^{\frac34}\left|qt-\tfrac{7}{15}\right|\left((1-t)|{}_{\rho}D_q\mathcal{X}(\rho)|+t|{}_{\rho}D_q\mathcal{X}(\xi)|\right){}_0d_qt\\&\quad\left.+\int_{\frac34}^{1}(1-qt)\left((1-t)|{}_{\rho}D_q\mathcal{X}(\rho)|+t|{}_{\rho}D_q\mathcal{X}(\xi)|\right){}_0d_qt\right)\\&=(\xi-\rho)\left(|{}_{\rho}D_q\mathcal{X}(\rho)|\int_0^{\frac14}qt(1-t)\,{}_0d_qt+|{}_{\rho}D_q\mathcal{X}(\xi)|\int_0^{\frac14}qt^2\,{}_0d_qt\right.\end{aligned}$$

$$
\begin{aligned}
&+\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tfrac{8}{15}-qt\right)(1-t)\,{}_0d_qt+\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tfrac{8}{15}-qt\right)t\,{}_0d_qt \qquad (3.10)\\
&+\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\tfrac{7}{15}\right|(1-t)\,{}_0d_qt+\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\tfrac{7}{15}\right|t\,{}_0d_qt\\
&+\left|{}_{a}D_qf(a)\right|\int_{\frac{3}{4}}^{1}(1-qt)(1-t)\,{}_0d_qt+\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|\int_{\frac{3}{4}}^{1}(1-qt)\,t\,{}_0d_qt\Bigg)\\
&=(\xi-\rho)\Bigg(\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|\Bigg(\int_0^{\frac{1}{4}}qt(1-t)\,{}_0d_qt+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tfrac{8}{15}-qt\right)(1-t)\,{}_0d_qt\\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\tfrac{7}{15}\right|(1-t)\,{}_0d_qt+\int_{\frac{3}{4}}^{1}(1-qt)(1-t)\,{}_0d_qt\Bigg)\\
&+\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|\Bigg(\int_0^{\frac{1}{4}}qt^2\,{}_0d_qt+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tfrac{8}{15}-qt\right)t\,{}_0d_qt+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\tfrac{7}{15}\right|t\,{}_0d_qt+\int_{\frac{3}{4}}^{1}(1-qt)\,t\,{}_0d_qt\Bigg)\Bigg)\\
&=(\xi-\rho)\left(\left(\Omega_1+\tfrac{64+205q+115q^2+36q^3-90q^4}{960(1+q)(1+q+q^2)}\right)\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|\right.\\
&\left.+\left(\Omega_2+\tfrac{304+231q+81q^2-208q^3-150q^4}{960(1+q)(1+q+q^2)}\right)\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|\right),
\end{aligned}
$$

where the right-hand side integrals of (3.10), whose values are as follows, were calculated using Definition 2.2 and Lemma 2.9.

$$
\int_0^{\frac{1}{4}}qt(1-t)\,{}_0d_qt=\tfrac{3q+3q^2+4q^3}{64(1+q)(1+q+q^2)}, \tag{3.11}
$$

$$
\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tfrac{8}{15}-qt\right)(1-t)\,{}_0d_qt=\tfrac{64+100q+55q^2-24q^3-45q^4}{960(1+q)(1+q+q^2)}, \tag{3.12}
$$

$$
\Omega_1=\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\tfrac{7}{15}\right|(1-t)\,{}_0d_qt=\begin{cases}\tfrac{28+39q-21q^2-64q^3-60q^4}{960(1+q)(1+q+q^2)}, & \text{if } 0<q<\tfrac{28}{45},\\ \tfrac{32928-49448q+129771q^2-273957q^3+21060q^4+51300q^5}{216000q(1+q)(1+q+q^2)}, & \text{if } \tfrac{28}{45}\leqslant q\leqslant\tfrac{14}{15},\\ \tfrac{-28-39q+21q^2+64q^3+60q^4}{960(1+q)(1+q+q^2)}, & \text{if } \tfrac{14}{15}<q<1,\end{cases} \tag{3.13}
$$

$$
\int_{\frac{3}{4}}^{1}(1-qt)(1-t)\,{}_0d_qt=\tfrac{4q+q^2-3q^4}{64(1+q)(1+q+q^2)}, \tag{3.14}
$$

$$
\int_0^{\frac{1}{4}}qt^2\,{}_0d_qt=\tfrac{q}{64(1+q+q^2)}, \tag{3.15}
$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \left(\tfrac{8}{15} - qt\right) t \,{}_0d_q t = \tfrac{64+36q+21q^2-28q^3-15q^4}{960(1+q)(1+q+q^2)}, \tag{3.16}$$

$$\Omega_2 = \int_{\frac{1}{2}}^{\frac{3}{4}} \left|qt - \tfrac{7}{15}\right| t \,{}_0d_q t = \begin{cases} \tfrac{84+5q-55q^2-124q^3-60q^4}{960(1+q)(1+q+q^2)}, & \text{if } 0 < q < \tfrac{28}{45}, \\ \tfrac{14112-7672q-166191q^2+244497q^3+53640q^4+2700q^5}{216000q(1+q)(1+q+q^2)}, & \text{if } \tfrac{28}{45} \leqslant q \leqslant \tfrac{14}{15}, \\ \tfrac{-84-5q+55q^2+124q^3+60q^4}{960(1+q)(1+q+q^2)}, & \text{if } \tfrac{14}{15} < q < 1, \end{cases} \tag{3.17}$$

$$\int_{\frac{3}{4}}^{1} (1 - qt) t \,{}_0d_q t = \tfrac{16+12q+3q^2-12q^3-9q^4}{64(1+q)(1+q+q^2)}. \tag{3.18}$$

The outcome is thus proved. □

Corollary 3.3. *By making* q *tend towards* 1^- *in Theorem 3.2, we obtain*

$$\left| \frac{8\mathcal{X}\left(\frac{3\rho+\xi}{4}\right) - \mathcal{X}\left(\frac{\rho+\xi}{2}\right) + 8\mathcal{X}\left(\frac{\rho+3\xi}{4}\right)}{15} - \frac{1}{\xi-\rho}\int_{\rho}^{\xi} \mathcal{X}(u)\, du \right| \leqslant \frac{17(\xi-\rho)}{240}\left(\left|\mathcal{X}'(\rho)\right| + \left|\mathcal{X}'(\xi)\right|\right). \tag{3.19}$$

To demonstrate the validity of the findings in Theorem 3.2, we provide the following example.

