J. Math. Computer Sci., 40 (2026), 542–547

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

An approach to solve nonlinear Caputo-Fabrizio fractional differential equations

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Abstract

This paper focuses on developing an approach for solving nonlinear Caputo-Fabrizio fractional differential equations (FDEs). In this approach, we use the exactness and integrating factors to solve nonlinear Caputo-Fabrizio FDE. The FDE is transformed to an ODE, and then the method of characteristics will generate an integrating factor for this ODE. Afterwards, using the exactness of differential equations concept, implicit analytical solutions of such equations are presented. We present an example to demonstrate how this approach facilitates the solution of equations that are generalized to results in previous studies.

Keywords: Caputo-Fabrizo fractional differential equations, exactness of ODEs, integrating factors, nonlinear differential equations.

2020 MSC: 26A33, 34A08.

1. Introduction

Differential and integral operators, and in general, operator theory, play important roles in Mathematics. The details for operator theory are discussed in [12, 24]. Recently, properties of difference operators and their applications, as inverse problems on difference equations and finding differential transforms for quotient of two functions, have been discussed in [7]. There are numerous approaches to solving differential equations, difference equations, and integral equations, as illustrated in [6]. Fractional calculus modeling has grown in popularity and is now widely used in a variety of engineering and applied science fields. For example, FDEs are handy tools for investigating models based on functional differential equations. For example, in [5], an application of fractional differential equations to RC-circuits was described. A thorough review of the applications of fractional calculus and fractional order derivative-based techniques in computer vision [8], work on sensors, analogue and digital filters [4], modeling of biological phenomena and diseases (see [16, 20]), modeling of diffusive type circuits (see [22, 23]).

1.1. Method of characteristics

The method of characteristics, see [21], is a powerful technique for solving partial differential equations (PDE), particularly first-order PDEs. It converts the PDE into a set of ordinary differential equations (ODEs) along specific curves known as characteristics, allowing the solution to be found more easily. The

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doi: 10.22436/jmcs.040.04.07

Received: 2025-04-06 *Revised:* 2025-04-29 *Accepted:* 2025-07-12

method is typically applied to first-order PDEs of the following form:

$$a(x,y,u)\frac{\partial u}{\partial x}+b(x,y,u)\frac{\partial u}{\partial y}=c(x,y,u), \tag{1.1}$$

where $u=u(x,y)$ is the unknown function, and $a(x,y,u), b(x,y,u)$, and $c(x,y,u)$ are known functions. To solve (1.1), set up the characteristic system of ODEs

$$\frac{dx}{dt}=a(x,y,u), \quad \frac{dy}{dt}=b(x,y,u), \quad \frac{du}{dt}=c(x,y,u).$$

Afterwards, solve this system of ODEs with the given initial/boundary conditions. Finally, eliminate the parameter t to find $u(x,y)$.

Example 1.1. The given PDE

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=x+y$$

is a linear first-order PDE, and it can be written in the form

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=c(x,y),$$

where $a(x,y)=1, b(x,y)=1$, and $c(x,y)=x+y$. The characteristic equations for this PDE are

$$\frac{dx}{ds}=1, \quad \frac{dy}{ds}=1, \quad \frac{du}{ds}=x+y.$$

Solving the characteristic equation $\frac{dx}{ds}=1$ implies

$$x(s)=s+x_0, \tag{1.2}$$

where x_0 is the initial value of x when $s=0$. Moreover, solving $\frac{dy}{ds}=1$ implies

$$y(s)=s+y_0, \tag{1.3}$$

where y_0 is the initial value of y when $s=0$. Finally, solving $\frac{du}{ds}=x+y$ and substituting the expressions for $x(s)$ and $y(s)$ will give

$$\frac{du}{ds}=(s+x_0)+(s+y_0)=2s+x_0+y_0.$$

Integrating with respect to s, the solution is written as

$$u(s)=s^2+(x_0+y_0)\,s+u_0. \tag{1.4}$$

The initial condition is $u(x,0)=0$. When $y=0$, we have $y_0=0$, and from the characteristic equations, $x=x_0$. Therefore, the initial value of u at $y=0$ is $u_0=0$. Thus, the solution for $u(s)$ becomes: $u(s)=s^2+x_0s$. By the characteristic equations (1.2) and (1.3) and since $y_0=0$, we have $y=s$. Therefore, $s=y$. By substituting this result into the equation (1.4), we conclude that $u(y)=y^2+x_0y$. The equation $x=s+x_0$ gives $x_0=x-y$. Now substituting this back into the solution for $u(y)$:

$$u(x,y)=y^2+(x-y)y=xy.$$

Thus, the solution to the PDE is $u(x,y)=xy$.