Example 3.4. Let us consider the function $\mathcal{X} : [0,1] \to \mathbb{R}$ defined by $\mathcal{X}(u) = u^3$. The q-derivative is ${}_0D_q\mathcal{X}(t) = \left(1+q+q^2\right)u^2$, which is convex. The result presented in Theorem 3.2 related to the function $\mathcal{X}$ under consideration by taking $\frac{14}{15} < q < 1$, we have

$${}_0D_q\mathcal{X}(u) = \tfrac{u^3-(qu)^3}{(1-q)u} = \tfrac{1-q^3}{1-q}u^2 = \left(1+q+q^2\right)u^2.$$

So, ${}_0D_q\mathcal{X}(0) = 0$ and ${}_0D_q\mathcal{X}(1) = \left(1+q+q^2\right)$. On the other hand, we have

$$\int_0^u \mathcal{X}(t)_0\, d_q t = (1-q)\, u \sum_{n=0}^{\infty} u^3 q^{4n} = \tfrac{1-q}{1-q^4}u^4 = \tfrac{1}{1+q+q^2+q^3}u^4.$$

Hence, $\int_0^1 \mathcal{X}(t)_0\, d_{\frac{1}{2}} t = \frac{1}{1+q+q^2+q^3}$. Thus

$$\begin{aligned} \text{LHS} &= \left| \tfrac{1}{15}\left(8\mathcal{X}\left(\tfrac{1}{4}\right) - \mathcal{X}\left(\tfrac{1}{2}\right) + 8\mathcal{X}\left(\tfrac{3}{4}\right)\right) - \int_0^1 \mathcal{X}(u)\, {}_0d_{\frac{1}{2}}u \right| \\ &= \left| \tfrac{1}{15}\left(\tfrac{8}{64} - \tfrac{1}{8} + \tfrac{216}{64}\right) - \tfrac{1}{1+q+q^2+q^3} \right| = \left| \tfrac{9}{40} - \tfrac{1}{1+q+q^2+q^3} \right| = \tfrac{1}{1+q+q^2+q^3} - \tfrac{9}{40} \end{aligned}$$

and

$$\text{RHS} = \tfrac{220+226q+139q^2-84q^3-90q^4}{960(1+q)}.$$

Figure 1 provides a graphical representation of the two expressions.

According to the representations provided by Figure 1, we observe that the right-hand side always exceeds the left-hand side, thereby confirming the validity of the result stated in Theorem 3.2.

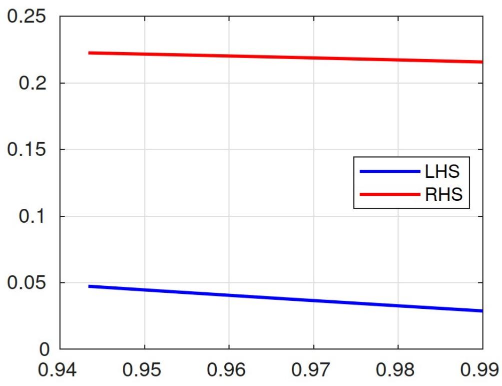

Figure 1: Graphical illustration of Theorem 3.2.

Theorem 3.5. *Let $0 < q < 1$, and let $\mathcal{X} : I = [\rho, \xi] \to \mathbb{R}$ be a q-differentiable function on I, such that ${}_\rho D_q \mathcal{X}$ is q-integrable on I. If $|{}_\rho D_q \mathcal{X}|^\zeta$ is convex, then we have*

$$\begin{aligned}
&\left| \frac{1}{15}\left(8\mathcal{X}\left(\frac{3\rho+\xi}{4}\right) - \mathcal{X}\left(\frac{\rho+\xi}{2}\right) + 8\mathcal{X}\left(\frac{\rho+3\xi}{4}\right)\right) - \frac{1}{\xi-\rho}\int_\rho^\xi \mathcal{X}(u)\, {}_\rho d_q u \right| \\
&\leqslant (\xi-\rho)\left(\frac{1-q}{q(1-q^{p+1})}\right)^{\frac{1}{p}}\left(\left(\frac{q}{4}\right)^{1+\frac{1}{p}}\left(\frac{3+4q}{16(1+q)}|{}_\rho D_q \mathcal{X}(\rho)|^\zeta + \frac{1}{16(1+q)}|{}_\rho D_q \mathcal{X}(\xi)|^\zeta\right)^{\frac{1}{\zeta}}\right. \\
&\quad + \left(\frac{(32-15q)^{p+1}-(32-30q)^{p+1}}{60^{p+1}}\right)^{\frac{1}{p}}\left(\frac{2+3q}{16(1+q)}|{}_\rho D_q \mathcal{X}(\rho)|^\zeta + \frac{2+q}{16(1+q)}|{}_\rho D_q \mathcal{X}(\xi)|^\zeta\right)^{\frac{1}{\zeta}} \\
&\quad + \left(\widetilde{\Lambda}\right)^{\frac{1}{p}}\left(\frac{1+2q}{16(1+q)}|{}_\rho D_q \mathcal{X}(\rho)|^\zeta + \frac{3+2q}{16(1+q)}|{}_\rho D_q \mathcal{X}(\xi)|^\zeta\right)^{\frac{1}{\zeta}} \\
&\quad \left. + \left(\frac{(4-3q)^{p+1}-(4-4q)^{p+1}}{4^{p+1}}\right)^{\frac{1}{p}}\left(\frac{q}{16(1+q)}|{}_\rho D_q \mathcal{X}(\rho)|^\zeta + \frac{4+3q}{16(1+q)}|{}_\rho D_q \mathcal{X}(\xi)|^\zeta\right)^{\frac{1}{\zeta}}\right),
\end{aligned}$$

where $\zeta > 1$ with $\frac{1}{p} + \frac{1}{\zeta} = 1$, and $\widetilde{\Lambda} = \frac{q(1-q^{p+1})}{1-q}\Lambda$ with Λ is defined by (3.22).

Proof. It is evident that (3.9) is obtained by applying the absolute value to both sides of (3.1). Now, using (3.9) and the Hölder's inequality, we get

$$\begin{aligned}
&\left| \frac{1}{15}\left(8\mathcal{X}\left(\frac{3\rho+\xi}{4}\right) - \mathcal{X}\left(\frac{\rho+\xi}{2}\right) + 8\mathcal{X}\left(\frac{\rho+3\xi}{4}\right)\right) - \frac{1}{\xi-\rho}\int_\rho^\xi \mathcal{X}(u)\, {}_\rho d_q u \right| \\
&\leqslant (\xi-\rho)\left(\left(\int_0^{\frac{1}{4}} (qt)^p\, {}_0 d_q t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{4}} |{}_\rho D_q \mathcal{X}((1-t)\rho+t\xi)|^\zeta\, {}_0 d_q t\right)^{\frac{1}{\zeta}}\right. \\
&\quad + \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{8}{15}-qt\right)^p\, {}_0 d_q t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}} |{}_\rho D_q \mathcal{X}((1-t)\rho+t\xi)|^\zeta\, {}_0 d_q t\right)^{\frac{1}{\zeta}} \qquad (3.20) \\
&\quad + \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left|qt-\frac{7}{15}\right|^p\, {}_0 d_q t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}} |{}_\rho D_q \mathcal{X}((1-t)\rho+t\xi)|^\zeta\, {}_0 d_q t\right)^{\frac{1}{\zeta}}
\end{aligned}$$

$$+\left(\int_{\frac{3}{4}}^{1}(1-qt)^{p}\,{}_{0}d_{q}t\right)^{\frac{1}{p}}\left(\int_{\frac{3}{4}}^{1}\left|{}_{\rho}D_{q}\mathcal{X}\left((1-t)\rho+t\xi\right)\right|^{\zeta}\,{}_{0}d_{q}t\right)^{\frac{1}{\zeta}}\Bigg).$$

From (3.20) and the convexity of $|{}_{\rho}D_{q}\mathcal{X}|^{\zeta}$, we arrive at

$$\left|\frac{1}{15}\left(8\mathcal{X}\left(\frac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\frac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\frac{\rho+3\xi}{4}\right)\right)-\frac{1}{\xi-\rho}\int_{\rho}^{\xi}\mathcal{X}(u)\,{}_{\rho}d_{q}u\right|$$
$$\leqslant(\xi-\rho)\left(\left(\int_{0}^{\frac{1}{4}}(qt)^{p}\,{}_{0}d_{q}t\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{4}}\left((1-t)\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}+t\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\right){}_{0}d_{q}t\right)^{\frac{1}{\zeta}}\right.$$
$$+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\frac{8}{15}-qt\right)^{p}\,{}_{0}d_{q}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left((1-t)\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}+t\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\right){}_{0}d_{q}t\right)^{\frac{1}{\zeta}}$$
$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\frac{7}{15}\right|^{p}\,{}_{0}d_{q}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left((1-t)\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}+t\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\right){}_{0}d_{q}t\right)^{\frac{1}{\zeta}}$$
$$\left.+\left(\int_{\frac{3}{4}}^{1}(1-qt)^{p}\,{}_{0}d_{q}t\right)^{\frac{1}{p}}\left(\int_{\frac{3}{4}}^{1}\left((1-t)\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}+t\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\right){}_{0}d_{q}t\right)^{\frac{1}{\zeta}}\right)$$
$$=(\xi-\rho)\left(\left(\frac{1-q}{q(1-q^{p+1})}\right)^{\frac{1}{p}}\left(\frac{q}{4}\right)^{1+\frac{1}{p}}\left(\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}\int_{0}^{\frac{1}{4}}(1-t)\,{}_{0}d_{q}t+\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\int_{0}^{\frac{1}{4}}t\,{}_{0}d_{q}t\right)^{\frac{1}{\zeta}}\right.\tag{3.21}$$
$$+\left(\frac{1-q}{q(1-q^{p+1})}\right)^{\frac{1}{p}}\left(\frac{(32-15q)^{p+1}-(32-30q)^{p+1}}{60^{p+1}}\right)^{\frac{1}{p}}$$
$$\times\left(\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}\int_{\frac{1}{4}}^{\frac{1}{2}}(1-t)\,{}_{0}d_{q}t+\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\int_{\frac{1}{4}}^{\frac{1}{2}}t\,{}_{0}d_{q}t\right)^{\frac{1}{\zeta}}$$
$$+(\Lambda)^{\frac{1}{p}}\left(\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}\int_{\frac{1}{2}}^{\frac{3}{4}}(1-t)\,{}_{0}d_{q}t+\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\int_{\frac{1}{2}}^{\frac{3}{4}}t\,{}_{0}d_{q}t\right)^{\frac{1}{\zeta}}+\left(\frac{1-q}{q(1-q^{p+1})}\right)^{\frac{1}{p}}\left(\frac{(4-3q)^{p+1}-(4-4q)^{p+1}}{4^{p+1}}\right)^{\frac{1}{p}}$$
$$\left.\times\left(\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}\int_{\frac{3}{4}}^{1}(1-t)\,{}_{0}d_{q}t+\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\int_{\frac{3}{4}}^{1}t\,{}_{0}d_{q}t\right)^{\frac{1}{\zeta}}\right)$$
$$=(\xi-\rho)\left(\frac{1-q}{q(1-q^{p+1})}\right)^{\frac{1}{p}}\left(\left(\frac{q}{4}\right)^{1+\frac{1}{p}}\left(\frac{3+4q}{16(1+q)}\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}+\frac{1}{16(1+q)}\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right.$$
$$+\left(\frac{(32-15q)^{p+1}-(32-30q)^{p+1}}{60^{p+1}}\right)^{\frac{1}{p}}\left(\frac{2+3q}{16(1+q)}\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}+\frac{2+q}{16(1+q)}\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\right)^{\frac{1}{\zeta}}$$