1.2. Exact differential equations

Definition 1.2. A differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{1.5}$$

is exact if there exists $\rho(x, y(x)) \in C^1(\mathbf{R}^2)$ such that

$$\frac{\partial \rho}{\partial x}(x,y) = M(x,y) \text{ and } \frac{\partial \rho}{\partial y}(x,y) = N(x,y).$$

Clearly,

$$\frac{\partial N}{\partial x}(x,y) = \frac{\partial^2 \rho}{\partial x \partial y}(x,y) = \frac{\partial^2 \rho}{\partial y \partial x}(x,y) = \frac{\partial M}{\partial y}(x,y).$$

Thus, if (1.5) is exact, then

$$\frac{\partial N}{\partial x}(x,y) = \frac{\partial M}{\partial y}(x,y).$$

Example 1.3. The differential equation $\left(2x + y^2\right)dx + 2xy\,dy = 0$ is exact.

Definition 1.4. A non-zero function $\tau(x, y(x)) \in C^1(\mathbf{R}^2)$ is a factor for (1.5) if

$$\tau(x,y)(M(x,y)dx + N(x,y)dy) = 0$$

is exact.

Example 1.5. The differential equation

$$\left(2x + 2y^2\right)dx + 2xy\,dy = 0 \tag{1.6}$$

is not exact. It is not hard to show that $\tau(x) = x$ is an integrating factor for (1.6).

The exactness of differential equations is discussed in many references. For example, see [4].

1.3. Caputo-Fabrizio fractional derivative

Definition 1.6 ([11])**.** For a differentiable function $h : [c, \infty) \to \mathbf{R}, \beta \in [0,1], c \in [-\infty, 0]$, and $x \geqslant 0$, the Caputo-Fabrizio fractional derivative is defined as

$$\left({}^{CF}D^{\beta}h\right)(x) = \begin{cases} \frac{1}{1-\beta}\int_c^x e^{-\frac{\beta}{1-\beta}(x-u)} f'(u)du, & 0 \leqslant \beta < 1, \\ h'(x), & \beta = 1. \end{cases}$$

For example, it is easy to show that if $h(t) = t^r; r > 0$, then

$$\left({}^{CF}D^{\beta}h\right)(t) = \frac{r}{1-\beta}\left(\frac{\beta}{\beta-1}\right)^{-r}\gamma\left(r, \frac{\beta t}{\beta - 1}\right)e^{-\frac{\beta t}{1-\beta}},$$

where γ is the incomplete gamma function, see [1].

Numerous researchers have thoroughly investigated new definitions and generalizations of fractional derivatives and integrals, along with their applications. Definition 1.6 was considered in [2], where the authors examined novel characteristics of the Caputo-Fabrizio derivative and devised an efficient method for converting classes of fractional differential equations into initial value problems with ordinary derivatives. It easy to notice that using Definition 1.6, if $h(x) = \left({}^{CF}D^{\beta}f\right)(x)$, then $h(x) = 0$ and $h(x)$ satisfies the following.

Proposition 1.7 ([3])**.** *Let D^{β} be the fractional derivative in Caputo-Fabrizio sense and let $h(x) = \left(D^{\beta}f\right)(x)$. Then $h(x)$ satisfies the differential equation $(1-\beta)h'(x) + \beta h(x) = f'(x)$.*

Proof. Since $h(x) = \left(D^{\beta}f\right)(x)$, then

$$(1-\beta)e^{\frac{\beta x}{1-\beta}}h(x) = \int_a^x e^{\frac{\beta u}{1-\beta}} f'(u)du. \tag{1.7}$$

Differentiating both sides of (1.7) implies $(1-\beta)e^{\frac{\beta x}{1-\beta}}h'(x) + \beta e^{\frac{\beta x}{1-\beta}}h(x) = e^{\frac{\beta x}{1-\beta}}f'(x)$. This proves the result. □

2. Solving nonlinear Caputo-Fabrizio fractional differential equations

The following result is a direct implementation of Proposition 1.7.