$$
+\left(\widetilde{\Lambda}\right)^{\frac{1}{p}}\left(\tfrac{1+2q}{16(1+q)}\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}+\tfrac{3+2q}{16(1+q)}\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\right)^{\frac{1}{\zeta}}
+\left(\tfrac{(4-3q)^{p+1}-(4-4q)^{p+1}}{4^{p+1}}\right)^{\frac{1}{p}}\left(\tfrac{q}{16(1+q)}\left|{}_{\rho}D_{q}\mathcal{X}(\rho)\right|^{\zeta}+\tfrac{4+3q}{16(1+q)}\left|{}_{\rho}D_{q}\mathcal{X}(\xi)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\Bigg),
$$

where Definition 2.2 and Lemma 2.9, were exploited when calculating the integrals of (3.21), whose values are

$$
\int_{\alpha}^{\beta} t\,{}_{0}d_{q}t=(\beta-\alpha)\,\tfrac{\beta+\alpha q}{1+q},
$$

$$
\int_{\alpha}^{\beta}(1-t)\,{}_{0}d_{q}t=(\beta-\alpha)\left(\tfrac{1-\beta+(1-\alpha)q}{1+q}\right),
$$

$$
\int_{0}^{\frac{1}{4}}(qt)^{p}\,{}_{0}d_{q}t=\tfrac{1-q}{q(1-q^{p+1})}\left(\tfrac{q}{4}\right)^{p+1},
$$

$$
\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tfrac{8}{15}-qt\right)^{p}\,{}_{0}d_{q}t=\tfrac{1-q}{q(1-q^{p+1})}\left(\tfrac{(32-15q)^{p+1}-(32-30q)^{p+1}}{60^{p+1}}\right),
$$

$$
\int_{\frac{3}{4}}^{1}(1-qt)^{p}\,{}_{0}d_{q}t=\tfrac{1-q}{q(1-q^{p+1})}\left(\tfrac{(4-3q)^{p+1}-(4-4q)^{p+1}}{4^{p+1}}\right),
$$

$$
\widetilde{\Lambda}=\tfrac{q(1-q^{p+1})}{1-q}\Lambda,
$$

with

$$
\Lambda=\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\tfrac{7}{15}\right|^{p}\,{}_{0}d_{q}t=\begin{cases}\tfrac{1-q}{q(1-q^{p+1})}\left(\tfrac{(28-30q)^{p+1}-(28-45q)^{p+1}}{60^{p+1}}\right), & \text{if } 0<q<\tfrac{28}{45},\\ \tfrac{1-q}{q(1-q^{p+1})}\left(\tfrac{(28-30q)^{p+1}+(45q-28)^{p+1}}{60^{p+1}}\right), & \text{if } \tfrac{28}{45}\leqslant q\leqslant\tfrac{14}{15},\\ \tfrac{1-q}{q(1-q^{p+1})}\left(\tfrac{(45q-28)^{p+1}-(30q-28)^{p+1}}{60^{p+1}}\right), & \text{if } \tfrac{14}{15}<q<1.\end{cases}\tag{3.22}
$$

The outcome is thus proved. □

Corollary 3.6. *By making q tend towards 1^{-} in Theorem 3.5, we obtain*

$$
\left|\tfrac{1}{15}\left(8\mathcal{X}\left(\tfrac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\tfrac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\tfrac{\rho+3\xi}{4}\right)\right)-\tfrac{1}{\xi-\rho}\int_{\rho}^{\xi}\mathcal{X}(u)\,du\right|
$$
$$
\leqslant\tfrac{\xi-\rho}{16}\left(\tfrac{1}{1+p}\right)^{\frac{1}{p}}\Bigg(\left(\tfrac{7|\mathcal{X}'(\rho)|^{\zeta}+|\mathcal{X}'(\xi)|^{\zeta}}{8}\right)^{\frac{1}{\zeta}}+\left(\tfrac{|\mathcal{X}'(\rho)|^{\zeta}+7|\mathcal{X}'(\xi)|^{\zeta}}{8}\right)^{\frac{1}{\zeta}}
+\left(\tfrac{(17)^{p+1}-(2)^{p+1}}{(15)^{p+1}}\right)^{\frac{1}{p}}\left(\left(\tfrac{5|\mathcal{X}'(\rho)|^{\zeta}+3|\mathcal{X}'(\xi)|^{\zeta}}{8}\right)^{\frac{1}{\zeta}}+\left(\tfrac{3|\mathcal{X}'(\rho)|^{\zeta}+5|\mathcal{X}'(\xi)|^{\zeta}}{8}\right)^{\frac{1}{\zeta}}\right)\Bigg).
$$

Moreover, if we use the discrete power mean inequality, we get

$$
\left|\tfrac{1}{15}\left(8\mathcal{X}\left(\tfrac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\tfrac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\tfrac{\rho+3\xi}{4}\right)\right)-\tfrac{1}{\xi-\rho}\int_{\rho}^{\xi}\mathcal{X}(u)\,du\right|
$$

$$\leqslant \frac{\xi-\rho}{8}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(1+\left(\frac{(17)^{p+1}-(2)^{p+1}}{(15)^{p+1}}\right)^{\frac{1}{p}}\right)\left(\frac{|\mathcal{X}'(\rho)|^{\zeta}+|\mathcal{X}'(\xi)|^{\zeta}}{2}\right)^{\frac{1}{\zeta}}.$$

Example 3.7. Let us consider the function $\mathcal{X}:[0,1]\to\mathbb{R}$ defined by $\mathcal{X}(u)=u^3$. The q-derivative is $|_0D_q\mathcal{X}(t)|^{\zeta}=\left(1+q+q^2\right)^{\zeta}u^{2\zeta}$, which is convex where $\zeta,p>1$ with $\frac{1}{\zeta}+\frac{1}{p}=1$. The result presented in Theorem 3.5 related to the function $\mathcal{X}$ under consideration by taking $\rho=0,\xi=1,q=\frac{1}{2}$ and $\zeta=p=2$, is given as follows:

$$\mathrm{LHS}=0.3083\leqslant 0.3619=\mathrm{RHS}.$$

Clearly, we have for $\rho=0,\xi=u$ and $\mathcal{X}(u)=u^3$,

$${}_0D_q\mathcal{X}(u)=\frac{u^3-(qu)^3}{(1-q)u}=\frac{1-q^3}{1-q}u^2=\left(1+q+q^2\right)u^2.$$

So,

$$|_0D_q\mathcal{X}(u)|^{\zeta}=\left(1+q+q^2\right)^{\zeta}\left(u^2\right)^{\zeta}$$

and

$$\int_0^u\mathcal{X}(t)_0\,d_qt=(1-q)\,u^4\sum_{n=0}^{\infty}q^{4n}=\frac{1-q}{1-q^4}u^4=\frac{1}{1+q+q^2+q^3}u^4.$$

Thus, if we put $\rho=0,\xi=1,q=\frac{1}{2}$, and $\zeta=p=2$, in Theorem 3.5, we obtain

$$\begin{aligned}\mathrm{LHS}&=\left|\frac{1}{15}\left(8\mathcal{X}\left(\tfrac{1}{4}\right)-\mathcal{X}\left(\tfrac{1}{2}\right)+8\mathcal{X}\left(\tfrac{3}{4}\right)\right)-\int_0^1\mathcal{X}(u)\,{}_0d_qu\right|\\&=\left|\frac{1}{15}\left(\frac{8}{64}-\frac{1}{8}+\frac{216}{64}\right)-\frac{1}{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}\right|=\left|\frac{27}{120}-\frac{8}{15}\right|=\frac{37}{120}\approx 0.3083.\end{aligned}$$

and RHS ≈ 0.3619.

Theorem 3.8. *Let $0<q<1$, and let $\mathcal{X}:I=[\rho,\xi]\to\mathbb{R}$ be a q-differentiable function on I, such that ${}_\rho D_q\mathcal{X}$ is q-integrable on I. If $|_\rho D_q\mathcal{X}|^{\zeta}$ is convex, then we have*

$$\begin{aligned}&\left|\frac{1}{15}\left(8\mathcal{X}\left(\frac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\frac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\frac{\rho+3\xi}{4}\right)\right)-\frac{1}{\xi-\rho}\int_\rho^\xi\mathcal{X}(u)\,{}_\rho d_qu\right|\\&\leqslant(\xi-\rho)\left(\left(\frac{q}{16(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\frac{3q+3q^2+4q^3}{64(1+q)(1+q+q^2)}|_\rho D_q\mathcal{X}(\rho)|^{\zeta}+\frac{q}{64(1+q+q^2)}|_\rho D_q\mathcal{X}(\xi)|^{\zeta}\right)^{\frac{1}{\zeta}}\right.\\&\quad+\left(\frac{32+2q-15q^2}{240(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\frac{64+100q+55q^2-24q^3-45q^4}{960(1+q)(1+q+q^2)}|_\rho D_q\mathcal{X}(\rho)|^{\zeta}\right.\\&\quad\left.+\frac{64+36q+21q^2-28q^3-15q^4}{960(1+q)(1+q+q^2)}|_\rho D_q\mathcal{X}(\xi)|^{\zeta}\right)^{\frac{1}{\zeta}}+(\Omega)^{1-\frac{1}{\zeta}}\left(\Omega_1^{\zeta}|_\rho D_q\mathcal{X}(\rho)|+\Omega_2^{\zeta}|_\rho D_q\mathcal{X}(\xi)|\right)^{\frac{1}{\zeta}}\\&\quad\left.+\left(\frac{4-3q^2}{16(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\frac{4q+q^2-3q^4}{64(1+q)(1+q+q^2)}|_\rho D_q\mathcal{X}(\rho)|^{\zeta}+\frac{16+12q+3q^2-12q^3-9q^4}{64(1+q)(1+q+q^2)}|_\rho D_q\mathcal{X}(\xi)|^{\zeta}\right)^{\frac{1}{\zeta}}\right),\end{aligned}$$

where $\zeta\geqslant 1$ and Ω,Ω_1, and Ω_2 are defined by (3.24), (3.13), and (3.17), respectively.