Proposition 2.1. *For* $K(x,y) \in C^1(\boldsymbol{R}^2)$, *the fractional differential equation* $\left(D^\beta y\right)(x) = K(x, y(x))$ *is equivalent to the ordinary differential equation*

$$\left((1-\beta)\frac{\partial K}{\partial y} - 1\right) y'(x) + (1-\beta)\frac{\partial K}{\partial x} + \beta K = 0. \tag{2.1}$$

Proof. Since $\left(D^\beta y\right)(x) = K(x, y(x))$, then (2.1) implies

$$(1-\beta)\frac{d}{dx}K(x,y) + \beta K(x,y) = y'. \tag{2.2}$$

Since $\frac{d}{dx}K(x,y) = \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}y'(x)$, then (2.2) becomes

$$(1-\beta)\left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}y'\right) + \beta K(x,y) = y'.$$

Rearranging the terms proves the desired result. □

In general, (2.1) is not exact,

$$\frac{\partial}{\partial x}\left((1-\beta)\frac{\partial K}{\partial y} - 1\right) = (1-\beta)\frac{\partial^2 K}{\partial x \partial y} \quad \text{and} \quad \frac{\partial}{\partial y}\left((1-\beta)\frac{\partial K}{\partial x} + \beta K\right) = (1-\beta)\frac{\partial^2 K}{\partial y \partial x} + \beta\frac{\partial K}{\partial y}.$$

Thus , if $\frac{\partial K}{\partial y} = 0$, then (2.1) is exact. If $\frac{\partial K}{\partial y} \neq 0$, multiply (2.1) by an integrating factor $\mu(x,y)$, the equation (2.1) becomes

$$\mu\left((1-\beta)\frac{\partial K}{\partial y} - 1\right) y'(x) + \mu\left((1-\beta)\frac{\partial K}{\partial x} + \beta K\right) = 0.$$

Since μ is an integrating factor, then

$$\frac{\partial}{\partial x}\left(\mu\left((1-\beta)\frac{\partial K}{\partial y} - 1\right)\right) = (1-\beta)\mu\frac{\partial^2 K}{\partial x \partial y} + \frac{\partial \mu}{\partial x}\left((1-\beta)\frac{\partial K}{\partial y} - 1\right),$$

and

$$\frac{\partial}{\partial y}\left(\mu\left((1-\beta)\frac{\partial K}{\partial x} + \beta K\right)\right) = (1-\beta)\mu\frac{\partial^2 K}{\partial y \partial x} + \beta\mu\frac{\partial K}{\partial y} + \frac{\partial \mu}{\partial y}\left((1-\beta)\frac{\partial K}{\partial x} + \beta K\right).$$

Therefore, μ satisfies the PDE

$$\left((1-\beta)\frac{\partial K}{\partial y} - 1\right)\frac{\partial \mu}{\partial x} - \left((1-\beta)\frac{\partial K}{\partial x} + \beta K\right)\frac{\partial \mu}{\partial y} = \beta\mu\frac{\partial K}{\partial y}.$$

The characteristic equations for this PDE are

$$\frac{dx}{ds} = \left((1-\beta)\frac{\partial K}{\partial y} - 1\right), \quad \frac{dy}{ds} = -\left((1-\beta)\frac{\partial K}{\partial x} + \beta K\right), \quad \frac{d\mu}{ds} = \beta\mu\frac{\partial K}{\partial y}. \tag{2.3}$$

Example 2.2. In this example, we will solve the nonlinear Caputo-Fabrizio fractional differential equation

$$\left(D^\beta y\right)(x) = Ae^{rx+\nu y}, \; 0 < r < 1, \; \nu \in \mathbf{R}. \tag{2.4}$$

Applying (2.1), the equation (2.4) is equivalent to the ODE

$$\left(A\nu(1-\beta)e^{rx+\nu y} - 1\right) y'(x) + \left(Ar + (1-r)A\beta\right) e^{rx+\nu y} = 0. \tag{2.5}$$

If μ is an integrating factor for (2.5), then the corresponding characteristic equations are

$$\frac{dx}{ds} = \left(A\nu(1-\beta)e^{rx+\nu y} - 1\right), \quad \frac{dy}{ds} = -\left(Ar + (1-r)A\beta\right)e^{rx+\nu y}, \quad \frac{d\mu}{ds} = A\nu\beta\mu e^{rx+\nu y}.$$

Therefore,

$$\frac{d\mu}{dy} = -\frac{A\nu\beta\mu}{Ar + (1-r)A\beta}. \tag{2.6}$$

By solving (2.6), $\mu(y) = e^{-\gamma y}$ is an integrating factor , where $\gamma = \frac{A\nu\beta}{Ar+(1-r)A\beta}$. It is easy to notice that $\nu - \gamma = \frac{A\nu r(1-\beta)}{Ar+(1-r)A\beta}$. Multiplying (2.5) by the integrating factor $\mu(y) = e^{-\gamma y}$ implies

$$\left(A\nu(1-\beta)e^{rx+(\nu-\gamma)y} - e^{-\gamma y}\right) y'(x) + \left(Ar + (1-r)A\beta\right) e^{rx+(\nu-\gamma)y} = 0 \tag{2.7}$$

is exact. Solving (2.7), the general solution of (2.4) is $\gamma\left(Ar + (1-r)A\beta\right)e^{rx+\nu y} + re^{-\gamma y} = C$.