Proof. It is evident that (3.9) is obtained by applying the absolute value to both sides of (3.1). Now, using (3.9) and the power mean inequality, we get

$$\begin{aligned}
&\left|\tfrac{1}{15}\left(8\mathcal{X}\left(\tfrac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\tfrac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\tfrac{\rho+3\xi}{4}\right)\right)-\tfrac{1}{\xi-\rho}\int_{\rho}^{\xi}\mathcal{X}(u)\,{}_{\rho}d_q u\right| \\
&\leqslant(\xi-\rho)\left(\left(\int_0^{\frac14} qt\,{}_0d_q t\right)^{1-\frac{1}{\zeta}}\left(\int_0^{\frac14} qt\left|{}_{\rho}D_q\mathcal{X}((1-t)\rho+t\xi)\right|^{\zeta}{}_0d_q t\right)^{\frac{1}{\zeta}}\right. \\
&\quad+\left(\int_{\frac14}^{\frac12}\left(\tfrac{8}{15}-qt\right){}_0d_q t\right)^{1-\frac{1}{\zeta}}\left(\int_{\frac14}^{\frac12}\left(\tfrac{8}{15}-qt\right)\left|{}_{\rho}D_q\mathcal{X}((1-t)\rho+t\xi)\right|^{\zeta}{}_0d_q t\right)^{\frac{1}{\zeta}} \\
&\quad+\left(\int_{\frac12}^{\frac34}\left|qt-\tfrac{7}{15}\right|{}_0d_q t\right)^{1-\frac{1}{\zeta}}\left(\int_{\frac12}^{\frac34}\left|qt-\tfrac{7}{15}\right|\left|{}_{\rho}D_q\mathcal{X}((1-t)\rho+t\xi)\right|^{\zeta}{}_0d_q t\right)^{\frac{1}{\zeta}} \\
&\quad\left.+\left(\int_{\frac34}^{1}(1-qt)\,{}_0d_q t\right)^{1-\frac{1}{\zeta}}\left(\int_{\frac34}^{1}(1-qt)\left|{}_{\rho}D_q\mathcal{X}((1-t)\rho+t\xi)\right|^{\zeta}{}_0d_q t\right)^{\frac{1}{\zeta}}\right).
\end{aligned}\tag{3.23}$$

From (3.23) and the convexity of $|{}_{\rho}D_q\mathcal{X}|^{\zeta}$, we get

$$\begin{aligned}
&\left|\tfrac{1}{15}\left(8\mathcal{X}\left(\tfrac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\tfrac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\tfrac{\rho+3\xi}{4}\right)\right)-\tfrac{1}{\xi-\rho}\int_{\rho}^{\xi}\mathcal{X}(u)\,{}_{\rho}d_q u\right| \\
&\leqslant(\xi-\rho)\left(\left(\int_0^{\frac14} qt\,{}_0d_q t\right)^{1-\frac{1}{\zeta}}\left(\int_0^{\frac14} qt\left((1-t)\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}+t\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\right){}_0d_q t\right)^{\frac{1}{\zeta}}\right. \\
&\quad+\left(\int_{\frac14}^{\frac12}\left(\tfrac{8}{15}-qt\right){}_0d_q t\right)^{1-\frac{1}{\zeta}}\left(\int_{\frac14}^{\frac12}\left(\tfrac{8}{15}-qt\right)\left((1-t)\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}+t\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\right){}_0d_q t\right)^{\frac{1}{\zeta}} \\
&\quad+\left(\int_{\frac12}^{\frac34}\left|qt-\tfrac{7}{15}\right|{}_0d_q t\right)^{1-\frac{1}{\zeta}}\left(\int_{\frac12}^{\frac34}\left|qt-\tfrac{7}{15}\right|\left((1-t)\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}+t\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\right){}_0d_q t\right)^{\frac{1}{\zeta}} \\
&\quad\left.+\left(\int_{\frac34}^{1}(1-qt)\,{}_0d_q t\right)^{1-\frac{1}{\zeta}}\left(\int_{\frac34}^{1}(1-qt)\left((1-t)\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}+t\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\right){}_0d_q t\right)^{\frac{1}{\zeta}}\right) \\
&=(\xi-\rho)\left(\left(\tfrac{1}{16(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}\int_0^{\frac14} qt(1-t)\,{}_0d_q t+\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\int_0^{\frac14} qt^2\,{}_0d_q t\right)^{\frac{1}{\zeta}}\right.
\end{aligned}$$

$$+\left(\tfrac{32+2q-15q^2}{240(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tfrac{8}{15}-qt\right)(1-t)\,{}_0d_qt+\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tfrac{8}{15}-qt\right)t\,{}_0d_qt\right)^{\frac{1}{\zeta}}$$

$$+(\Omega)^{1-\frac{1}{\zeta}}\left(\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\tfrac{7}{15}\right|(1-t)\,{}_0d_qt+\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\tfrac{7}{15}\right|t\,{}_0d_qt\right)^{\frac{1}{\zeta}}$$

$$\left.+\left(\tfrac{4-3q^2}{16(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}\int_{\frac{3}{4}}^{1}(1-qt)(1-t)\,{}_0d_qt+\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\int_{\frac{3}{4}}^{1}(1-qt)t\,{}_0d_qt\right)^{\frac{1}{\zeta}}\right)$$

$$=(\xi-\rho)\left(\left(\tfrac{q}{16(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\tfrac{3q+3q^2+4q^3}{64(1+q)(1+q+q^2)}\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}+\tfrac{q}{64(1+q+q^2)}\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right.$$

$$+\left(\tfrac{32+2q-15q^2}{240(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\tfrac{64+100q+55q^2-24q^3-45q^4}{960(1+q)(1+q+q^2)}\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}+\tfrac{64+36q+21q^2-28q^3-15q^4}{960(1+q)(1+q+q^2)}\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\right)^{\frac{1}{\zeta}}$$

$$+(\Omega)^{1-\frac{1}{\zeta}}\left(\Omega_2\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}+\Omega_1\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\right)^{\frac{1}{\zeta}}$$

$$\left.+\left(\tfrac{4-3q^2}{16(1+q)}\right)^{1-\frac{1}{\zeta}}\left(\tfrac{4q+q^2-3q^4}{64(1+q)(1+q+q^2)}\left|{}_{\rho}D_q\mathcal{X}(\rho)\right|^{\zeta}+\tfrac{16+12q+3q^2-12q^3-9q^4}{64(1+q)(1+q+q^2)}\left|{}_{\rho}D_q\mathcal{X}(\xi)\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right),$$

where we have used

$$\int_0^{\frac{1}{4}}qt\,{}_0d_qt=\tfrac{q}{16(1+q)},$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\tfrac{8}{15}-qt\right){}_0d_qt=\tfrac{32+2q-15q^2}{240(1+q)},$$

$$\int_{\frac{1}{2}}^{\frac{3}{4}}\left(qt-\tfrac{7}{15}\right){}_0d_qt=\begin{cases}\tfrac{28+28q-45q-30q^2}{240(1+q)}, & \text{if } 0\leqslant q\leqslant\tfrac{28}{45}\\ \tfrac{784-17360q+345q^2+900q^3}{3600q(1+q)}, & \text{if } \tfrac{28}{45}\leqslant q\leqslant\tfrac{14}{15}\\ \tfrac{-28+17q+30q^2}{240(1+q)}, & \text{if } \tfrac{14}{15}\leqslant q\leqslant 1,\end{cases}$$

$$\Omega=\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\tfrac{7}{15}\right|{}_0d_qt=\begin{cases}\tfrac{28-17q-30q^2}{240(1+q)}, & \text{if } 0<q<\tfrac{28}{45},\\ \tfrac{784-1736q+345q^2+900q^3}{3600q(1+q)}, & \text{if } \tfrac{28}{45}\leqslant q\leqslant\tfrac{14}{15},\\ \tfrac{-28+17q+30q^2}{240(1+q)}, & \text{if } \tfrac{14}{15}<q<1,\end{cases}\tag{3.24}$$

$$\int_{\frac{3}{4}}^{1}(1-qt)\,{}_0d_qt=\tfrac{4-3q^2}{16(1+q)},$$

and (3.11)-(3.18). The outcome is thus proved. □

Corollary 3.9. *By making* q *tend towards* 1^- *in Theorem 3.8, we obtain*

$$\left|\tfrac{1}{15}\left(8\mathcal{X}\left(\tfrac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\tfrac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\tfrac{\rho+3\xi}{4}\right)\right)-\tfrac{1}{\xi-\rho}\int_{\rho}^{\xi}\mathcal{X}(u)\,du\right|$$

$$\leqslant \frac{\xi-\rho}{32}\left(\left(\left(\frac{5|\mathcal{X}'(\rho)|^{\zeta}+|\mathcal{X}'(\xi)|^{\zeta}}{6}\right)^{\frac{1}{\zeta}}+\left(\frac{|\mathcal{X}'(\rho)|^{\zeta}+5|\mathcal{X}'(\xi)|^{\zeta}}{2}\right)^{\frac{1}{\zeta}}\right)\right.$$
$$\left.+\frac{19}{15}\left(\left(\frac{25|\mathcal{X}'(\rho)|^{\zeta}+13|\mathcal{X}'(\xi)|^{\zeta}}{38}\right)^{\frac{1}{\zeta}}+\left(\frac{13|\mathcal{X}'(\rho)|^{\zeta}+25|\mathcal{X}'(\xi)|^{\zeta}}{38}\right)^{\frac{1}{\zeta}}\right)\right).$$

Moreover, if we use the discrete power mean inequality, we get

$$\left|\frac{8\mathcal{X}\left(\frac{3\rho+\xi}{4}\right)-\mathcal{X}\left(\frac{\rho+\xi}{2}\right)+8\mathcal{X}\left(\frac{\rho+3\xi}{4}\right)}{15}-\frac{1}{\xi-\rho}\int_{\rho}^{\xi}\mathcal{X}(u)\,du\right|\leqslant\frac{17(\xi-\rho)}{120}\left(\frac{|\mathcal{X}'(\rho)|^{\zeta}+|\mathcal{X}'(\xi)|^{\zeta}}{2}\right)^{\frac{1}{\zeta}}. \tag{3.25}$$

4. Applications

4.1. Corrected dual Euler-Simpson quadrature formula

Let Υ be the partition of the points $\rho=x_0<x_1<\cdots<x_n=\xi$ of the interval $[\rho,\xi]$, and consider the quadrature formula

$$\int_{\rho}^{\xi}\mathcal{X}(u)\,du=\lambda(\mathcal{X},\Upsilon)+R(\mathcal{X},\Upsilon),$$

where

$$\lambda(\mathcal{X},\Upsilon)=\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{15}\left(8\mathcal{X}\left(\frac{3x_i+x_{i+1}}{4}\right)-\mathcal{X}\left(\frac{x_i+x_{i+1}}{2}\right)+8\mathcal{X}\left(\frac{x_i+3x_{i+1}}{4}\right)\right)$$

and the corresponding approximation error is shown by $R(\mathcal{X},\Upsilon)$.

Proposition 4.1. *Let $\mathcal{X}:[\rho,\xi]\to\mathbb{R}$ be a differentiable function on (ρ,ξ), where $0\leqslant\rho<\xi$ and $n\in\mathbb{N}$ with $\mathcal{X}'\in L[\rho,\xi]$. If $|\mathcal{X}'|$ is a convex function, then we have*

$$|R(f,\Upsilon)|\leqslant\sum_{i=0}^{n-1}\frac{17(x_{i+1}-x_i)^2}{240}\left(\left|\mathcal{X}'(x_i)\right|+\left|\mathcal{X}'(x_{i+1})\right|\right).$$