Example 2.3. In this example, we solve the following generalized form of fractional logistic differential equation

$$D^{\beta}x(t) = x(t)\left(1 - \frac{x(t)}{a}\right)\left(1 - \frac{x(t)}{b}\right). \tag{2.8}$$

If $a = 1$ and $b \to \infty$, (2.8) will be the fractional version of the logistic differential equation

$$D^{\beta}x(t) = x(t)\Big(1 - x(t)\Big) \tag{2.9}$$

that has been studied in [17]. Using (2.1), (2.8) is equivalent to the ODE

$$\left((1-\beta)\left(\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right) - \frac{x}{a}\left(1-\frac{x}{b}\right) - \frac{x}{b}\left(1-\frac{x}{a}\right)\right) - 1\right)\frac{dx}{dt} + \beta x\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right) = 0. \tag{2.10}$$

Equation 2.10 is not exact. (2.3) implies that the characteristics equations corresponding to (2.10) are

$$\begin{aligned}
\frac{dt}{ds} &= \left((1-\beta)\left(\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right) - \frac{x}{a}\left(1-\frac{x}{b}\right) - \frac{x}{b}\left(1-\frac{x}{a}\right)\right) - 1\right),\\
\frac{dx}{ds} &= -\beta x\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right),\\
\frac{1}{\mu}\frac{d\mu}{ds} &= \beta\left(\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right) - \frac{x}{a}\left(1-\frac{x}{b}\right) - \frac{x}{b}\left(1-\frac{x}{a}\right)\right).
\end{aligned}$$

Therefore,

$$\frac{d}{dx}\ln\mu(x) = \frac{1}{\mu}\frac{d\mu}{dx} = -\frac{\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right) - \frac{x}{a}\left(1-\frac{x}{b}\right) - \frac{x}{b}\left(1-\frac{x}{a}\right)}{x\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right)} = -\frac{d}{dx}\left(\ln\left(x\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right)\right)\right).$$

Consequently, $\mu(x) = \frac{1}{x\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right)}$ is an integrating factor for (2.10). Thus, the ODE

$$\frac{(1-\beta)\left(\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right) - \frac{x}{a}\left(1-\frac{x}{b}\right) - \frac{x}{b}\left(1-\frac{x}{a}\right)\right) - 1}{x\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right)}\frac{dx}{dt} + \beta = 0 \tag{2.11}$$

is exact. By partial fractions decomposition, (2.11) can be rewritten as

$$\left(\frac{(1-\beta)\left(\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right) - \frac{x}{a}\left(1-\frac{x}{b}\right) - \frac{x}{b}\left(1-\frac{x}{a}\right)\right)}{x\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right)} + \frac{1}{x} + \frac{\frac{1}{a(1-\frac{a}{b})}}{1-\frac{x}{a}} + \frac{\frac{1}{b(1-\frac{b}{a})}}{1-\frac{x}{b}}\right)dx = -\beta dt. \tag{2.12}$$

Integrating both sides of (2.12) we get

$$(1-\beta)\ln\left(x(1-\frac{x}{a})(1-\frac{x}{b})\right)+\ln(x)-\frac{1}{1-\frac{a}{b}}\ln(1-\frac{x}{a})-\frac{1}{1-\frac{b}{a}}\ln(1-\frac{x}{b})=-\beta t+C. \tag{2.13}$$

Equation (2.13) implies the general solution of FDE (2.8) is

$$x^{2-\beta}\left(1-\frac{x}{a}\right)^{1-\beta-\frac{1}{1-\frac{a}{b}}}\left(1-\frac{x}{b}\right)^{1-\beta-\frac{1}{1-\frac{b}{a}}}=Ce^{-\beta t},$$

when if $a=1$ and $b\to\infty$ the solution of (2.9) is $x^{2-\beta}\,(1-x)^{-\beta}=Ce^{-\beta t}$. Therefore, the solution of (2.9) is $\frac{x-x^2}{(1-x)^{\frac{2}{\beta}}}=Ce^{t}$, which coincides with the result in [17].

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