Proof. Applying Corollary 3.9 on the subintervals $[x_i,x_{i+1}]$ ($i=0$ to $n-1$) of the partition Υ, we get

$$\left|\frac{1}{15}\left(8\mathcal{X}\left(\frac{3x_i+x_{i+1}}{4}\right)-\mathcal{X}\left(\frac{x_i+x_{i+1}}{2}\right)+8\mathcal{X}\left(\frac{x_i+3x_{i+1}}{4}\right)\right)-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\mathcal{X}(u)\,du\right|$$
$$\leqslant\frac{17(x_{i+1}-x_i)}{240}\left(\left|\mathcal{X}'(x_i)\right|+\left|\mathcal{X}'(x_{i+1})\right|\right).$$

The desired outcome is produced by multiplying both sides of the aforementioned inequality by $(x_{i+1}-x_i)$, adding up the resulting inequalities for $i=0$ to $n-1$, and applying the triangular inequality. □

4.2. Application to special means

Let ρ,ξ be a real numbers. $A(\rho,\xi)=\frac{\rho+\xi}{2}$, represents the arithmetic mean. For $p\in\mathbb{R}\backslash\{-1,0\}$ and $\rho,\xi>0$ with $\rho\neq\xi$, $L_p(\rho,\xi)=\left(\frac{\xi^{p+1}-\rho^{p+1}}{(p+1)(\xi-\rho)}\right)^{\frac{1}{p}}$ is the p-logarithmic mean.

Proposition 4.2. *Let $0<q<\frac{28}{45}$ and $\rho,\xi\in\mathbb{R}$ with $0<\rho<\xi$, then for $n>2$ we have*

$$\left|16A\left(\left(\frac{3\rho+\xi}{4}\right)^n,\left(\frac{\rho+3\xi}{4}\right)^n\right)-A^n(\rho,\xi)-\frac{15(n+1)(1-q)}{1-q^{n+1}}L_n^n(\rho,\xi)\right|$$
$$\leqslant\frac{\xi-\rho}{64}\left(\frac{92+244q+94q^2-28q^3-150q^4}{(1+q)(1+q+q^2)}\left|n\rho^{n-1}\right|+\frac{388+236q+26q^2-332q^3-210q^4}{960(1+q)(1+q+q^2)}\left|\frac{\xi^n-(q(\xi-\rho)+\rho)^n}{(1-q)(\xi-\rho)}\right|\right).$$

Proof. Applying Theorem 3.2 to the function $\mathcal{X}(u) = u^n$ yields the claim. □

Proposition 4.3. *Let* $\frac{28}{45} \leqslant q \leqslant \frac{14}{15}$, $\zeta > 1$ *with* $\frac{1}{p} + \frac{1}{\zeta} = 1$, *and* $\rho, \xi \in \mathbb{R}$ *with* $0 < \rho < \xi$, *then for* $n > 2$ *we have*

$$\begin{aligned}
&\left|16A\left(\left(\tfrac{3\rho+\xi}{4}\right)^n, \left(\tfrac{\rho+3\xi}{4}\right)^n\right) - A^n(\rho,\xi) - \tfrac{15(n+1)(1-q)}{1-q^{n+1}} L_n^n(\rho,\xi)\right| \\
&\leqslant 15(\xi-\rho)\left(\tfrac{1-q}{q(1-q^{p+1})}\right)^{\frac{1}{p}}\left(\left(\tfrac{q}{4}\right)^{1+\frac{1}{p}}\left(\tfrac{3+4q}{16(1+q)}\left|n\rho^{n-1}\right|^{\zeta} + \tfrac{1}{16(1+q)}\left|\tfrac{\xi^n-(q(\xi-\rho)+\rho)^n}{(1-q)(\xi-\rho)}\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right. \\
&\quad + \left(\tfrac{(32-15q)^{p+1}-(32-30q)^{p+1}}{60^{p+1}}\right)^{\frac{1}{p}}\left(\tfrac{2+3q}{16(1+q)}\left|n\rho^{n-1}\right|^{\zeta} + \tfrac{2+q}{16(1+q)}\left|\tfrac{\xi^n-(q(\xi-\rho)+\rho)^n}{(1-q)(\xi-\rho)}\right|^{\zeta}\right)^{\frac{1}{\zeta}} \\
&\quad + \left(\tfrac{(28-30q)^{p+1}+(45q-28)^{p+1}}{60^{p+1}}\right)^{\frac{1}{p}}\left(\tfrac{1+2q}{16(1+q)}\left|n\rho^{n-1}\right|^{\zeta} + \tfrac{3+2q}{16(1+q)}\left|\tfrac{\xi^n-(q(\xi-\rho)+\rho)^n}{(1-q)(\xi-a)}\right|^{\zeta}\right)^{\frac{1}{\zeta}} \\
&\quad \left. + \left(\tfrac{(4-3q)^{p+1}-(4-4q)^{p+1}}{4^{p+1}}\right)^{\frac{1}{p}}\left(\tfrac{q}{16(1+q)}\left|n\rho^{n-1}\right|^{\zeta} + \tfrac{4+3q}{16(1+q)}\left|\tfrac{\xi^n-(q(\xi-\rho)+\rho)^n}{(1-q)(\xi-\rho)}\right|^{\zeta}\right)^{\frac{1}{\zeta}}\right).
\end{aligned}$$

Proof. Applying Theorem 3.5 to the function $\mathcal{X}(u) = u^n$ yields the claim. □

5. Conclusion

In this study, we have constructed a new quantum identity. Based on this equality, we have established some quantum counterparts of the corrected dual Euler-Simpson type integral inequalities for functions with convex q-derivatives. They have deduced the classical instances. We presented some applications of our result to inequalities involving special-means and numerical integration. In the future, we hope to expand this study to a few other well-known inequalities as well as their fractional quantum counterparts. We anticipate that our findings open a new area for more study in this area and will be helpful and inspire scholars to investigate in this field.

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Conflict of interest

The authors have no conflicts of interest to declare.

Author contributions

All authors contributed to the content and writing of the main manuscript. All authors reviewed the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this research.

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